BULL. AUSTRAL. MATH. SOC. 20D10, 20D20
VOL. 9 (1973), 239-248.

On saturated formations whose projectors are complemented

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It is shown that a saturated formation F has the property that in each group G each F-projector of G is complemented if and only if F is the formation of finite soluble π-groups for some set π of primes.

All groups considered here are finite and soluble. A *formation* is a class of groups which is closed under taking homomorphisms and subdirect products. A formation F is said to be *saturated* if a group $G \in F$ whenever $G/\Phi(G) \in F$, where $\Phi(G)$ denotes the Frattini subgroup of G. A subgroup F of a group G is called an F-*projector* of G if

(1) $F \in F$ and

(2) $FN = H$ whenever $F \leq H \leq G$, $N \triangleleft H$ and $H/N \in F$.

Gaschütz showed in [5] that if F is saturated, then the F-projectors of G always exist and constitute a single conjugacy class of subgroups of G. Corresponding to a saturated formation F, G also has another canonical conjugacy class of subgroups called the F-*normalizers* of G (see Carter and Hawkes [1]).

In [2] the authors have shown that if F is a saturated formation, then F has the property that in each group G each F-projector has a single conjugacy class of complements if and only if $F = S_{\pi}$, the class of all soluble π-groups, for some set π of primes. In Section 2 of the present note we show that one can drop the assumption that each projector

Received 1 May 1973. The author is indebted to Professor Trevor Hawkes for his encouragement and also for many stimulating discussions.

has a single conjugacy class of complements and assume only that each projector is complemented. As will be seen, our methods yield the same result if we assume that in each group G each F-normalizer of G is complemented.

Throughout F will denote a saturated formation and $\{F(p)\}$ the family of full, integrated formations, one for each prime p, which defines F locally (see Section 2 in Doerk [4]). Moreover, π will always denote the set of all those primes p such that $F(p) \neq \emptyset$, the empty set. For convenience, F will be said to have *the property* C if in each group G each F-projector of G is complemented.

1. Some lemmas

In this section, we will prove four elementary lemmas which we require in the next section for the proof of our main result. We begin with the following

LEMMA 1.1. *Let G be a group with a unique minimal normal subgroup N which is complemented in G, and let $|N| = p^{\alpha}$, $\alpha > 0$, p a prime number. Then there exists a faithful $GF(p)[G]$-module W with the properties:*

(i) W has a unique irreducible submodule U;

(ii) U is trivial for G; and

(iii) U_N, the restriction of U to N, is the unique irreducible submodule of W_N.

Proof. Let K be a complement of N in G, U the trivial, irreducible $GF(p)[G]$-module and $W = (U_K)^G$. Since $\mathrm{core}_G(K) = 1$, W is clearly a faithful $GF(p)[G]$-module. Also, by a result of Mackey (see Satz V.16.9 in Huppert [6]), W_N is the regular $GF(p)[N]$-module. Hence, W_N has a unique irreducible submodule V, say. In fact, $V = U_N$. For, by a result of Nakayama (see Satz V.16.6 (a) in Huppert [6]) and the irreducibility of U, U is a submodule of W. Thus, since V is the unique irreducible submodule of W_N, we have $U_N = V$. The rest of the lemma now follows.

Next, we show

LEMMA 1.2. *Suppose* $\mathfrak{F}$ *has the property* C *and let* $H \in \mathfrak{F}$. *If* V *is a faithful, irreducible* $GF(r)[H]$*-module for some* $r \in \pi$ *and if* $(r, |H|) = 1$, *then the semidirect product* $K = [V]H$ *of* V *by* H *belongs to* $\mathfrak{F}$.

Proof. Suppose to the contrary that $K \notin \mathfrak{F}$. Then, since V is a minimal normal subgroup of K , clearly H is an $\mathfrak{F}$-projector of K . Also, $H \notin \mathfrak{F}(r)$. Let $L = H^{\mathfrak{F}(r)}$, the $\mathfrak{F}(r)$-residual of H . Since V is the unique minimal normal subgroup of K which is complemented in K and has order r^{β} , $\beta > 0$, K has, by Lemma 1.1, a faithful $GF(r)[K]$-module W with the properties:

(i) W has a unique irreducible submodule U ;

(ii) U is trivial for K ; and

(iii) U_V is the unique irreducible submodule of W_V .

