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CONJUGATIONS ON BANACH ALGEBRAS

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An example is given of a complex unital Banach algebra carrying a continuum of conjugations, for each of which the selfconjugate subalgebra is strictly real.

1. INTRODUCTION

A *conjugation* on a complex Banach algebra $\mathfrak{B}$ is a map $c\colon \mathfrak{B} \to \mathfrak{B}$, $x \mapsto x^c$, with the properties: for all $x, y \in \mathfrak{B}$ and $\lambda \in \mathbb{C}$, (i) $(x+y)^c = x^c + y^c$, (ii) $(\lambda x)^c = \overline{\lambda} x^c$, (iii) $(xy)^c = x^c y^c$, (iv) $x^{cc} = x$, (v) c is continuous. (Here $\overline{\lambda}$ denotes the complex conjugate of λ.) When $\mathfrak{B}$ is unital with unit e, necessarily $e^c = e$. When $\mathfrak{B}$ is commutative, c is a star operator. Given such a conjugation, the set

$$\mathfrak{B}^{(c)} := \{x \in \mathfrak{B} : x^c = x\}$$

is a real subalgebra of $\mathfrak{B}$, the *selfconjugate subalgebra for* c. If d is another conjugation on $\mathfrak{B}$ distinct from c, then $\mathfrak{B}^{(c)}$ and $\mathfrak{B}^{(d)}$ are distinct.

An arbitrary complex Banach algebra $\mathfrak{B}$ carries a conjugation if and only if it is equivalent (that is, isomorphic and renormable) to the complexification of a real Banach algebra. If c is a conjugation on $\mathfrak{B}$ then $\mathfrak{B}$ is equivalent to $\left(\mathfrak{B}^{(c)}\right)_{\mathbb{C}}$, the complexification of $\mathfrak{B}^{(c)}$ (see [**2**, **4**]).

A real Banach algebra $\mathfrak{A}$ is called *strictly real* if $\mathrm{Sp}_{\mathfrak{A}}(a) \subset \mathbb{R}$ for every $a \in \mathfrak{A}$. Recall that $\mathrm{Sp}_{\mathfrak{A}}(a)$, the spectrum of a in $\mathfrak{A}$, is by definition the spectrum of the image of a in $\mathfrak{A}_{\mathbb{C}}$ under the canonical embedding of $\mathfrak{A}$ in its complexification $\mathfrak{A}_{\mathbb{C}}$. Many aspects of Banach algebra theory take a simpler form when the algebra is strictly real; in particular, the algebra carries a partial order and its closure preorder, and much real-type analysis is possible on it: there are noticeable simplifications to the behaviour of the elementary functions; the resolvent is a Pick function; there are integral representation theorems in the dual space. See [1], [3]. As a consequence, there is special interest also in complex Banach algebras which are complexifications of strictly real algebras. The question arises: Do there exist complex Banach algebras $\mathfrak{B}$ which are expressible as a complexification of a strictly real Banach algebra in more than one way. In [**2**] an example was given of an algebra $\mathfrak{B}$ carrying two distinct but commuting conjugations

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for each of which the selfconjugate subalgebra is strictly real. The purpose of the present note is, by a modification and extension of that example, to give an example of a commutative unital Banach algebra $\mathfrak{B}$ carrying *a continuum* of conjugations, for each of which the selfconjugate subalgebra is strictly real.

2. The example

$\mathfrak{B}$ is defined to be the unital Banach algebra with a single generator s for which $s^2 = 0$; each element has a unique representation in the form $x = \mu e + \nu s$ $(\mu, \nu \in \mathbb{C})$, and we can take $\|x\| := |\mu| + |\nu|$. Define

$$\mathfrak{Z} = \{\mu e + \nu is : \mu, \nu \in \mathbb{R}\}.$$

$\mathfrak{Z}$ is a real subalgebra of $\mathfrak{B}$; its complexification $\mathfrak{Z}_{\mathbb{C}} = \mathfrak{Z} \times \mathfrak{Z}$, with elements written (z_1, z_2), is complex isomorphic to $\mathfrak{B}$ by the continuous map $\theta\colon (z_1, z_2) \mapsto z_1 + iz_2$, and if $\mathfrak{Z}_{\mathbb{C}}$ is renormed by writing $|||(z_1, z_2)||| := \|z_1 + iz_2\|$ then θ becomes an isometry. Henceforth we write $\mathfrak{B}$ in place of $\mathfrak{Z}_{\mathbb{C}}$, for convenience. The set of all conjugations on $\mathfrak{B}$ is given by the following lemma.

Lemma 1. *A map $d\colon \mathfrak{B} \to \mathfrak{B}$ is a conjugation on $\mathfrak{B}$ if and only if there exists a function $\delta\colon \mathfrak{B} \to \mathfrak{Z}$, such that:*

(a) $(z_1, z_2)^d = (\delta(z_1, z_2), \delta(-z_2, z_1))$ *for all* $z_1, z_2 \in \mathfrak{Z}$,

(b) δ *is real linear,*

(c) $\delta(e, 0) = e$, $\delta(0, e) = 0$; *and for some real numbers* τ_1, τ_2 *with* $\tau_1^2 + \tau_2^2 = 1$, $\delta(is, 0) = \tau_1 is$, $\delta(0, is) = \tau_2 is$.

