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ON THE POINT-ARBORICITY OF A GRAPH AND ITS COMPLEMENT

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1. Introduction. The *point-arboricity* $\rho(G)$ of a graph G is defined as the minimum number of subsets into which the point set $V(G)$ of G may be partitioned so that each subset induces an acyclic subgraph. Equivalently, the point-arboricity of G may be defined as the least number of colours needed to colour the points of G so that no cycle of G has all of its points coloured the same. This term was introduced by Chartrand, Geller, and Hedetniemi [**1**], although the concept was first considered by Motzkin [**4**].

As with the chromatic number of a graph G, which we denote by $\chi(G)$, there is no explicit formula for the point-arboricity of a graph. However, Nordhaus and Gaddum [**5**] have shown that if G is a graph with p points, then

$$2\sqrt{p} \leqq \chi(G) + \chi(\bar{G}) \leqq p + 1 \quad \text{and} \quad p \leqq \chi(G) \cdot \chi(\bar{G}) \leqq (\tfrac{1}{2}(p+1))^2,$$

where $\bar{G}$ denotes the complement of G. We prove a result for point-arboricity analogous to the Nordhaus-Gaddum theorem.

2. An analogue to the Nordhaus-Gaddum theorem. We begin with the following theorem which compares the chromatic number of a graph with its point-arboricity.

THEOREM 1. *For any graph G,*

$$\tfrac{1}{2}\chi(G) \leqq \rho(G) \leqq \chi(G).$$

Proof. The point set of G can be partitioned into $\chi(G)$ sets such that each set induces a graph with no lines. Each of these sets induces an acyclic graph which implies that $\rho(G) \leqq \chi(G)$.

There is a partition of the point set of G into $\rho(G)$ sets each of which induces an acyclic subgraph of G. The chromatic number of an acyclic graph never exceeds two. This implies that $\chi(G) \leqq 2\rho(G)$, which completes the proof.

We observe that the bounds given in Theorem 1 are sharp in the sense that for any positive integer m, there exist graphs G and H with point-arboricity m such that $\chi(G) = m$ and $\chi(H) = 2m$. To verify this we let H be the complete graph on $2m$ points, denoted K_{2m}, and let G be the complete m-partite graph $K(2m, 2m, \ldots, 2m)$. Clearly the chromatic number of G is m. In order to see that $\rho(G) = m$, we assume that $\rho(G) \leqq m - 1$. Since $(2m+2)(m-1) = 2m^2-2$ is less than the order of G, any partition of the point set of G into $m - 1$ sets

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includes a set S with at least $2m + 2$ points. However, this set induces a graph with a cycle. This contradiction implies that $\rho(G) = m$.

We are now in a position to prove the aforementioned analogue to the Nordhaus-Gaddum theorem.

THEOREM 2. *If G is a graph of order p, then*

$$(p)^{\frac{1}{2}} \leqq \rho(G) + \rho(\bar{G}) \leqq \tfrac{1}{2}(p+3) \tag{1}$$

and

$$\tfrac{1}{4}p \leqq \rho(G) \cdot \rho(\bar{G}) \leqq (\tfrac{1}{4}(p+3))^2. \tag{2}$$

Proof. The Nordhaus-Gaddum theorem together with Theorem 1 imply the lower bounds of (1) and (2).

We use induction to verify the right inequality of (1). Clearly this inequality holds for $p = 1$ or 2; thus assume that it holds for all graphs with fewer than p points, $p \geqq 3$. Let H be a graph of order p. It is easily verified that every graph of order two or more contains two distinct points of equal degree. Let u and v be two such points in H, say with $\deg u = \deg v = d$.

