INTERPOLATING SEQUENCES FOR THE DERIVATIVES OF BLOCH FUNCTIONS

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0. Abstract. We prove that sufficiently separated sequences are interpolating sequences for $f'(z)(1-|z|^2)$ where f is a Bloch function. If the sequence $\{z_n\}$ is an η net then the boundedness of $f'(z)(1-|z|^2)$ on $\{z_n\}$ is a sufficient condition for f to be a Bloch function. The essential norm of a Hankel operator with a conjugate analytic symbol $\bar{f}$ acting on the Bergman space is shown to be equivalent to $\limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2)$.

1. Introduction. Let D denote the open unit disc in the complex plane C and let L_a^p denote the collection of holomorphic functions f on D for which $\|f\|_p^p = \int |f|^p\, dA$ is finite. Here and elsewhere dA denotes the Lebesgue area measure on D and the unadorned integral is always taken over D. As usual H^∞ will denote the space of bounded analytic functions on D. For $w \in D$ let $\phi_w(z) = (w-z)(1-\bar{w}z)^{-1}$, $z \in D$ be the Möbius map taking w to 0. For $z, w \in D$ the pseudo-hyperbolic distance between z and w is defined by $\rho(z,w) = |\phi_w(z)|$. For $0<\eta<1$, let $D_w(\eta)$ denote the pseudo-hyperbolic disc of center w and radius η. A sequence $\{z_n\}$ in D is called η separated if $D_{z_n}(\eta) \cap D_{z_m}(\eta) = \varnothing$ whenever $n \neq m$ and $\{z_n\}$ is said to be η dense if $D = \cup D_{z_n}(\eta)$.

A holomorphic function f on D is called a *Bloch function* if $\sup_{z\in D} |f'(z)|\,(1-|z|^2)$ is finite. The space of Bloch functions together with the norm

$$\|f\| = |f(0)| + \sup_{z \in D} |f'(z)|\,(1-|z|^2)$$

is denoted by $\mathscr{B}$. An important and useful property of Bloch functions is the Möbius invariance of the norm:

$$\|f \circ \phi_w - f(w)\| = \|f - f(0)\|, \qquad f \in \mathscr{B}.$$

The "little Bloch space" $\mathscr{B}_0$ is the subspace of $\mathscr{B}$ consisting of functions f such that $f'(z)(1-|z|^2) \to 0$ as $|z| \to 1$. It is not hard to verify that $\mathscr{B}_0$ is the closure of the polynomials in $\mathscr{B}$.

A sequence $\{z_n\}$ in D is called an *interpolating sequence for the drivatives of Bloch functions* (or simply an *interpolating sequence*) if for each bounded sequence of complex numbers c_n, $f'(z_n)(1-|z_n|^2) = c_n$, $n = 1, 2, \ldots$, for some Bloch function f. One may note in passing that $f'(z)(1-|z|^2)$ is the derivative of $f: D \to C$ when D is endowed with the pseudo-hyperbolic metric. We will show that a sequence $\{z_n\}$ is interpolating only if it is *sufficiently* separated and that interpolating sequences are necessarily separated. As an application we will estimate the distance from a Bloch function f to the little Bloch space and then we will estimate the essential norm of a Hankel operator (see Section 3 for the definition of this Hankel operator) from the Bergman space L_a^2 to $L^2(D, dA)$.

2. Interpolating sequences. We begin by organizing in Proposition 1 some well known facts about Bloch functions.

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PROPOSITION 1. *For a holomorphic function f on D the following are equivalent*:
(1) $f \in \mathcal{B}$;
(2) $|f \circ \phi_w(z) - f(w)| \leq c \log \frac{(1+|z|)}{(1-|z|)}$ *for some constant c independent of both z and w*;
(3) $f \in L_a^2$ *and* $\sup_{w \in D} \int |f \circ \phi_w - f(w)|^2 \, dA$ *is finite*;
(4) $\{f \circ \phi_w - f(w)\}_{w \in D}$ *is a normal family.*

Proof (sketch). Condition 2 is the usual point estimate for Bloch functions. The claim $1 \Rightarrow 2$ follows from an integration of the growth condition of f' and the Möbius invariance of the Bloch norm. Clearly $2 \Rightarrow 3$. The assertion $3 \Rightarrow 4$ follows from the general result that any norm bounded family in L_a^2 is uniformly bounded on compact subsets of the disc, and therefore, by Montel's theorem, is a normal family. Finally $4 \Rightarrow 1$ is a theorem of Pommerenke [**8**].