Proof: It can be verified that any such δ determines a conjugation d. Conversely, let d be any given conjugation on $\mathfrak{B}$. It will determine two functions $\delta, \varepsilon\colon \mathfrak{B} \to \mathfrak{Z}$, by the formula $(z_1, z_2)^d = (\delta(z_1, z_2), \varepsilon(z_1, z_2))$. Property (i) of Section 1 shows that δ and ε are additive, and (ii) shows further that δ is real linear and $\varepsilon(z_1, z_2) = \delta(-z_2, z_1)$: this gives (a) and (b). Since d preserves the unit $(e, 0)$ of $\mathfrak{B}$, $(\delta(e, 0), \delta(0, e)) = (e, 0)$; the first two equations of (c) follow. Supposing that

$$\delta(is, 0) = \rho_1 e + \tau_1 is, \qquad \delta(0, is) = \rho_2 e + \tau_2 is$$

and substituting in $\left((is, 0)^2\right)^d = \left((is, 0)^d\right)^2$, we find that $\rho_1 = \rho_2 = 0$. Property (iv) with (a) gives $\delta(\delta(z_1, z_2), \delta(-z_2, z_1)) = z_1$ for all $z_1, z_2 \in \mathfrak{Z}$, and the choice $z_1 = is$, $z_2 = 0$ here shows that $\tau_1^2 + \tau_2^2 = 1$. ∎

With d, δ and the τ's as in the lemma we find that, for all $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{R}$,

$$(*) \qquad (\mu_1 e + \nu_1 is, \mu_2 e + \nu_2 is)^d = (\mu_1 e + (\tau_1\nu_1 + \tau_2\nu_2)is, -\mu_2 e + (\tau_2\nu_1 - \tau_1\nu_2)is).$$

Lemma 2. *The selfconjugate subalgebra of $\mathfrak{B}$ for the conjugation d of Lemma 1 is, if $-1 < \tau < 1$,*

$$\mathfrak{B}^{(d)} = \{\left(\mu e + \nu i s,\ \tau_2^{-1}(1-\tau_1)\nu i s\right) : \mu,\ \nu \in \mathbb{R}\},$$

or $\mathfrak{B}^{(d)} = \{(\mu e, \nu i s) : \mu, \nu \in \mathbb{R}\}$ *if* $\tau_1 = -1$, $\mathfrak{B}^{(d)} = \{(\mu e + \nu i s, 0) : \mu, \nu \in \mathbb{R}\}$ *if* $\tau_1 = 1$.

Proof: Verification, from (*). □

It remains to show

Theorem. *$\mathfrak{B}^{(d)}$ in Lemma 2 is strictly real.*

Proof: This involves working in the complexification $\left(\mathfrak{B}^{(d)}\right)_{\mathbb{C}}$. A typical element of this algebra has the form

$$p = \left(\left(\mu_1 e + \nu_1 i s,\ \tau_2^{-1}(1-\tau_1)\nu_1 i s\right),\ \left(\mu_2 e + \nu_2 i s,\ \tau_2^{-1}(1-\tau_1)\nu_2 i s\right)\right),$$

(considering first the case $\tau_2 \neq 0$). Suppose that $\lambda = \alpha + i\beta$ belongs to the resolvent set of p; this is so if and only if there exists an element q in the algebra such that

$$(\lambda((e, 0), (0, 0)) - p)q = ((e, 0), (0, 0)).$$

This equation when reduced to its components gives a system of linear equations in the 4 real coefficients appearing in q, the equations being independent of τ_1 and τ_2, and solvable if and only if $(\alpha - \mu_1)^2 + (\beta - \mu_2)^2 \neq 0$. But we are concerned only with the case when p is the embedded image in $\left(\mathfrak{B}^{(d)}\right)_{\mathbb{C}}$ of an element of $\mathfrak{B}^{(d)}$, that is, when $\mu_2 = \nu_2 = 0$; in this case the spectrum of p is $\{\mu_1\}$, which is real. This proves that $\mathfrak{B}^{(d)}$ is strictly real when $\tau_2 \neq 0$, and the case $\tau_2 = 0$ is verified in the same way. □

Thus each pair of reals τ_1, τ_2 with $\tau_1^2 + \tau_2^2 = 1$ determines an algebra of the sought kind, and these algebras are pairwise distinct.

The union of all these algebras is the set of all elements of $\mathfrak{B}$ with real spectrum; any two distinct algebras intersect in $\mathbb{R}e$. If σ_1, σ_2 with $\sigma_1^2 + \sigma_2^2 = 1$ is another parameter pair distinct from τ_1, τ_2, the two corresponding conjugations commute on $\mathfrak{B}$ if and only if $\sigma_1 = -\tau_1$ and $\sigma_2 = -\tau_2$.

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