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MULTIPLE SERIES MANIPULATIONS AND GENERATING FUNCTIONS

BY
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Let γ be an increasing function on the real numbers such that $\gamma(0)=0$ (which, by translation of axes, is no restriction) and suppose that $\gamma(n)$ is a positive integer if n is a positive integer. Let γ^- denote the inverse function of γ. Furthermore, let $L(x)$ be the least integer $\geq x$; let $[x]$ be the greatest integer $\leq x$, and suppose that $c_0, c_1, \ldots$ is an arbitrary sequence of numbers. Finally, the q-Eulerian function $H_k(x \mid q_1, \ldots, q_k)$ may be defined symbolically by $H^k = x^{-1}\pi_{j=1}^k(1+q_jH)$ if $k \geq 1$ and $H_0=1$; see [1] and [5].

In [4] we proved a special case of the following and indicated that the general proof is similar:

$$(1)\quad \sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=0}^{\gamma(\min(n_1,\ldots,n_k))} c_j q_1^{n_1} \cdots q_k^{n_k} z^t$$
$$= (1-z)^{-1}(-1)^k q_1^{-1} \cdots q_k^{-1} H_k(z \mid q_1^{-1}, \ldots, q_k^{-1}) \sum_{j=0}^{\infty} c_j (q_1 \cdots q_k z)^{L(\gamma^-(j))}$$

As an application we found a product formula for Eulerian functions and some new properties of Eulerian numbers. The proof of (1) makes use of some results of Roselle [5], which we shall use here also:

$$(2)\qquad xH_k(x \mid q_1, \ldots, q_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \cdots q_k^{n_1} x^{-\max(n_1,\ldots,n_k)},$$

$$(3)\qquad H_k(x^{-1} \mid q_1^{-1}, \ldots, q_k^{-1}) = (-1)^k x q_1 \cdots q_k H_k(xq_1, \ldots, q_k).$$

In this note we shall derive a result that is complementary to (1) and similar to those of [3]. In addition, it generalizes some results of Gould and Moser [2]. The formula is

$$(4)\quad \sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=0}^{[\gamma^-(\min(n_1,\ldots,n_k))]} c_j q_1^{n_1} \cdots q_k^{n_k} z^t$$
$$= (1-z)^{-1}(-1)^k q_1^{-1} \cdots q_k^{-1} H_k(z \mid q_1^{-1}, \ldots, q_k^{-1}) \sum_{j=0}^{\infty} c_j (q_1 \cdots q_k z)^{\gamma(j)}$$

Note that (4) does not follow from (1) by simply making replacements such

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as γ by γ^-; γ^- is not necessarily integer valued on the integers as is γ; also note the different uses of [ ] and L.

Before proving (4) it is instructive to list some elementary summation formulas that are either needed in the proof or are useful elsewhere.

$$(5)\quad \sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=0}^{\gamma(\min(j_1,\ldots,j_n))} f(k, j_1, \ldots, j_n) = \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n=L(\gamma^-(k))}^{\infty} f(k, j_1, \ldots, j_n)$$

$$(6)\quad \sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=\gamma(\max(j_1,\ldots,j_n))}^{\infty} f(k, j_1, \ldots, j_n) = \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n=0}^{[\gamma^-(k)]} f(k, j_1, \ldots, j_n)$$

$$(7)\quad \sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=0}^{[\gamma^-(\min(j_1,\ldots,j_n))]} f(k, j_1, \ldots, j_n) = \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n=\gamma(k)}^{\infty} f(k, j_1, \ldots, j_n)$$

$$(8)\quad \sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=L(\gamma^-(\max(j_1,\ldots,j_n))}^{\infty} f(k, j_1, \ldots, j_n) = \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n=0}^{\gamma(k)} f(k, j_1, \ldots, j_n)$$

In the case that $\gamma(x) = x$, (5) and (6) are obvious; otherwise, the proofs are similar but simpler than that of (4).

