10

Symplectic invariance of CCR in finite-dimensions

This is the first chapter devoted to the symplectic invariance of the CCR. In this chapter we restrict ourselves to regular CCR representations over finite-dimensional symplectic spaces.

In an infinite-dimensional symplectic space there is no distinguished topology. This problem is absent in a finite-dimensional space. This motivates a separate discussion of the finite-dimensional case.

The chapter is naturally divided in two parts. In the first three sections we consider symplectic invariance without invoking any conjugation on the symplectic space. We consider an arbitrary irreducible regular CCR representation over a finite-dimensional symplectic space and do not explicitly use the Schrödinger representation.

In the last two subsections we fix a conjugation, so that our symplectic space can be written as $\mathcal{Y} = \mathcal{X}^{\#} \oplus \mathcal{X}$, and we consider the Schrödinger representation on $L^2(\mathcal{X})$.

10.1 Classical quadratic Hamiltonians

Throughout this section $(\mathcal{Y}, \omega)$ is a finite-dimensional symplectic space. Recall that $(\mathcal{Y}^{\#}, \omega^{-1})$ is also a symplectic space. As before we denote by y the generic element of $\mathcal{Y}$ and by v the generic element of $\mathcal{Y}^{\#}$.

Remark 10.1 *It is natural to consider the two symplectic spaces $\mathcal{Y}$ and $\mathcal{Y}^{\#}$ in parallel. It is a little difficult to decide which space should be viewed as the principal one: $\mathcal{Y}^{\#}$ is perhaps more important from the point of view of classical mechanics, since it plays the role of the phase space, whereas the dual phase space $\mathcal{Y}$ is more natural quantum mechanically, since we use it in the CCR relations.*

Recall that $\zeta \in L_{\rm s}(\mathcal{Y}^{\#}, \mathcal{Y})$ iff $\zeta \in L(\mathcal{Y}^{\#}, \mathcal{Y})$ and $\zeta^{\#} = \zeta$. We write $\zeta \geq 0$ if $v{\cdot}\zeta v \geq 0$, $v \in \mathcal{Y}^{\#}$. We write $\zeta > 0$ if in addition $\operatorname{Ker}\zeta = \{0\}$.

The following section is a preparation for the next two where we consider a regular CCR representation over $\mathcal{Y}$.

10.1.1 Symplectic transformations

Let $r \in L(\mathcal{Y})$. Recall that $r \in Sp(\mathcal{Y})$ iff

$$r^{\#}\omega r = \omega. \tag{10.1}$$

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This is equivalent to $r^{\#} \in Sp(\mathcal{Y}^{\#})$, which means

$$r\omega^{-1}r^{\#} = \omega^{-1}. \tag{10.2}$$

We have an isomorphism of groups,

$$Sp(\mathcal{Y}) \ni r \mapsto \omega r \omega^{-1} = (r^{\#})^{-1} \in Sp(\mathcal{Y}^{\#}).$$

Let $a \in L(\mathcal{Y})$. Recall that $a \in sp(\mathcal{Y})$ iff $a^{\#}\omega + \omega a = 0$. This is equivalent to $a^{\#} \in sp(\mathcal{Y}^{\#})$, which means $a\omega^{-1} + \omega^{-1}a^{\#} = 0$. Note that

$$sp(\mathcal{Y}) \ni a \mapsto \omega a \omega^{-1} = -a^{\#} \in sp(\mathcal{Y}^{\#})$$

is an isomorphism of Lie algebras.

10.1.2 Poisson bracket

Definition 10.2 *For $b_1, b_2 \in C^1(\mathcal{Y}^{\#})$ we define the* Poisson bracket

$$\{b_1, b_2\}(v) := \omega\nabla b_1(v)\cdot\nabla b_2(v) = -\nabla b_1(v)\cdot\omega\nabla b_2(v).$$

$C^\infty(\mathcal{Y}^{\#})$ equipped with $\{\cdot,\cdot\}$ is a Lie algebra.