Let $G = H - u - v$ so that $\bar{G} = \bar{H} - u - v$. By the induction hypothesis, we have

$$\rho(G) + \rho(\bar{G}) \leqq \tfrac{1}{2}(p+1). \tag{3}$$

Since any two points of a graph induce an acyclic subgraph, we have

$$\rho(H) \leqq \rho(G) + 1 \tag{4}$$

and

$$\rho(\bar{H}) \leqq \rho(\bar{G}) + 1. \tag{5}$$

If strict inequality holds in either (4) or (5), then by using (3) we have

$$\rho(H) + \rho(\bar{H}) \leqq \rho(G) + \rho(\bar{G}) + 1 \leqq \tfrac{1}{2}(p+3),$$

which is the desired result.

Suppose then that equality holds in both (4) and (5). Let $\rho(G) = r$ and $V_1, \ldots, V_r$ a partition of $V(G)$ into acyclic sets. From (4) we conclude that either adding u to G or v to $G + u$ increases the point-arboricity by one. Thus either u is adjacent to two points in each V_i, $i = 1, 2, \ldots, r$, or v is adjacent to two points in each set of any partition of $V(G + u)$ into r acyclic sets. Hence,

$$d = \deg_H u = \deg_H v \geqq 2\rho(G). \tag{6}$$

Similarly, (5) implies that

$$p - d - 1 = \deg_{\bar{H}} u = \deg_{\bar{H}} v \geqq 2\rho(\bar{G}). \tag{7}$$

Adding (6) and (7) we obtain

$$p - 1 \geqq 2\rho(G) + 2\rho(\bar{G}).$$

Thus,

$$\rho(H) + \rho(\bar{H}) = \rho(G) + \rho(\bar{G}) + 2 \leqq \tfrac{1}{2}(p - 1) + 2 = \tfrac{1}{2}(p + 3),$$

which is the upper bound of (1).

The upper bound of (2) follows from the right side of (1) and the fact that the geometric mean never exceeds the arithmetic mean. This completes the proof of Theorem 2.

3. The sharpness of the bounds. Finck [**3**] and Stewart [**6**] independently established that for positive integers k, k', and p such that $k + k' \leqq p + 1$ and $p \leqq k \cdot k'$, there is a graph G with p points such that $\chi(G) = k$ and $\chi(\bar{G}) = k'$. We prove an analogous result for point-arboricity. First, we find it convenient to introduce some additional notation and results. For any real number r, $[r]$ and $\{r\}$ denote the greatest integer not exceeding r and the least integer not less than r, respectively. The subgraph induced by a set S of points of a graph is denoted by $\langle S\rangle$.

For $i = 1, 2, \ldots, n$ let G_i have point set V_i and line set E_i. The *union* $G = \bigcup_{i=1}^{n} G_i$ of the graphs $G_1, \ldots, G_n$ is the graph whose point set is $\bigcup_{i=1}^{n} V_i$ and which has line set $\bigcup_{i=1}^{n} E_i$. Two graphs are *disjoint* if their point sets are disjoint. For $n \geqq 2$, if $G_1, \ldots, G_n$ are mutually disjoint graphs, $\sum_{i=1}^{n} G_i$ is the graph which consists of $\bigcup_{i=1}^{n} G_i$ together with all possible lines joining points in G_i to points in G_j, for $i \neq j$.

LEMMA 1. *If G is a path with p points, then $\rho(\bar{G}) = \{\tfrac{1}{4}p\}$.*

Proof. If G is a path with less than five points, $\bar{G}$ is acyclic and

$$\rho(\bar{G}) = 1 = \{\tfrac{1}{4}p\}.$$

Suppose that G is the path $v_1, v_2, \ldots, v_p$, $p \geqq 5$. Partition $V(\bar{G})$ into $m = \{\tfrac{1}{4}p\}$ subsets $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{v_5, v_6, v_7, v_8\}, \ldots, V_m = \{v_{4(m-1)+1}, \ldots, v_p\}$. Since each V_i induces an acyclic subgraph of $\bar{G}$, $\rho(\bar{G}) \leqq \{\tfrac{1}{4}p\}$. The fact that any set of five points of $\bar{G}$ induces a subgraph with a cycle implies that $\rho(\bar{G}) = \{\tfrac{1}{4}p\}$.