The dual of the Bergman space L_a^1 can be identified with the Bloch space. There are several versions of the pairing of this identification [**2**, Theorem 2.4]; [**4**, Theorem 2.6]; [**5**, Lemma 5.1]; here is the version that we will use in this note:

$$\langle f, g \rangle = \int f(z)\bar{g}'(z)(1-|z|^2)\, dA(z), \qquad f \in L_a^1 \quad \text{and} \quad g \in \mathcal{B}.$$

The little Bloch space is the pre-dual of the Bergman space L_a^1 [**2**, Theorem 2.4] with the pairing

$$\langle f, g \rangle = \int f'(z)(1-|z|^2)\bar{g}(z)\, dA(z) \qquad f \in \mathcal{B}_0 \quad \text{and} \quad g \in L_a^1.$$

None of the above identifications are isometries; they are only Banach space isomorphisms.

To derive a necessary condition for a sequence to be an interpolating sequence, we first prove a Schwarz–Pick lemma type inequality for Bloch functions.

LEMMA 2. *Let $f \in \mathcal{B}$. Then*

$$|f'(z)(1-|z|^2) - f'(w)(1-|w|^2)| \leq 9\, \|f\|\, \rho(z, w), \qquad z \text{ and } w \in D.$$

Proof. Fix $f \in \mathcal{B}$. Then

$$|f''(z)|(1-|z|)^2 \leq 4\, \|f - f(0)\|, \qquad z \in D,$$

(see for example [**3**, Lemma 2.1, lines 2.2]). Let $z \in D$ and pick α such that $e^{-i\alpha}z = |z|$. Then

$$\begin{aligned} |f'(z) - f'(0)| &\leq \int_0^{|z|} |f''(te^{i\alpha})|\, dt \\ &\leq 4\, \|f - f(0)\| \int_0^{|z|} (1-t)^{-2}\, dt \\ &\leq 4\, \|f - f(0)\|\, |z|(1-|z|)^{-1} \end{aligned}$$

so that

$$|f'(z)(1-|z|^2) - f'(0)| \leq 8\, \|f - f(0)\|\, |z| + \|f - f(0)\|\, |z|^2 \leq 9\, \|f - f(0)\|\, |z|.$$

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Let $w \in D$ and replace f by $f \circ \phi_w$. Recalling the Möbius invariance of the Bloch norm and that $|\phi_w'(z)|\,(1-|z|^2) = 1 - |\phi_w(z)|^2$, we have

$$|f' \circ \phi_w(z)(1-|\phi_w(z)|^2) - f'(w)(1-|w|^2)| \le \|f - f(0)\|\,|z|,$$

from which the desired result follows by replacing z by $\phi_w(z)$.

COROLLARY 3. *Let $\{z_n\}$ be an interpolating sequence. Then $\{z_n\}$ is separated.*

Proof. The usual proof for a sequence to be an interpolating sequence for bounded analytic functions [**6**, page 285] is also applicable here. Indeed, let $T: \mathcal{B} \to l^\infty$ be the operator map $T(f) = \{f'(z_n)(1-|z_n|^2)\}$. Clearly T is bounded and if $\{z_n\}$ is interpolating, then T is *onto*, so by the open mapping theorem there exists a constant c such that for each sequence $\{a_n\}$ in l^∞,

$$f'(z_n)(1-|z_n|^2) = a_n, \qquad n = 1, 2, \ldots$$

for some $f \in \mathcal{B}$ with $\|f\| \le c\,\|\{a_n\}\|_\infty$. Let n and m be distinct positive integers. Then there exists $f \in \mathcal{B}$ such that $\|f\| \le c$ and $f'(z_n)(1-|z_n|^2) = 1$ but $f'(z_m)(1-|z_m|^2) = 0$. Now by Lemma 2 clearly the sequence $\{z_n\}$ is separated.

We now prove the main proposition of this section. It asserts that if a sequence is sufficiently separated then it is an interpolating sequence. The author wishes to thank the referee for suggesting the following statement of the proposition.

PROPOSITION 4. *There exists a constant c_1 such that, given any η separated sequence $\{z_n\}$ and any $\{c_n\} \in l^\infty$, there exists an $f \in \mathcal{B}$ such that $f'(z_n)(1-|z_n|^2) = c_n$, $n = 1, 2, \ldots$, and $\|f\| \le c_1 c(\eta)(1 - c(\eta)o(1-\eta))^{-1}\,\|c_n\|_\infty$, where $c(\eta) = (1+\eta)^4\pi^{-1}\eta^{-2}$ and $o(1-\eta) \to 0$ as $\eta \to 1-$. Moreover, such an f must have norm $\ge \|c_n\|_\infty$.*