Definition 10.3 *By a* quadratic*, resp.* purely quadratic polynomial *we will mean a polynomial of degree ≤ 2, resp. $= 2$.*

Recall that the space of complex quadratic, resp. purely quadratic polynomials on $\mathcal{Y}^{\#}$ is denoted $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$, resp. $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{2}(\mathcal{Y}^{\#})$. Both are Lie subalgebras of $C^\infty(\mathcal{Y}^{\#})$ w.r.t. the Poisson bracket. More precisely, if $\lambda_i \in \mathbb{C}$, $y_i \in \mathbb{C}\mathcal{Y}$, $\zeta_i \in \mathbb{C}L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$ and $\chi_i(v) := \lambda_i + y_i v + \frac{1}{2}v\cdot\zeta_i v$, then

$$\begin{aligned}\{\chi_1, \chi_2\}(v) = -y_1\cdot\omega y_2 + (\zeta_2\omega y_1 - \zeta_1\omega y_2)\cdot v \\ + \frac{1}{2}v\cdot(-\zeta_1\omega\zeta_2 + \zeta_2\omega\zeta_1)v.\end{aligned}$$

If $\chi \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$, so that $\chi(v) = \lambda + y\cdot v + \frac{1}{2}v\cdot\zeta v$ with $\lambda \in \mathbb{C}$, $y \in \mathbb{C}\mathcal{Y}$ and $\zeta \in \mathbb{C}L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$, then

$$\mathbb{C}\mathcal{Y}^{\#} \ni v \mapsto \omega\nabla\chi(v) = \omega y + \omega\zeta v \in \mathbb{C}\mathcal{Y}^{\#}$$

is an affine transformation on $\mathbb{C}\mathcal{Y}^{\#}$. We have surjective homomorphisms of Lie algebras

$$\begin{aligned}\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#}) \ni \chi \mapsto \omega\nabla\chi \in asp(\mathbb{C}\mathcal{Y}^{\#}), \\ \mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#}) \ni \chi \mapsto \omega\nabla\chi \in asp(\mathcal{Y}^{\#})\end{aligned}$$

(see Def. 1.102 for the definition of $asp(\mathbb{C}\mathcal{Y}^{\#})$ and $asp(\mathcal{Y}^{\#})$).

Definition 10.4 *If $(w, a^{\#}) \in asp(\mathcal{Y}^{\#})$ and $\omega\nabla\chi(v) = w + a^{\#}v$, then we say that χ is a* Hamiltonian *of $(w, a^{\#})$.*

Clearly, every element of $asp(\mathcal{Y}^{\#})$ has a one parameter family of Hamiltonians χ differing by a constant. We will usually demand that $\chi(0) = 0$, which fixes the choice of a Hamiltonian in a canonical way. With this choice,

$$\chi(v) = (\omega^{-1}w)\cdot v + \frac{1}{2}v\cdot\omega^{-1}a^{\#}v.$$

Let $\chi \in \mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#}, \mathbb{R})$, and let v_t solve

$$\frac{\mathrm{d}}{\mathrm{d}t}v_t = \omega\nabla\chi(v_t), \quad v_0 = v.$$

Clearly, $v_t = \mathrm{e}^{t\omega\nabla\chi}v$. Moreover, if $b \in C^1(\mathcal{Y}^{\#})$ and if we set $b_t(v) = b(v_t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}b_t(v) = \{\chi, b_t\}(v) = \{\chi, b\}(v_t). \tag{10.3}$$

10.1.3 Spectrum of symplectic transformations

Recall that a subspace $\mathcal{Y}_1$ of $\mathcal{Y}$ is called symplectic iff ω restricted to $\mathcal{Y}_1$ is non-degenerate. The following proposition is immediate:

Proposition 10.5 *Let $\mathcal{Y} = \mathcal{Y}_1 \oplus \cdots \oplus \mathcal{Y}_k$, and let $\mathcal{Y}_1, \dots, \mathcal{Y}_k$ be mutually ω-orthogonal subspaces. Then all $\mathcal{Y}_i$, $i = 1, \dots, k$, are symplectic.*

Definition 10.6 *An element $r \in Sp(\mathcal{Y})$ such that $\mathrm{Ker}(r + \mathbb{1}) = \{0\}$ will be called* regular.