Proof of (4) is given as follows where the reason for each step is given in parentheses at the end of each step.

$$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=0}^{[\gamma^-(\min(n_1,\ldots,n_k))]} c_j q_1^{n_1} \cdots q_k^{n_k} z^t$$

$$= \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{t \ge \max(n_1,\ldots,n_k)} \sum_{j=0}^{[\gamma^-(\min(n,\ldots,n_k))]} c_j q_1^{n_k} \cdots q_k^{n_k} z^t$$

(formula (5) where $\gamma(x) = x$)

$$= (1-z)^{-1} \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{j=0}^{[\gamma^-(\min(n_1,\ldots,n_k))]} c_j q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1} \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{j \le [\gamma^-(n_1)],\ldots,[\gamma^-(n_k)]} c_j q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

(properties of γ, min, [ ])

$$= (1-z)^{-1} \sum_{j=0}^{\infty} c_j \sum_{n_1,\ldots,n_k \ge \gamma(j)}^{\infty} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

(property of γ)

$$= (1-z)^{-1} z^{-1} H_k(z^{-1} \mid q_1, \ldots, q_k) \sum_{j=0}^{\infty} c_j (q_1 \cdots q_k z)^{\gamma(j)} \quad \text{(equation (2))}$$

$$= (1-z)^{-1}(-1)^k q_k^{-1} \cdots q_k^{-1} H_k(z \mid q_1^{-1}, \ldots, q_k^{-1}) \sum_{j=0}^{\infty} c_j (q_1 \cdots q_k z)^{\gamma(j)}$$

This completes the proof of (4).

Application 1. In (c) put $\gamma(j)=j^m$, $c_0=0$, $c_j=1 (j\geq 1)$. Then

$$(9)\quad \sum_{t=0}^{\infty}\sum_{n_1,\ldots,n_k=0}^{t}[\min(n_1,\ldots,n_k)^{1/m}]q_1^{n_1}\cdots q_k^{n_k}z^t$$
$$=(1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z\mid q_1^{-1},\ldots,q_k^{-1})\sum_{j=1}^{\infty}(q_1\cdots q_k z)^{j^m}.$$

In the case that $k=1$, (9) may be compared with some results in [2] and [6] and with other similar formulas of number theoretic interest.

Application 2. Let $f(n)$ be a non-decreasing sequence of positive integers and define the distribution function, $D(f)$, by

$$D(f(n))=\text{card}\{k\mid f(k)\leq n;\qquad k=1,2,\ldots\}.$$

Note that

$$(10)\quad \sum_{n=1}^{\infty}D(f(n))x^n=\sum_{n=1}^{\infty}\sum_{f(k)\leq n}x^n=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}x^{n+f(k)}=(1-x)^{-1}\sum_{k=1}^{\infty}x^{f(k)}.$$

Therefore, if we put $c_0=0$ and $c_j=1$ in (4) and then use (10), we have

$$\sum_{t=0}^{\infty}\sum_{n_1,\ldots,n_k=0}^{t}[\gamma^{-}(\min(n_1,\ldots,n_k))]q_1^{n_1}\cdots q_k^{n_k}z^t$$
$$=(1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z\mid q^{-1},\ldots,q_k^{-1})\sum_{j=1}^{\infty}(q_1\cdots q_k z)^{\gamma(j)}$$
$$=(1-q_1q_2\cdots q_k z)(1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z\mid q_1^{-1},\ldots,q_k^{-1})$$
$$\times\sum_{j=1}^{\infty}D(\gamma(j))(q_1\cdots q_k z)^j.$$

This gives an interesting q-generating formula for distribution functions:

$$\sum_{j=1}^{\infty}D(\gamma(j))(q_1\cdots q_k z)^j$$
$$=(1-q_1q_2\cdots q_k z)^{-1}(1-z)(-1)^k q_1\cdots q_k(H_k(z\mid q_1^{-1},\ldots,q_k^{-1}))^{-1}$$
$$\sum_{t=0}^{\infty}\sum_{n_1,\ldots,n_k=0}^{t}[\gamma^{-}(\min(n_1,\ldots,n_k))]q_1^{n_1}\cdots q_k^{n_k}z^t.$$

References

1. L. Carlitz, *q-Bernoulli and Eulerian number*, Trans. Amer. Math. Soc. **76** (1954), 333–350.

2. H. W. Gould, *Generalization of a bracket function of L. Moser*, Canadian Math. Bull. **6** (1963), 275–277; Editor's Comment. 277–278.

3. R. C. Grimson, *Some partition generating functions*, Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing, 1973, 299–308.

4. R. C. Grimson, *A summation formula and some properties of Eulerian functions*, Proc. Amer. Math. Soc. **53** (1975), 290–292.

5. D. P. Roselle, *Generalized Eulerian functions and a problem in partitions*, Duke Math. J. **33** (1966), 293–304.

6. Zeller, *Über Summen von grössten Janzen bei arithmetischen Reihen*, Gött Nachr. **197,** 243–268.

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