Proof. Let $\{z_n\}$ be a η, $0 < \eta < 1$, separated sequence. Define the operator map $T: L_a^1 \to l^1$ by $T(f) = \{f(z_n)(1-|z_n|^2)^2\}$. Then T is bounded with $\|T\| \le (1+\eta)^4\pi^{-1}\eta^{-2}$ (see for example, [**7**, page 96, line 4.5]). The adjoint $T^*: l^\infty \to \mathcal{B}$ is given by

$$(T^*(\{a_n\}))' = \sum_n a_n \frac{(1-|z_n|^2)^2}{(1-\bar{z}_n z)^3}.$$

Let S be the operator defined on l^∞ by

$$S(\{a_n\}) = \{(T^*(\{a_n\}))'(z_k)(1-|z_k|^2)\}.$$

Let I be the identity operator on l^∞. Then the kth term of $(S-I)(\{a_n\})$ is

$$\sum_{n \ne k} a_n (1-|z_n|^2)^2 (1-|z_k|^2)(1-\bar{z}_n z_k)^{-3}.$$

Let us temporarily write $D_n(\eta)$ for $D_{z_n}(\eta)$ and ϕ_n for ϕ_{z_n}. The following estimates in (1) and (2) are well known and the calculations presented are standard:

$$(1+\eta)^4(1-|z_n|^2)^2 \le \text{area}(D_n) \le (1-\eta)^{-4}(1-|z_n|^2)^2 \tag{1}$$

and, if f is analytic on D, then

$$\begin{aligned}\int_{D_n(\eta)} |f(z)|\, dA(z) &= \int_{|z|<\eta} |f\circ\phi_n(z)|\,|\phi_n'(z)|^2\, dA(z)\\ &\geq (1+\eta)^{-4}(1-|z_n|^2)^2 \int_{|z|<\eta} |f\circ\phi_n(z)|\, dA(z)\\ &\geq \pi\eta^2(1+\eta)^{-4}(1-|z_n|^2)^2\,|f(z_n)|. \end{aligned} \tag{2}$$

Thus writing $c(\eta)=(1+\eta)^4/\pi\eta^2$ and applying (2) to $f(z)=(1-\bar{z}_k z)^{-3}$, $z\in D$, we have

$$\begin{aligned}\|(S-I)(\{a_n\})\|_\infty &\leq c \sup_k \sum_{n\neq k} |a_n|(1-|z_k|^2)\int_{D_n(\eta)} |1-\bar{z}_k z|^{-3}\, dA(z)\\ &\leq c\,\|\{a_n\}\|_\infty \sup_k (1-|z_k|^2)\int_{D-D_k(\eta)} |1-\bar{z}_k z|^{-3}\, dA(z).\end{aligned}$$

By the change of variable z to $\phi_{z_k}(z)$ we get the last integral to be equal to

$$\begin{aligned} & c\,\|\{a_n\}\|_\infty \sup_k \int_{|z|>\eta} |1-\bar{z}_k z|^{-1}\, dA(z)\\ &= c\,\|\{a_n\}\|_\infty \sup_k \int_\eta^1 \left(\int_0^{2\pi} |(1-|z_k|\, re^{i\theta})|^{-1}\, d\theta/2\pi\right) r\, dr\\ &\leq c\,\|\{a_n\}\|_\infty \int_{|z|>\eta} |1-z|^{-1}\, dA(z).\end{aligned}$$

In deducing the last inequality we used the fact that the mean modulus of an analytic function on the circle $|z|=r$ is an increasing function of r. Thus when η is sufficiently close to 1, S is invertible on l^∞, and $\|S^{-1}\|\leq(1-\|I-S\|)^{-1}$. Given $\{c_n\}\in l^\infty$, write $f=T^*(\{a_n\})$ where $f'(z_n)(1-|z_n|^2)=S(\{a_n\})=c_n$. Then, clearly $\|f\|\geq\|\{c_n\}\|_\infty$ and

$$\|f\|\leq\frac{\|T^*\|}{(1-\|I-S\|)}\|\{c_n\}\|_\infty,$$

from which the proposition follows.

Corollary 5. *Every separated sequence is the finite union of interpolating sequences.*

Proof. The decomposition given in [**1**, page 718], shows that a separated sequence is the finite union of sequences which are sufficiently separated to be interpolating sequences.

Let c_0 denote the space of all null sequences. A sequence $\{z_n\}$ is called an interpolating sequence for the derivative of functions in $\mathcal{B}_0$ if for every sequence $\{c_n\}$ in c_0 there exists a function f in $\mathcal{B}_0$ such that $f'(z_n)(1-|z_n|^2)=c_n$, $n=1,2,\ldots$.