Proposition 10.7 *Let $r \in Sp(\mathcal{Y})$.*

(1) $\mathrm{spec}\, r_{\mathbb{C}}$ *is invariant under $\mathbb{C} \ni z \mapsto z^{-1} \in \mathbb{C}$.*
(2) *For $\lambda \in \mathrm{spec}\, r_{\mathbb{C}} \cap \{\mathrm{Im}\, z \geq 0, \ \ |z| \geq 1\} =: \Lambda_r$ set $P_\lambda := \mathbb{1}_{\{\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\}}(r_{\mathbb{C}})$. Then P_λ are real projections, constitute a partition of unity, commute with r and $P_\lambda^{\#}\omega_{\mathbb{C}} = \omega_{\mathbb{C}}P_\lambda$.*
(3) *If we set $\mathcal{Y}_\lambda := P_\lambda\mathcal{Y}$, then $\mathcal{Y}_\lambda$ are symplectic, mutually ω-orthogonal, invariant for r and $\mathcal{Y} = \underset{\lambda\in\Lambda_r}{\oplus} \mathcal{Y}_\lambda$.*
(4) *Set $\mathcal{Y}_{\mathrm{sg}} := \mathcal{Y}_{-1}$ and $\mathcal{Y}_{\mathrm{reg}} := \underset{\lambda\in\Lambda_r\setminus\{-1\}}{\oplus} \mathcal{Y}_\lambda$. Then $\mathcal{Y} = \mathcal{Y}_{\mathrm{sg}} \oplus \mathcal{Y}_{\mathrm{reg}}$. If we set $\kappa := (-\mathbb{1}) \oplus \mathbb{1}$, then*

$$r = \kappa r_0 = r_0\kappa, \tag{10.4}$$

κ is a symplectic involution and $r_0 \in Sp(\mathcal{Y})$ is regular.

Proof $r^{\#}\omega = \omega r^{-1}$ implies (1). We also obtain

$$\omega_{\mathbb{C}}(z\mathbb{1} - r_{\mathbb{C}}^{-1})^{-1} = (z\mathbb{1} - r_{\mathbb{C}}^{\#})^{-1}\omega_{\mathbb{C}}.$$

Hence

$$\omega_{\mathbb{C}}\mathbb{1}_{\{\lambda^{-1}\}}(r_{\mathbb{C}}) = \omega_{\mathbb{C}}\mathbb{1}_{\{\lambda\}}(r_{\mathbb{C}}^{-1}) = \mathbb{1}_{\{\lambda\}}(r_{\mathbb{C}}^{\#})\omega_{\mathbb{C}}.$$

Therefore,

$$\omega_{\mathbb{C}}\mathbb{1}_{\{\lambda,\lambda^{-1}\}}(r_{\mathbb{C}}) = \mathbb{1}_{\{\lambda,\lambda^{-1}\}}(r_{\mathbb{C}}^{\#})\omega_{\mathbb{C}} = \mathbb{1}_{\{\lambda,\lambda^{-1}\}}(r_{\mathbb{C}})^{\#}\omega_{\mathbb{C}}.$$

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If $|\lambda| = 1$, then $P_\lambda = 1\!\!1_{\{\lambda,\lambda^{-1}\}}(r_{\mathbb{C}})$. If $|\lambda| \neq 1$, then

$$P_\lambda = 1\!\!1_{\{\lambda,\lambda^{-1}\}}(r_{\mathbb{C}}) + 1\!\!1_{\{\overline{\lambda},\overline{\lambda}^{-1}\}}(r_{\mathbb{C}}).$$

In both cases, P_λ is real and can be restricted to $\mathcal{Y}$. This proves (2).

(2) implies (3), which yields (4). □

There exists a classification of quadratic forms in a symplectic case due to Williamson. The following proposition is its special case for a positive semi-definite quadratic form, which is all that we need. Note that it would have a much simpler proof if we assumed that the form is positive definite.