Thus, if $M = [W]K$ and $C = C_W(L)$, then clearly $C \geq U > 1$ and U is the socle of WV . Moreover, by Hilfssatz 2.6 in Doerk [4] and by Theorem 5.6 of Carter and Hawkes [1], $F = CH$ is an $\mathfrak{F}$-projector as well as an $\mathfrak{F}$-normalizer of M . Hence, also $C < W$ since otherwise $H \cong M/WV \in \mathfrak{F}(r)$, WV being the intersection of the centralizers of the r-chief factors of M , and we have a contradiction. Let X be a complement of F in M . Since X is an r-subgroup of M and since WV is the normal Sylow r-subgroup of M , $X \leq WV$. Therefore, since F avoids WV/W , X covers the latter. Also, since $C < W$, we have $|X| > |V|$, and so, $X \cap W > 1$. But then $X \cap W \geq U$ since U is the unique minimal normal subgroup of WV and $X \cap W$ is V-invariant. Hence $X \cap F \neq 1$, contrary to X being a complement of F . With this contradiction the lemma is proved.

The following lemma is a consequence of Lemma 3.2 in [7], but for the sake of completeness we will prove it here.

LEMMA 1.3. *Let* G *be a group whose Fitting subgroup* F *is an elementary abelian* p*-group and let* S *be an elementary abelian* *s-subgroup of* G *of order* s^n *,* $n > 0$ *, where* $s \neq p$ *. Then* $|F : C_F(S)| \geq p^{nm}$ *, where* m *is the smallest positive integer such that* $p^m \equiv 1(s)$ *.*

Proof. We will prove this lemma by induction on n . Since $C_G(F) = F$, we observe first that $C_F(S) < F$. Thus if $n = 1$, the lemma is certainly true in view of Maschke's Theorem and Satz II.3.10 in Huppert [6]. Hence assume $n > 1$ and let T be a maximal element in the set

$$\{S^* \mid S^* < S \text{ and } C_F(S) < C_F(S^*)\} .$$

By Lemma 3.2 in [7], S/T is cyclic of order s . Hence, $|T| = s^{n-1}$, and so, by the induction hypothesis, $|F : C_F(T)| \geq p^{(n-1)m}$. But then, since $|C_F(T) : C_F(S)| \geq p^m$ as before, we already have $|F : C_F(S)| \geq p^{nm}$, as required.

Finally, we show

LEMMA 1.4. *Let* G *be a group,* $M \triangleleft G$ *,* $M \leq Z(G)$ *, the centre of* G *, and* H *a permutation group on the finite set* Ω *. Let* $W = G \sim H$ *, the wreath product of* G *by* H *according to the given permutation representation of* H *, and let* D *be the subgroup of the base group* B *of* W *generated by all elements* $(f, 1)$ *such that* $f(i) = \left(f(i+1)\right)^{-1} \in M$ *for some* $i \in \Omega$ *and* $f(k) = 1$ *if* $k \in \Omega$ *and* $i \neq k \neq i+1$ *. Then* $D \triangleleft W$ *and* B/D *is isomorphic to the central product with respect to* M *of* $|\Omega|$ *copies of* G *.*

