Thus $\frac{1}{1\cdot04}+\frac{1}{1\cdot08}+\frac{1}{1\cdot12}+\ldots+\frac{1}{1\cdot32}$

$$=\frac{(\log 1\cdot32-\log 1\cdot04)\times 23}{\frac{\cdot04}{\cdot1}}+\frac{\frac{1}{1\cdot04}+\frac{1}{1\cdot32}}{2}$$

$$=5\cdot957+\cdot4807+\cdot3787=\underline{6\cdot8164}$$

The result correct to 3 places obtained from Tables $=\underline{6\cdot822}$.

We may find the sum of

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\ldots+\frac{1}{10,000}$$

It is $=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}+\frac{1}{10}\left(\frac{1\cdot}{1\cdot1}+\frac{1}{1\cdot2}+\frac{1}{1\cdot3}+\ldots+\frac{1}{1000}\right)$.

But $1+\frac{1}{2}+\ldots+\frac{1}{10}$ from Table of Reciprocals $=2\cdot92897$, and the remaining part $=\frac{1}{10}\left\{23\log 1000-23\log 1\cdot1+\frac{1}{2}\left(\frac{1}{1\cdot1}+\frac{1}{1000}\right)\right\}$

$$=\frac{1}{10}(68\cdot5028)=6\cdot85028.$$

$\therefore$ The sum of the series $=2\cdot92897+6\cdot85028=9\cdot77925$.

This exceeds $\log_e 10000$ or $9\cdot21$ by $\cdot57$.

(Euler's constant or $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log_e n$ when n is infinite $=\cdot57721\ldots$).

C. M'Leod.

Certain Processes in the Theory of Equations illustrated Geometrically.—

The Derived Function: In what follows by $f(x)$ is meant a rational integral function of x of degree n with the coefficient of x^n unity. By the "roots of $f(x)$" is meant the roots of $f(x)=0$; by $f'(x)$ is meant $df(x)/dx$.

If $$f(x)=(x-a_1)(x-a_2)\ldots(x-a_n),$$

then $$f'(x)=\frac{f(x)}{x-a_1}+\frac{f(x)}{x-a_2}+\ldots+\frac{f(x)}{x-a_n},$$

which may be written shortly

$$\prod_{s=1}^{s=n}(x-a_s)\left\{\sum_{r=1}^{r=n}\frac{1}{x-a_r}\right\}.$$

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Now if $f(x)$ has all its n roots real, $y=f(x)$ cuts the x-axis in n points B_1, B_2, ..., B_n, and $y=f'(x)$ cuts the x axis in $(n-1)$ points D_1, D_2, ..., D_{n-1}, where D_r lies between B_r and B_{r+1}.

This shows that if $f(x)$ has all its roots real, then will $f'(x)$ have all its $(n-1)$ roots real, each root of $f'(x)$ lying between an adjacent pair of the roots of $f(x)$.

From the above we have the interesting fact that the equation $\prod_{s=1}^{s=n}(x-a_s)\left\{\sum_{r=1}^{r=n}\frac{1}{x-a_r}\right\}=0$, where the a's are all real, has all its roots real.

We can prove this directly when $n=3$; we have to prove that

$$(x-\alpha)(x-\beta)+(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)=0$$

has real roots.

The equation

$$3x^2-2(\alpha+\beta+\gamma)x+(\alpha\beta+\beta\gamma+\gamma\alpha)=0$$

has real roots if

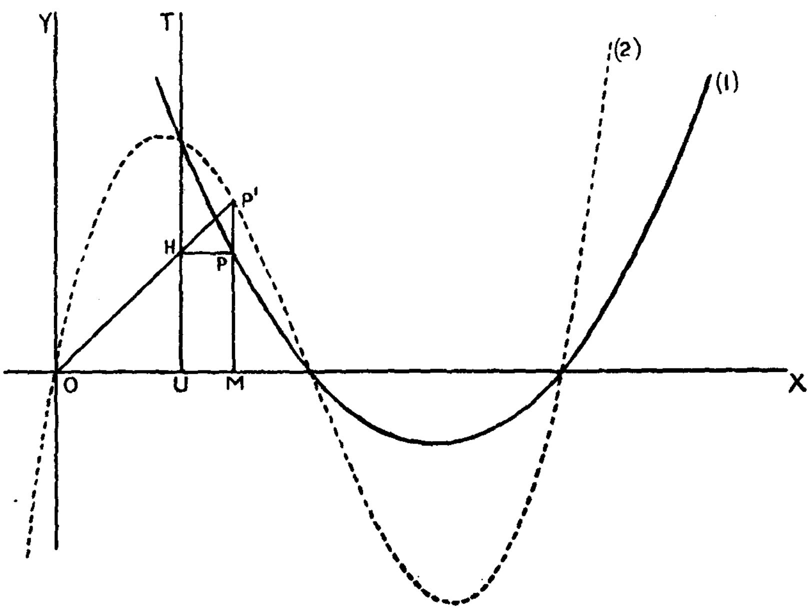
$$(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha) \text{ is } +ve,$$