Proposition 10.8 *Let $\zeta \in L_{\rm s}(\mathcal{Y}^\#, \mathcal{Y})$ and $\zeta \geq 0$. Then we can find $p \leq m \leq d$, $\lambda_1, \dots, \lambda_p > 0$ and a basis $(e_1, \dots, e_{2d})$ in $\mathcal{Y}$ so that, if the corresponding dual basis of $\mathcal{Y}^\#$ is $(e^1, \dots, e^{2d})$, then*

$$\omega e_{2j-1} = -e^{2j}, \quad \omega e_{2j} = e^{2j-1}, \; j = 1, \dots, d; \tag{10.5}$$

$$\zeta e^{2j-1} = \lambda_j e_{2j-1}, \; \zeta e^{2j} = \lambda_j e_{2j}, \; j = 1, \dots, p; \tag{10.6}$$

$$\zeta e^{2j-1} = e_{2j-1}, \quad \zeta e^{2j} = 0, \quad j = p+1, \dots, m; \tag{10.7}$$

$$\zeta e^{2j-1} = 0, \quad \zeta e^{2j} = 0, \quad j = m+1, \dots, d.$$

Consequently, $\operatorname{spec} \omega\zeta \subset {\rm i}\mathbb{R}$. *Besides,* $\operatorname{spec} \big(-(\omega\zeta)^2\big) \subset]0, \infty[$ *and* $(\omega\zeta)^2$ *is diagonalizable. If $\zeta > 0$, then $\omega\zeta$ is diagonalizable as well.*

Note that we have two forms on $\mathcal{Y}^\#$: ζ and ω^{-1}. The complements of $\mathcal{V} \subset \mathcal{Y}^\#$ w.r.t. these forms have standard symbols $\mathcal{V}^{\zeta\perp}$ and $\mathcal{V}^{\omega^{-1}\perp}$. For brevity, we will write $\mathcal{V}^\perp$ for the former and $\mathcal{V}^\circ$ for the latter.

For the proof of Prop. 10.8 we need two lemmas. We set

$$\mathcal{V}_1 := \operatorname{Ker} \zeta, \; \mathcal{V}_2 := \mathcal{V}_1^\circ, \; \mathcal{V}_3 := (\mathcal{V}_2)^\perp, \; \mathcal{V}_4 := \mathcal{V}_3^\circ.$$

Lemma 10.9 *We have a direct sum decomposition,*

$$\mathcal{Y}^\# = \mathcal{V}_3 \oplus \mathcal{V}_4,$$

which is both ω^{-1}- and ζ-orthogonal, and ζ is non-degenerate on $\mathcal{V}_4$.

Proof We have

$$\mathcal{V}_1 = (\mathcal{Y}^\#)^\perp \subset \mathcal{V}_2^\perp = \mathcal{V}_3, \tag{10.8}$$

hence

$$\mathcal{V}_4 = \mathcal{V}_3^\circ \subset \mathcal{V}_1^\circ = \mathcal{V}_2. \tag{10.9}$$

Clearly,

$$\mathcal{V}_2 \subset (\mathcal{V}_2^\perp)^\perp = \mathcal{V}_3^\perp. \tag{10.10}$$

From (10.9) and (10.10), we get

$$\mathcal{V}_4 \subset \mathcal{V}_3^\perp. \tag{10.11}$$

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Let us show that $\mathcal{V}_3 \cap \mathcal{V}_4 = \{0\}$. Assume that $v \in \mathcal{V}_3 \cap \mathcal{V}_4$ and $v \neq 0$. By (10.11), we have $v\cdot\zeta v = 0$, hence $v \in \mathcal{V}_1$.

By the non-degeneracy of ω^{-1}, there exists v' such that $v'\cdot\omega v \neq 0$. Let us fix a basis $(e^1, \dots, e^q)$ of $\mathcal{V}_2$ such that

$$\begin{aligned} e^i \cdot \zeta e^j &= 0, \quad \text{for } i \neq j, \\ e^i \cdot \zeta e^i &= 1, \quad \text{for } 1 \leq i \leq p, \\ e^i \cdot \zeta e^i &= 0, \quad \text{for } p+1 \leq i \leq q. \end{aligned}$$

We set $v'' = v' - \sum_{i=1}^{p}(v'\cdot\zeta e^i)e^i$ so that $v''\cdot\zeta e^i = 0$ for $1 \leq i \leq q$, and hence $v'' \in \mathcal{V}_2^{\perp} = \mathcal{V}_3$. Since $v'' - v' \in \mathcal{V}_2 = \mathcal{V}_1^{\circ}$, we have $v''\cdot\omega^{-1}v = v'\cdot\omega^{-1}v \neq 0$. Therefore, $v \notin \mathcal{V}_3^{\circ} = \mathcal{V}_4$, which is a contradiction.