Proof. Let $\Omega = \{1, 2, \ldots, n\}$ and, for each $i = 1, 2, \ldots, n-1$, let D_i be the subgroup of B consisting of all elements $(f, 1)$, where $f(i) = \left(f(i+1)\right)^{-1}$ and $f(k) = 1$ if $k \in \Omega$ and $i \neq k \neq i+1$. Then $D = D_1 \times D_2 \times \ldots \times D_{n-1}$, and, for each $i = 1, \ldots, n-1$, $D_i \cong M$. Let $(f, 1) \in D_i$, $(e, h) \in H \leq W$, and $(f, 1)^{(e,h)} = (g, 1)$, where $g(j) = f\left(j^{h^{-1}}\right)$ for each $j \in \Omega$. Clearly,

https://doi.org/10.1017/S0004972700043136 Published online by Cambridge University Press

$$(g, 1) = (f_l, 1)(f_{l+1}, 1) \ldots (f_{k-1}, 1) ,$$

where $l = \min\{i^h, (i+1)^h\}$, $k = \max\{i^h, (i+1)^h\}$ and $(f_m, 1) \in D_m$ for $m = l, l+1, \ldots, k-1$. Moreover, $f_l(l) = f(i)$ or $f(i+1)$ according as $l = i^h$ or $l = (i+1)^h$, and $f_m(m) = (f_{m+1}(m+1))$ for $m = l, l+1, \ldots, k-1$. Thus, D is normalized by H . Since also $D \leq Z(B)$, it follows now that $D \triangleleft W$. Finally, one can easily check that $G_i \cong G_iD/D$ for each $i \in \Omega$, and, furthermore, if M^Ω is the direct product of n copies of M in B then $M^\Omega = DM_i$ for each $i \in \Omega$ and $G_iD \cap G_jD = D$ for each pair $i, j \in \Omega$, $i \neq j$. This final remark completes the proof.

2. The main result

In this last section we will prove our main result of this paper, namely:

THEOREM. F *has the property* C *if and only if* $F = S_\pi$.

Proof. Assume first that F has the property C . In order to show $F = S_\pi$ it will suffice to show that $F(p) = F$ for each $p \in \pi$. Suppose to the contrary that $F(p) \subset F$ for some $p \in \pi$ and let $G \in F \backslash F(p)$ be of minimal order. Since $F(p)$ is a formation, G has a unique minimal normal subgroup N , say. Let $|N| = q^\alpha$, $\alpha > 0$. Since $F(p)$ is full, certainly $q \neq p$. We consider now two cases:

Case (a). π is not the set of all primes.

Let s be a prime number not in π and let S_1 be a cyclic group of order s . Since $s \notin \pi$ and $G \in F$, we have $s \nmid |G|$ and, in particular, $s \neq p$. Let $H = S_1 \times G$, the direct product of S_1 and G . Then H has at most two distinct minimal normal subgroups. Also, $O_p(H) = 1$, where $O_p(H)$ is the largest normal p-subgroup of H . Hence, by Hilfssatz 1.3 of Doerk [4], H has a faithful, irreducible $GF(p)[H]$-module M , say. Let $K = [M]H$. Clearly M is the unique minimal normal

p-subgroup of K which is complemented in K. Hence, by Lemma 1.1, there exists a faithful $\mathrm{GF}(p)[K]$-module W with the properties:

(i) W has a unique irreducible submodule W_0,

(ii) W_0 is trivial for K, and

(iii) $(W_0)_M$ is the unique irreducible submodule of W_M.

Next, let S_2 be another cyclic group of order s and U a faithful, irreducible $\mathrm{GF}(p)[S_2]$-module. Let $A = S_2 \times K$ and let $B = [V]A$, where $V = U \# W$, the outer tensor product of U and W, which is clearly a faithful $\mathrm{GF}(p)[A]$-module (see Section 43 in Curtis and Reiner [3]). Since

$$V_K = \underbrace{W \oplus \ldots\ldots\ldots \oplus W}_{\leftarrow \dim_{\mathrm{GF}(p)}(U) \rightarrow},$$

the direct sum of $\dim_{\mathrm{GF}(p)}(U)$ copies of W, and since $C_W(K) \geq W_0 > 1$, we have $C_V\left(G^{F(p)}\right) = C_V(N) > 1$. Thus, $E = C_V\left(G^{F(p)}\right)G > G$. Also, since $C_M\left(G^{F(p)}\right) = 1$, it follows, by Hilfssatz 2.6 in Doerk [4], that G is an F-projector of K, and, hence, E is an F-projector of B. Let X be a complement of E in B which contains $S = S_1 \times S_2$. We will show that $Y = X \cap VMG$ covers MV/V and $Y \cap V \neq 1$.