Corollary 6. *If a sequence is sufficiently separated, then it is an interpolating sequence for the derivatives of functions in $\mathcal{B}_0$.*

Proof. Let $\{z_n\}$ be an interpolating sequence. Then we will show that it also has the claimed interpolating property. Define the operator map $T:\mathcal{B}_0\to c_0$ by $T(f)=\{f'(z_n)(1-|z_n|^2)\}$. Then $T^*:l^1\to L_a^1$ and $T^{**}:\mathcal{B}\to l^\infty$. The operator T^{**} can be verified

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to be the natural extension of T, thus T^{**} is *onto*. In particular T^{**} has closed range, so T has closed range. Now a standard argument involving the annihilator of the range of operators and the kernels of the adjoints shows that T is *onto*.

Since sufficiently separated sequences are interpolating, a sequence close enough to a given sequence, which is sufficiently separated to be interpolating, is also an interpolating sequence. The following proposition shows that this is true for *any* interpolating sequence, whether it is sufficiently separated or not. (See [7, Theorem 5.1] for a similar theorem.)

PROPOSITION 7 (Stability of Interpolating Sequences). *Suppose $\{z_n\}$ is an interpolating sequence. Then there exists a number δ depending on the sequence $\{z_n\}$ such that if $\rho(z_n, w_n) < \delta$, $n = 1, 2, \ldots$, then the sequence $\{w_n\}$ is also an interpolating sequence.*

Proof. Let $\{z_n\}$ be an interpolating sequence. Then the operator $T:\mathcal{B}\to l^\infty$ defined by $T(f) = \{f'(z_n)(1-|z_n|^2)\}$ is *onto*, so $T^*:(l^\infty)^*\to\mathcal{B}^*$ is one–one and (since T has closed range) has closed range. Therefore T^* is left-invertible. Recall that the left-invertible elements in a Banach algebra form an open set. Let $S:\mathcal{B}\to l^\infty$ be an operator. Now by a standard argument one can show that there exists a number $\delta = \delta(T)$ such that if $\|T - S\| < \delta$, then S is also *onto*. Let $\{w_n\}$ be a sequence such that $\rho(z_n, w_n) < \delta/9$ and let S be the operator $S(f) = \{f'(w_n)(1-|w_n|^2)\}$. By Lemma 2, $\|T - S\| < \delta$, so S is *onto*.

3. Essential norm of Hankel operators. A sequence which is η/a separated and $a\eta$ dense, for some $a > 0$, is called an η net. When computing the Bloch norm it suffices to sup only over a given η net as the following proposition shows. But first we note a well known lemma which is the Paley-Littlewood formula in the context of the Bergman space.

LEMMA 8. *Let f be a holomorphic function on D with $f(0) = 0$. Then*

$$\tfrac{1}{3}\|f\|_2^2 \le \|f'(z)(1-|z|^2)\|_2^2 \le \|f\|_2^2.$$

Proof. Compute the norms.

PROPOSITION 9. *Let $\{z_n\}$ be an η net. A holomorphic function on D is a Bloch function if and only if*

$$\sup_n |f'(z_n)|\,(1-|z_n|^2) < \infty.$$

Proof. Suppose $\sup_n |f'(z_n)|\,(1-|z_n|^2) = \sqrt{M} < \infty$. Let $w \in D$. Then

$$\begin{aligned}\int |f\circ\phi_w - f(w)|^2\,dA &\le 3\int |(f\circ\phi_w)'(z)|^2\,(1-|z|^2)^2\,dA(z) \quad \text{(by Lemma 8)}\\ &= 3\int |f'(\phi_w(z))|^2\,|\phi_w'(z)|^2\,(1-|z|^2)^2\,dA(z)\\ &= 3\int |f'(z)|^2\,(1-|\phi_w(z)|^2)^2\,dA(z)\end{aligned}$$

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$$\leq c \sum |f'(z_n)|^2(1-|z_n|^2)^2(1-|\phi_w(z_n)|^2)^2$$

(by [7, Theorem 4.1])

$$\leq cM \sum (1-|\phi_w(z_n)|^2)^2.$$

Write $t_n = \phi_w(z_n)$. Then, by the conformal invariance of the pseudo-hyperbolic metric, $\{t_n\}$ is also a η/a separated sequence. Thus, for some constant $c = c(\eta/a)$,

$$\sum (1-|t_n|^2)^2 \leq c \sum \text{area}(D_n(\eta/a)) \leq c\,\text{area}(D).$$

Hence by Proposition 1, we get f to be a Bloch function.

We can now estimate the distance from a Bloch function to the little Bloch space.