Hence, $\mathcal{V}_3 \cap \mathcal{V}_4 = \{0\}$ and $\mathcal{Y}^{\#} = \mathcal{V}_3 \oplus \mathcal{V}_4$. The direct sum is clearly ω^{-1}-orthogonal, and also ζ-orthogonal by (10.11). Finally, $\mathcal{V}_4 \cap \mathcal{V}_1 \subset \mathcal{V}_4 \cap \mathcal{V}_3 = \{0\}$ by (10.8). Hence, ζ is non-degenerate on $\mathcal{V}_4$. □

Lemma 10.10 *There exists a direct sum decomposition*

$$\mathcal{Y}^{\#} = \mathcal{V}_8 \oplus \mathcal{V}_7 \oplus \mathcal{V}_4$$

which is both ω^{-1}- and ζ-orthogonal, such that ζ is non-degenerate on $\mathcal{V}_4$, $\mathrm{Ker}\,\zeta \cap \mathcal{V}_7$ *is Lagrangian in $\mathcal{V}_7$, and $\zeta = 0$ on $\mathcal{V}_8$.*

Proof Let $\mathcal{V}_5 \subset \mathcal{V}_3$ be a maximal subspace on which ζ is non-degenerate. By (10.8), $\mathcal{V}_1 \subset \mathcal{V}_3$, so $\mathcal{V}_3 = \mathcal{V}_1 \oplus \mathcal{V}_5$. Set

$$\mathcal{V}_6 := \mathcal{V}_1 \cap \mathcal{V}_2, \ \mathcal{V}_7 := \mathcal{V}_6 + \mathcal{V}_5, \ \ \mathcal{V}_8 := \mathcal{V}_7^{\circ} \cap \mathcal{V}_3.$$

We claim first that

$$\mathcal{V}_5 \cap (\mathcal{V}_1 + \mathcal{V}_2) = \{0\}, \tag{10.12}$$

$$\mathcal{V}_6 \cap \mathcal{V}_5^{\circ} = \{0\}. \tag{10.13}$$

In fact

$$\mathcal{V}_5 \cap (\mathcal{V}_1 + \mathcal{V}_2) \subset \mathcal{V}_3 \cap (\mathcal{V}_1 + \mathcal{V}_2) = \mathcal{V}_2^{\perp} \cap (\mathcal{V}_1 + \mathcal{V}_2) \subset \mathcal{V}_1.$$

Hence, $\mathcal{V}_5 \cap (\mathcal{V}_1 + \mathcal{V}_2) \subset \mathcal{V}_5 \cap \mathcal{V}_1 = \{0\}$.

Similarly,

$$\mathcal{V}_6 \cap \mathcal{V}_5^{\circ} \subset \mathcal{V}_2 \cap \mathcal{V}_5^{\circ} = \mathcal{V}_1^{\circ} \cap \mathcal{V}_5^{\circ} = (\mathcal{V}_1 + \mathcal{V}_5)^{\circ} = \mathcal{V}_3^{\circ} = \mathcal{V}_4.$$

Hence, $\mathcal{V}_6 \cap \mathcal{V}_5^{\circ} \subset \mathcal{V}_3 \cap \mathcal{V}_4 = \{0\}$.