LEMMA 2. *Let $G_1, G_2, \ldots, G_m$ be mutually disjoint paths, where G_1 has $k \geqq 2$ points and each of $G_2, \ldots, G_m$ has at least two and at most k points. If $G = \sum_{i=1}^{m} G_i$, then*

$$\rho(G) = m \tag{8}$$

and

$$\rho(\bar{G}) = \{\tfrac{1}{4}k\}. \tag{9}$$

Proof. Since each of $G_1, \ldots, G_m$ is acyclic, $\rho(G) \leqq m$. The subgraph H of G which consists of two adjacent points from each G_i is the complete graph on $2m$ points. This implies that $\rho(G) \geqq \rho(H) = m$, which proves (8).

In order to verify (9), we see that if $k = 2$ or $k = 3$, then $\bar{G}$ has no cycles. If $k > 3$, then $\bar{G}$ has at least m components. Let H be a component of $\bar{G}$ with a maximum number of lines. Then H is the complement of a path with k points and according to Lemma 1, $\rho(H) = \{\frac{1}{4}k\}$. However, the point-arboricity of a graph is the maximum of the point-arboricity of its components. Thus, $\rho(\bar{G}) = \{\frac{1}{4}k\}$, which completes the proof.

In [**2**] Chartrand, Kronk, and Wall showed that if G is the complete n-partite graph $k(p_1, \ldots, p_n)$, then

$$\rho(G) = n - \max\left\{k: \sum_{i=0}^{k} p_i \leqq n - k\right\},$$

where p_0 is defined as zero. This result is used in proving Theorem 3.

THEOREM 3. *For any positive integers a, a', and p such that*

$$a + a' \leqq \tfrac{1}{2}(p + 3) \qquad \textit{and} \qquad \tfrac{1}{4}p \leqq a \cdot a'$$

there exists a graph G with p points such that $\rho(G) = a$ and $\rho(\bar{G}) = a'$.

Proof. Without loss of generality, we suppose that $a' \leqq a$ and consider a number of different cases.

Case (i). $p + 2 \leqq 2a + 2a' \leqq p + 3$. If $2a + 2a' = p + 3$, let $G_1 = K_{2a-1}$ and $G_i = K_1$ for $i = 2, \ldots, 2a' - 1$ be mutually disjoint graphs. If $2a + 2a' = p + 2$, let $G_1 = K_{2a}$ and $G_i = K_1$ for $i = 2, 3, \ldots, 2a' - 1$ be mutually disjoint graphs. In either case, denote $\bigcup_{i=1}^{2a'-1} G_i$ by G. Then $\rho(G) = \rho(G_1) = a$ and $\rho(\bar{G}) = \rho(K_{2a'-1}) = a'$.

Case (ii). $2a + 2a' \leqq p + 1$ *and* $p \leqq 2aa' + a'$. We form the following mutually disjoint graphs. For $i = 1, 2, \ldots, a'$, let $G_i = K_1$, let $G_{2a'} = K_{2a}$, and for $i = a' + 1, \ldots, 2a' - 1$ let G_i be a complete graph with at least one point and at most $2a$ points such that exactly p points are used in these $2a'$ graphs. This is possible since $2a + 2a' - 1 \leqq p$ and $2aa' + a' \geqq p$.

Denote the union of these $2a'$ graphs by G. Then $\rho(G) = \rho(G_{2a'}) = a$ and $\bar{G} = K(p_1, p_2, \ldots, p_{2a'})$, where $p_i = 1$ for $i = 1, \ldots, a'$. Hence,

$$\rho(\bar{G}) = 2a' - \max\left\{k: \sum_{i=0}^{k} p_i \leqq 2a' - k\right\} = 2a' - a' = a'.$$

Case (iii). $2a + 2a' \leqq p + 1$ *and* $p > 2aa' + a'$.