PROPOSITION 10. *Let f be a Bloch function. Then, for some constant c,*

$$\limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2) \leq \text{dist}(f, \mathcal{B}_0) \leq c \limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2).$$

Proof. The left-hand side of the inequality follows immediately from the definitions of $\mathcal{B}$ and $\mathcal{B}_0$. To prove right-hand side of the inequality, fix an η net which is sufficiently separated to be an interpolating sequence. Let n be a positive integer. Then by Corollary 6, there exists a $g \in \mathcal{B}_0$ such that $g'(z_j)(1-|z_j|^2) = f'(z_j)(1-|z_j|^2)$ if $1 \leq j \leq n$, and $g'(z_j)(1-|z_j|^2) = 0$ otherwise. Then by Proposition 9,

$$\|f-g\| \leq c \sup_{j>n} |f'(z_j)|\,(1-|z_j|^2),$$

from which the desired result follows easily.

Let $\langle f, g\rangle_2 = \int f\bar{g}\,dA/2\pi$ denote the usual inner product in the Hilbert space $L^2(D, dA)$. Let $(L_a^2)^\perp$ denote the orthogonal complement of L_a^2 in $L^2(D, dA)$ and let Q denote the orthogonal projection of $L^2(D, dA)$ to $(L_a^2)^\perp$. For a function $f \in L_a^2$ the densely defined operator from L_a^2 to $(L_a^2)^\perp$

$$H_{\bar{f}}(g) = Q(\bar{f}g), \qquad g \in H^\infty$$

is called a Hankel operator. In a 1986 paper Sheldon Axler [2] proved that the operator norm $\|H_{\bar{f}}\|$ is equivalent to the Bloch norm of f and that $H_{\bar{f}}$ is compact if and only if $f \in \mathcal{B}_0$. After all the hard work of Axler, in the following proposition we estimate the essential norm $\|H_{\bar{f}}\|_e$ of this Hankel operator.

PROPOSITION 11. *Let $f \in \mathcal{B}$. Then, for some constant c,*

$$c^{-1} \limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2) \leq \|H_{\bar{f}}\|_e \leq c \limsup_{|z|\to 1} |f'(z)|\,(1-|z|^2).$$

Proof. Let $g \in \mathcal{B}_0$, then $H_{\bar{g}}$ is compact [2, Theorem 7], so by Axler's theorem [2, Theorem 6], we have,

$$\|H_{\bar{f}}\|_e \leq \|H_{\bar{f}} - H_{\bar{g}}\| \leq c\,\|f-g\|.$$

Thus, $\|H_{\bar{f}}\|_e \leq c\,\text{dist}(f, \mathcal{B}_0)$. Now the right-hand side of the inequality in Proposition 11 follows from Proposition 10. To establish the reverse inequality, consider the kernel

$l_w(z) = (1 - \bar{w}z)^{-3/2}$, $z \in D$. Then

$$\pi(1 - |w|^2)^{-1} \le \|l_w\|_2^2 \le 2\pi(1 + |w|)(1 - |w|^2)^{-1}.$$

Moreover by differentiating the identity

$$h(w) = \pi^{-1} \int h(z)(1 - \bar{z}w)^{-2}\, dA(z), \qquad h \in L_a^2,$$

we have $h'(w) = 2\pi^{-1} \int h\bar{z}l_w^2\, dA$, $h \in L_a^2$.

Let $K : L_a^2 \to (L_a^2)^\perp$ be a compact operator. Since $\{\|l_w\|^{-1} l_w\}$ goes to zero weakly in L_a^2 as $|w| \to 1$, $\{K(\|l_w\|^{-1} l_w)\}$ goes to zero strongly in L_a^2. Thus,

$$\begin{aligned}
\|H_{\bar{f}} - K\| &\ge \limsup_{|w| \to 1} \|H_{\bar{f}}(\|l_w\|_2^{-1}\, l_w)\| \\
&\ge \limsup_{|w| \to 1} \|l_w\|_2^{-2}\, |\langle H_{\bar{f}}(l_w), \bar{z}\bar{l}_w \rangle_2| \\
&= \limsup_{|w| \to 1} \|l_w\|_2^{-2}\, |\langle \bar{f}l_w, Q(\bar{z}\bar{l}_w) \rangle_2| \\
&= \limsup_{|w| \to 1} \|l_w\|_2^{-2}\, |\langle \bar{f}l_w, \bar{z}\bar{l}_w \rangle_2| \\
&\ge 8^{-1} \limsup_{|w| \to 1} |f'(w)|\, (1 - |w|^2),
\end{aligned}$$

completing the proof.

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