Recall that if E_1, E_2, F are subspaces of E, then

$$(E_1 + E_2) \cap F = E_1 \cap F + E_2 \cap F, \text{ if } E_i \subset F \text{ for } i = 1 \text{ or } 2. \tag{10.14}$$

Let us prove that

$$\mathcal{V}_7 \cap \mathcal{V}_8 = \{0\}. \tag{10.15}$$

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In fact,

$$\mathcal{V}_8 = \mathcal{V}_6^\circ \cap \mathcal{V}_5^\circ \cap \mathcal{V}_3 = (\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}_5^\circ \cap \mathcal{V}_3,$$

hence

$$\mathcal{V}_7 \cap \mathcal{V}_8 = (\mathcal{V}_5 + \mathcal{V}_6) \cap (\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}_5^\circ.$$

Since $\mathcal{V}_6 \subset \mathcal{V}_1 + \mathcal{V}_2$, we have

$$(\mathcal{V}_5 + \mathcal{V}_6) \cap (\mathcal{V}_1 + \mathcal{V}_2) = \mathcal{V}_5 \cap (\mathcal{V}_1 + \mathcal{V}_2) + \mathcal{V}_6 \cap (\mathcal{V}_1 + \mathcal{V}_2) = \mathcal{V}_6,$$

using (10.12). Next, $\mathcal{V}_6 \cap \mathcal{V}_5^\circ = \{0\}$ by (10.13), which proves (10.15).

It follows that $\mathcal{V}_3 = \mathcal{V}_7 \oplus \mathcal{V}_8$, and that this decomposition is ω^{-1}-orthogonal. Since $\mathcal{V}_8 \subset (\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}_3 \subset \mathcal{V}_1$, the decomposition is also ζ-orthogonal and $\zeta = 0$ on $\mathcal{V}_8$.

Finally, $\mathrm{Ker}\,\zeta \cap \mathcal{V}_7 = \mathcal{V}_1 \cap \mathcal{V}_7 = \mathcal{V}_1 \cap \mathcal{V}_2$ and

$$\begin{aligned}(\mathcal{V}_1 \cap \mathcal{V}_2)^\circ \cap \mathcal{V}_7 &= (\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{V}_7 = (\mathcal{V}_1 + \mathcal{V}_2) \cap (\mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{V}_5) \\ &= \mathcal{V}_1 \cap \mathcal{V}_2 + (\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}_5 = \mathcal{V}_1 \cap \mathcal{V}_2,\end{aligned}$$

by (10.12). Hence, $\mathrm{Ker}\,\zeta$ is Lagrangian in $\mathcal{V}_7$. □

Proof of Prop. 10.8. We first consider separately three cases:
Case 1: ζ is non-degenerate. We can treat $\mathcal{Y}$ as a Euclidean space and apply Corollary 2.85 to the anti-symmetric operator $\omega\zeta$ with a trivial kernel.
Case 2: $\mathrm{Ker}\,\zeta$ *is a Lagrangian subspace of* $\mathcal{Y}^\#$. Let $\mathcal{V}$ a maximal subspace of $\mathcal{Y}^\#$ on which ζ is non-degenerate. We check that $\mathcal{V}$ is Lagrangian and $\mathcal{Y}^\# = \mathcal{V} \oplus \mathrm{Ker}\,\zeta$. We choose a ζ-orthogonal basis $(e^1, \dots, e^d)$ of $\mathcal{V}$ and complete it to a symplectic basis of $\mathcal{Y}^\#$ by setting $e^{2j} = \omega\zeta e^j$, for $1 \leq j \leq d$.
Case 3: ζ=0. We choose any symplectic basis of $\mathcal{Y}^\#$.

In the general case we use Lemma 10.10 and apply Case 1 to $\mathcal{V}_4$, Case 2 to $\mathcal{V}_7$ and Case 3 to $\mathcal{V}_8$. The remaining statements of the proposition are immediate. □

Proposition 10.11 *Let* $\zeta \in \mathbb{C}L_{\mathrm{s}}(\mathcal{Y}^\#, \mathcal{Y})$*, with* $\mathrm{Re}\,\zeta > 0$. *Then* $\mathrm{spec}\,(\omega\zeta) \subset \mathbb{C}\backslash\mathbb{R}$.