Suppose that $a' \geqq 2$. We form mutually disjoint graphs $G_1, \ldots, G_{a'}$, where G_1 is a path with $4a$ points and $G_2, \ldots, G_{a'}$ are paths with at most $4a$ points and at least two points. In this way we use at most $4aa'$ points, and by hypothesis, $4aa' \geqq p$. Also we use at least $4a + (a' - 1)2$ points. Induction on a' can be used to show that for any positive integers a and a', $4a + (a' - 1)2 \leqq 2aa' + a'$. However, $2aa' + a' < p$ by the hypothesis for this case. Thus $G_1, \ldots, G_{a'}$ can be chosen so that exactly p points are used.

Let $\bar{G} = \sum_{i=1}^{a'} G_i$; then Lemma 2 implies that $\rho(\bar{G}) = a'$ and

$$\rho(G) = \{4a/4\} = a.$$

Assume that $a' = 1$; then we have $2a + 2 \leqq p \leqq 4a$. Let G_1 be a path with four points and let $G_2, \ldots, G_a$ be paths with two, three, or four points such that the G_i are mutually disjoint. Since this procedure uses at most $4a \geqq p$ points and at least $4 + (a - 1)2 \leqq p$ points, we can choose G_i such that exactly p points are used. Denote $\sum_{i=1}^{a} G_i$ by G; then according to Lemma 2, $\rho(G) = a$ and $\rho(\bar{G}) = 1 = a'$. Thus, in all cases the theorem is proved.

This theorem can now be used to show that each of the bounds in Theorem 1 is the best possible for infinitely many values of p.

COROLLARY 1. *For any positive integer p there are graphs G and H with p points such that*

$$\rho(G) + \rho(\bar{G}) = [\tfrac{1}{2}(p + 3)] \tag{10}$$

and

$$\rho(H) \cdot \rho(\bar{H}) = \{\tfrac{1}{4}p\}. \tag{11}$$

Proof. For equation (10), we let $a = [\frac{1}{2}(p + 3)] - 1$ and $a' = 1$; then $a \cdot a' = [\frac{1}{2}(p + 1)] \geqq \{\frac{1}{4}p\}$. Thus by Theorem 3 there is a graph G with p points such that $\rho(G) = a$ and $\rho(\bar{G}) = a'$.

In order to prove (11), we let $a = \{\frac{1}{4}p\}$ and $a' = 1$. Then

$$a + a' = \{\tfrac{1}{4}p\} + 1 = \{\tfrac{1}{4}(p + 4)\} \leqq [\tfrac{1}{2}(p + 3)].$$

Again, by applying Theorem 3 there exists a graph H with $\rho(H) = \{\frac{1}{4}p\}$ and $\rho(\bar{H}) = 1$, so that (11) is satisfied.

COROLLARY 2. *There are infinite sets P_1 and P_2 of integers with the property that, for every p in P_1 there is a graph G of order p such that $\rho(G) + \rho(\bar{G}) = (p)^{\frac{1}{2}}$ and for every p in P_2 there is a graph H of order p such that*

$$\rho(H) \cdot \rho(\bar{H}) = (\tfrac{1}{4}(p + 3))^2.$$

Proof. Let $P_1 = \{p: p = 4n^2, n = 1, 2, 3, \ldots\}$. Then for any p in P_1 let $a = a' = \frac{1}{2}(p)^{\frac{1}{2}}$, which is a positive integer. Then a and a' satisfy the inequalities of Theorem 3, and hence there is a graph G with p points such that $\rho(G) = \frac{1}{2}(p)^{\frac{1}{2}} = \rho(\bar{G})$.

We define P_2 as the set $\{p: p = 4n + 1, n = 1, 2, \ldots\}$. For any p in P_2 let $a = \frac{1}{4}(p + 3) = a'$. Then according to Theorem 3 there is a graph H with p points such that $\rho(H) = \frac{1}{4}(p + 3) = \rho(\bar{H})$.

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