i.e. if $\quad 2(\alpha+\beta+\gamma)^2-6(\alpha\beta+\beta\gamma+\gamma\alpha)$,, ,,

,, ,, $\quad 2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$,, ,,

,, ,, $\quad (\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$,, ,,

which is true.



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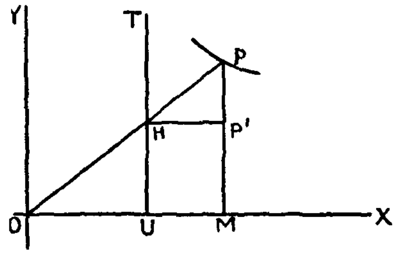
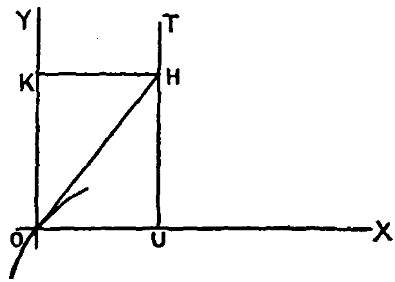
Reduction of Equations : From the curve $y=f(x)$ to get the curve $y=xf(x)$. Measure $OU=1$ (see fig. on p. 101) along OX. Draw UT perpendicular to OX. Let P be a point on $y=f(x)$ Draw PH perpendicular to UT. Join OH cutting the ordinate MP in P′. From similar triangles $MP'/OM=UH/OU=UH=MP$. Therefore $MP'=OM \,.\, MP = x \,.\, y = xf(x)$. The figure shows the graphs of $y=x^2-6x+8$ and $y=x(x^2-6x+8)$.

From the curve $y=f(x)$ to get the curve $y=\frac{1}{x}.f(x)$. Let P be a point on $y=f(x)$. Join OP cutting UT in H. Draw HP′ parallel to OX cutting the ordinate MP in P′. The locus of P′ is the curve $f(x)/x$.

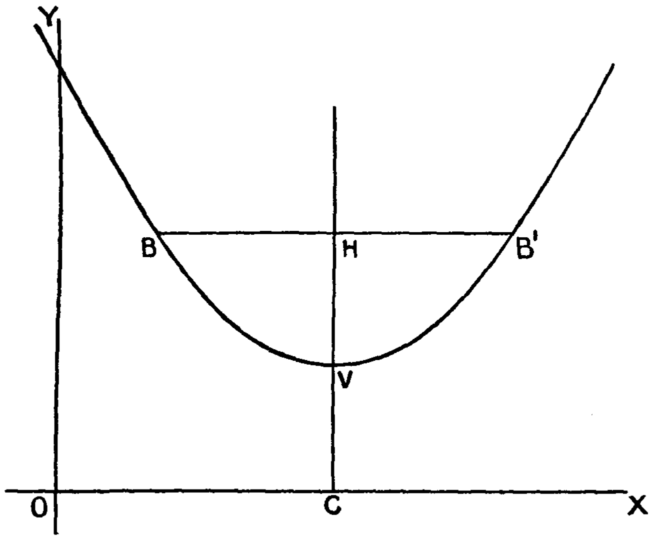




If $y=f(x)$ passes through the origin, then $(1/x).f(x)$ will be of the form 0/0 at the origin. The point where $y=(1/x).f(x)$ cuts the y-axis is found thus :—Draw OH the tangent at O cutting UT in H. Draw HK parallel to OX cutting OY in K. Then K is the required point. (When squared paper is used, we can pass readily from P to P′.)