Since $VMG \triangleleft B$, clearly $[Y, S] \leq Y$. Moreover, $[Y, S] \leq [YMV, S] \leq MV$. Hence, $[Y, S] \leq MV \cap Y$, and so, $Y = (MV \cap Y)C_Y(S)$. Thus $VMG = YE = (MV \cap Y)C_Y(S)E$. But $C_B(S) = G \times S$. For clearly $C_B(S) \geq G \times S$. However, since

$$V_{S_2} = \underbrace{U \oplus \ldots\ldots\ldots \oplus U}_{\leftarrow \dim_{\mathrm{GF}(p)}(W) \rightarrow}$$

and $C_U\left(S_2\right) = 1$, we have $C_V\left(S_2\right) = 1$ and, therefore, $C_V(S) \leq C_V\left(S_2\right) = 1$. Similarly, since $C_M\left(S_1\right) = 1$, we have $C_M(S) \leq C_M\left(S_1\right) = 1$. Thus, $C_B(S) = G \times S$, as claimed, and hence $VMG = (MV \cap Y)E$. Since $Y \cap E = 1$, an order argument now shows that

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$MV \cap Y = Y$. Therefore, since E avoids MV/V and $E \cap V < V$ (the latter since $V = O_{p'p}(VG)$, the largest normal p-nilpotent subgroup of VG , and $G \notin \mathfrak{F}(p)$) it follows that Y covers MV/V and $Y \cap V \neq 1$, as required. In particular, $Y \cap V$ is a non-trivial $\mathrm{GF}(p)[M]$-submodule of V_M , and so,

$$Y \cap \Big(\underbrace{W_0 \oplus \ldots\ldots \oplus W_0}_{\leftarrow \dim_{\mathrm{GF}(p)}(U) \rightarrow}\Big) \neq \{0\} .$$

But then $Y \cap E \neq 1$ since

$$\Big(\underbrace{W_0 \oplus \ldots\ldots \oplus W_0}_{\leftarrow \dim_{\mathrm{GF}(p)}(U) \rightarrow}\Big) \subseteq C_V\big(G^{\mathfrak{F}(p)}\big) ,$$

and we have arrived at a contradiction. Thus, we must have $\mathfrak{F}(p) = \mathfrak{F}$ in this case.

Case (b). π is the set of all primes.

Let p^β be the order of a Sylow p-subgroup of G . We observe first that $\beta > 0$. For, since $O_p(G) = 1$ and G has a unique minimal normal subgroup, it follows, by Hilfssatz 1.3 of Doerk [4], that G has a faithful, irreducible $\mathrm{GF}(p)[G]$-module U . Thus, if $\beta = 0$, then, by Lemma 1.2, $H = [U]G \in \mathfrak{F}$, and so, $G \cong H/U = H/O_{p'p}(H) \in \mathfrak{F}(p)$, a contradiction. Therefore, $\beta > 0$, as claimed. Next, let r be a prime such that

(i) $(r, |G|) = 1$,

(ii) $r > 2\beta$ and

(iii) the order m of p modulo r is greater than 2β .

Since $O_r(G) = 1$ and G has a unique minimal normal subgroup, it follows, by Hilfssatz 1.3 of Doerk [4] that G has a faithful, irreducible $\mathrm{GF}(r)[G]$-module T . Let $K = [T]G$. By Lemma 1.2, $K \in \mathfrak{F}$. Also, since $O_p(K) = 1$ and K has a unique minimal normal subgroup, another application of Hilfssatz 1.3 of Doerk [4] yields that K has a faithful, irreducible $\mathrm{GF}(p)[K]$-module V .

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Let $L = [V]K$. Since V is the unique minimal normal subgroup of L which is complemented in L and is of p-power order, it follows, by Lemma 1.1, that L has a faithful $GF(p)[L]$-module W with the properties:

(i) W has a unique irreducible submodule M,

(ii) M is trivial for L, and

(iii) M_V is the unique irreducible submodule of W_V.

