Representation Theory of the Symmetric Group, by G. de B. Robinson. Toronto, University of Toronto Press, 1961. viii + 204 pages. $6.00.

The symmetric groups provide a display window for the representation theory of finite groups. Their theory is intricate and fascinating, and its original methods and outlook may well lead to future developments in the general theory of groups. There are applications to other parts of mathematics and physics. Professor Robinson has ample reason, and qualifications, for writing on the subject. His book goes beyond the previous ones in treating both modular and ordinary theories. In spirit it owes much to A. Young's fundamental work.

The initial exposé of general representation theory (Ch. I) is often terse, giving the impression of haste to get on. There is one error (the account of reduction (mod $\mathfrak{p}$) on p. 22) and some oversights (the modular irreducibles cannot be as neatly parcelled down the diagonal as 12.11 indicates; the mapping on p. 7, l. 3 is not an anti-automorphism as claimed). Such lapses are unfortunate because they may deter the reader from going on to the later, much better, chapters. It remains true, however, that throughout the book a number of explanations are rather unclear or over-concise. This is partly due to the combinatorial nature of the arguments, which are difficult to communicate in words. In this regard, the many numerical illustrations are very helpful, often forming an essential supplement to the formal proof.

Chapters II, III are concerned with the ordinary representation theory of S_n. This is developed on the basis of two theorems of Young, one giving the semi-normal form for the irreducible representations and the other the analysis of certain natural permutation representations into irreducible parts. The marvellous aptness of the Young diagrams (both "right" and "skew") in this work is clearly put into evidence; indeed it is one of the main themes of the book. There is a rather sketchy account of the relation between the representation of S_n and those of the full linear group GL(n). Various combinations of representations which arise from this relation (Kronecker product, "outer products", plethysm of S-functions, etc.) are treated in some detail.

The remaining four chapters deal essentially with the modular representation theory of S_n and are perhaps the most interesting and novel in the book. Let p be a fixed prime. A given ordinary representation R of S_n can be transformed so that the coefficients

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in the representing matrices are all rational integers. Replacing these coefficients by their residues (mod p), one gets the corresponding (p-)modular representation $\tilde{R}$ of S_n over the Galois field GF(p). Suppose now that $R_1, \ldots, R_u$ are the ordinary irreducible representations, $S_1, \ldots, S_v$ the modular irreducible representations. Then

$$(1) \qquad \tilde{R}_i = \sum_j d_{ij} S_j$$

in the sense of the reduction of representations, and $D = (d_{ij})$ is called the decomposition matrix. For a suitable arrangement of the indices,

$$D = D_1 \dotplus \ldots \dotplus D_t ,$$

where no D_i admits a further direct decomposition. The representations, both ordinary and modular, corresponding to the rows and columns of D_i form the "i-th block" and D_i is the decomposition matrix of this block. The problems dealt with in this book are (a) the determination of the blocks and (b) the calculation of their decomposition matrices.

The solution to (a), conjectured by Nakayama and proved by Brauer and the author, is simple and striking. Two ordinary representations are in the same block if, and only if, their Young diagrams yield the same "p-core" on successive removal of "p-hooks"; further, the number of modular representations in a block is the number of ordinary representations in the block whose diagrams have no p rows of equal length. E.g., the removal of 3-hooks indicated below shows that the ordinary representations (4, 3, 2, 1) and $(4, 3^2)$ of S_{10} belong to the same 3-block:

The proofs of these results are anything but simple, and the combinatorial considerations needed are extremely interesting. Just as the "raising" and "lowering" of nodes in diagrams govern the ordinary theory, so here the raising and lowering of nodes "of a given residue class (mod p)" play a crucial part. An interesting point is that many of the combinatorial arguments do not depend on the primeness of p, and one wonders what this means in terms of representation theory.

The solution to problem (b), due to the author, Johnson and Taulbee, is less easy to describe though still expressible in terms of an algorithm on the diagrams. The proof is formidable and is achieved by actually carrying out the reduction (1) on Young's semi-normal form. Although the general picture is convincing, I was unable to follow some parts of the proof (e.g., the proof of indecomposability on p. 162). Moreover, the solution is incorrect for $p = 2$, as Professor Robinson points out in his note below. Further research is needed here to make the methods more perspicuous and to clarify the case $p = 2$.

There is a good bibliography and a useful Appendix tabulating the D-matrices for $p = 2, 3$, $n \leq 10$.

I found this book rather difficult, sometimes exasperatingly so, but finally rewarding. It contains much essential wisdom on its subject and should be closely studied by those interested in representation theory.

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Professor Robinson has written to the reviewer as follows:

"I have recently noticed that the proofs of Theorems 8.36 and 8.41 are not valid for $p = 2$, since the last column of the matrix 8.262 reduces to yield zeros above the $-1 \equiv 1 \pmod 2$ while the third column does not. Thus table 2-7 for core [1] should read

	1	13	20
[7]	1		
[5, 2]	1	1	
$[5, 1^2]$	2	1	
[4, 2, 1]	2	1	1
$[3^2, 1]$	1	0	1
$[3, 2^2]$	1	0	1
$[3, 2, 1^2]$	2	1	1
$[3, 1^4]$	2	1	
$[2^2, 1^3]$	1	1	
$[1^7]$	1		

in agreement with Osima (Can. J. Math. 6(1954), p. 518).

Corresponding changes should be made in tables 2-8, 2-9, 2-10."