If we know one root of an equation, the degree of the equation can be reduced by unity. Let $f(x)=0$ be the equation, then $f(x)=(x-a)\,.\,\phi(x)$, and the equation is reduced to $\phi(x)=0$. The geometrical equivalent is as follows :—Let $y=f(x)$ cut the axis at A. Move the origin to A Let $y=x\,.\,\psi(x)$ be the equation of the curve referred to the origin A. Draw the curve $y=(1/x)\,.\,\{x\,.\,\psi(x)\}$. Then this curve is $y=\psi(x)$ where $\psi(x)$ is one degree lower than $f(x)$; $\psi(x)$ becomes $\phi(x)$ on moving the origin back to O. (See figure on p. 104 illustrating the imaginary roots of a cubic).

Imaginary Roots: Imaginary roots of the Quadratic.—A common method of solving a quadratic, *when it has real roots*, is by means of a graph. The roots of a quadratic when imaginary can be readily obtained from a graph.



Let $x^2+2a_1x+a_2$ have imaginary roots. Draw the curve $y=x^2+2a_1x+a_2$. From V the vertex draw VC perpendicular to OX. From CV produced cut off VH = CV. Draw BHB′ parallel to OX meeting the curve in B, B′. Then the roots are $\mathrm{OC}+\mathrm{BH}\sqrt{-1}$, $\mathrm{OC}-\mathrm{BH}\sqrt{-1}$. The figure shows the graph of $y=x^2-22x+171$; the roots are $11+7{\cdot}07\sqrt{-1}$, $11-7{\cdot}07\sqrt{-1}$.

Imaginary Roots of the Cubic: Let

$$x^3+3a_1x^2+3a_2x+a_3$$

have two imaginary roots. Draw the curve $y=x^3+3a_1x^2+3a_2x+a_3$ cutting the x-axis in A. Move the origin to A. Let $y=x\phi(x)$ be the equation of the curve referred to A. Draw the curve $y=\phi(x)$. This is a quadratic whose roots $\mathrm{AC}+\mathrm{BH}\sqrt{-1}$, $\mathrm{AC}-\mathrm{BH}\sqrt{-1}$ can be found by the above method. Hence the roots of the cubic are

$$\mathrm{OA},\ \mathrm{OC}+\mathrm{BH}\sqrt{-1},\ \mathrm{OC}-\mathrm{BH}\sqrt{-1}.$$

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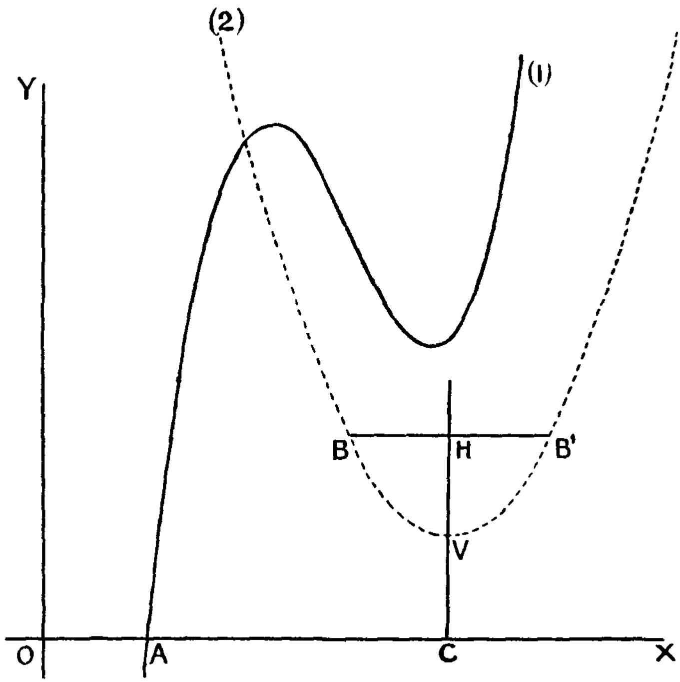

The figure shows the curves

$$y=(x-1)(x^2-8x+17),$$

and $$y= \qquad x^2-8x+17,$$

and the roots are

$$1,\ 4+\sqrt{-1},\ 4-\sqrt{-1}.$$

T. D. SALMON.

The Arithmetic Mean of a number of real positive numbers is not less than their Geometric Mean.—The subjoined proof is not given in the current text books, but is handed on by oral tradition.

The usual proof requires *in general* the assumption of an infinite series of operations (with the consequent limit theorems involved), as all the n given numbers tend to equality. Let us take an arithmetical example, and let us tabulate the sequences involved in the way suggested in my paper on "The Teaching of Limits and Convergence to Scholarship Candidates" in the May (1911) issue of the *Mathematical Gazette.*