Let $B = [W]L$ and let $\Omega = \{1, 2, \ldots, n\}$, where $n = |K : G|$. Then K is a primitive subgroup of S_Ω, the symmetric group on Ω. Let $R = B \wr K$, the wreath product of B by K according to the given permutation representation of K and let Y be the subgroup of the base group B^* of R generated by all elements $(f, 1)$ of R such that $f(i) = (f(i+1))^{-1} \in M$ for some $i \in \Omega$ and $f(k) = 1$ if $k \in \Omega$ and $i \neq k \neq i+1$. By Lemma 1.4, $Y \triangleleft R$ and B^*/Y is isomorphic to the central product with respect to M of n copies of B. Let $W^* = W_1 \times \ldots \times W_n \leq B^*$, let $V^* = V_1 \times \ldots \times V_n \leq B^*$ and let $M^* = M_1 \times \ldots \times M_n \leq B^*$. Let D be the diagonal subgroup of B^* and let $A/Y = (W^*V^*DK)/Y \leq R/Y$.

Since $(n, p) = 1$, we have $M^*/Y \leq DY/Y$, and so, $M^*/Y \leq Z(A/Y)$. Thus, if $\overline{F}/Y = Y(\overline{K}\times K)/Y$, where $\overline{K}$ is a subgroup of D isomorphic to K, if E/Y is the $\mathfrak{F}(p)$-residual of $\overline{F}/Y$ and if $C/Y = C_{W^*/Y}(E/Y)$, then $C/Y \geq M^*/Y$, and hence, also $F/Y = (C/Y)(\overline{F}/Y) \geq M^*/Y$. On the other hand, since T is the unique minimal normal subgroup of K which is non-central and since $K \notin \mathfrak{F}(p)$, we have $E/Y \geq TY/Y$, and therefore, since T is transitive on Ω, $C/Y \leq DY/Y$. Similarly, $E/Y \geq \overline{T}Y/Y$, where $\overline{T}$ is the subgroup of $\overline{K}$ isomorphic to T in K, and so, $C_{V^*Y/Y}(E/Y) = 1$. Hence, since $\overline{F}/Y \in \mathfrak{F}$, it follows, by Hilfssatz 2.6 in Doerk [4], that $\overline{F}/Y$ is an $\mathfrak{F}$-projector of $V^*\overline{F}/Y$, and therefore, by the same result, F/Y is an $\mathfrak{F}$-projector of A/Y.

Let X/Y be a complement of F/Y in A/Y. Clearly X/Y is a p-subgroup of A/Y of order $p^\delta|V^*|\cdot|W|^{n-1}/p^{n-1}$, where

$p^\delta = |(W^* \cap DY)/Y : C/Y|$. Since $E/Y \geq \overline{T}Y/Y$, it follows, by Lemma 1.3, applied to $(W^* \cap DY)\overline{T}/Y$, that $p^\delta > p^{2\beta}$. Thus, certainly X/Y does not avoid W_iY/Y for each $i = 1, 2, \ldots, n$. Moreover, since $n > r > 2\beta$ and since $|X/Y \cap W^*/Y| \leq p^\delta |W|^{n-1}/p^{n-1}$, it follows that X/Y covers $\overline{V}_j = W^*(V_1 \times \ldots \times V_j)/W^*(V_1 \times \ldots \times V_{j-1})$ for some j , $1 \leq j \leq n$. Hence, if $Q = [W_jY/Y]\overline{V}_j$, then $X/Y \cap W_jY/Y$ is a non-trivial $\overline{V}_j$-invariant subgroup of Q , and so, $X/Y \cap W_jY/Y \geq M_jY/Y = M^*/Y$ since M_jY/Y is the unique minimal normal subgroup of Q . But $M^*/Y \leq F/Y$, whence $F/Y \cap X/Y > 1$ and we have arrived at a contradiction. Hence, $F(p) = F$ in this case too.

Finally, since, by the well-known results of P. Hall, F has the property C if $F = S_\pi$, the proof of the theorem is complete.

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