Proof Set $\zeta = \zeta_1 + \mathrm{i}\zeta_2$, with ζ_1, ζ_2 real and $\zeta_1 > 0$. Let $w \in \mathbb{C}\mathcal{Y}^\#$ with $\omega\zeta w = \lambda w$ be an eigenvector of $\omega\zeta$ for a real eigenvalue λ. Let $w = w_1 + \mathrm{i}w_2$, with $w_1, w_2 \in \mathcal{Y}$. Then,

$$\begin{aligned}2\lambda\mathrm{i}\langle\omega^{-1}w_1|w_2\rangle &= \lambda\langle\omega^{-1}\overline{w}|w\rangle \\ &= \langle\omega^{-1}\overline{w}|\omega\zeta w\rangle = -\langle\overline{w}|\zeta w\rangle \\ &= -\langle w_1|\zeta w_1\rangle + \langle w_2|\zeta w_2\rangle.\end{aligned}$$

Since $\lambda \in \mathbb{R}$, taking the real part yields $\langle w_1|\zeta_1 w_1\rangle + \langle w_2|\zeta_1 w_2\rangle = 0$, and hence $w = 0$. □

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10.1.4 Poisson bracket on charged symplectic spaces

Recall that a complex space $\mathcal{Y}$ equipped with a non-degenerate anti-Hermitian form $\overline{y}_1\cdot\omega y_2$ is called a charged symplectic space. Its realification $\mathcal{Y}_\mathbb{R}$ is equipped with an anti-involution given by the imaginary unit denoted by j and the symplectic form

$$y_1\cdot\omega_\mathbb{R}y_2 := \mathrm{Re}\,\overline{y}_1\cdot\omega y_2 = \frac{1}{2}(\overline{y}_1\cdot\omega y_2 + y_1\cdot\overline{\omega y}_2). \tag{10.16}$$

Recall that v denotes the generic variable of $\mathcal{Y}^\#$ (which in this subsection is complex). Recall from Subsect. 4.1.5 that every function F on $\mathcal{Y}^\#$ has the usual derivative $\nabla_v^\mathbb{R}F(v)\in\mathbb{C}\mathcal{Y}_\mathbb{R}$, the holomorphic derivative $\nabla_vF(v)\in\mathcal{Y}$ and the anti-holomorphic derivative $\nabla_{\overline{v}}F(v)\in\overline{\mathcal{Y}}$, related by the identities

$$\begin{aligned}
u\cdot\nabla_v &= \frac{1}{2}\left(u\cdot\nabla_v^\mathbb{R} - \mathrm{i}(\mathrm{j}u)\cdot\nabla_v^\mathbb{R}\right),\\
\overline{u}\cdot\nabla_{\overline{v}} &= \frac{1}{2}\left(u\cdot\nabla_v^\mathbb{R} + \mathrm{i}(\mathrm{j}u)\cdot\nabla_v^\mathbb{R}\right),\\
u\cdot\nabla_v^\mathbb{R} &= u\cdot\nabla_v + \overline{u}\cdot\nabla_{\overline{v}}, \qquad u\in\mathcal{Y}^\#.
\end{aligned} \tag{10.17}$$

The symplectic form $\omega_\mathbb{R}$ allows us to define a Poisson bracket. Its expression in terms of real derivatives is

$$\{F,G\}(v) = -\nabla_v^\mathbb{R}F(v)\cdot\omega_\mathbb{R}\nabla_v^\mathbb{R}G(v).$$

Proposition 10.12 *The Poisson bracket expressed in terms of the holomorphic and anti-holomorphic derivative is*

$$\{F,G\}(v) = -\frac{1}{2}\nabla_{\overline{v}}F(v)\cdot\omega\nabla_vG(v) - \frac{1}{2}\nabla_vF(v)\cdot\overline{\omega}\nabla_{\overline{v}}G(v).$$

Proof We can write $\mathbb{C}\mathcal{Y}_\mathbb{R}\simeq\mathcal{Y}\oplus\overline{\mathcal{Y}}$; see (1.33). By (10.17), $\nabla_v^\mathbb{R}F = (\nabla_vF, \nabla_{\overline{v}}F)$, $\nabla_v^\mathbb{R}G = (\nabla_vG, \nabla_{\overline{v}}G)$, as elements of $\mathcal{Y}\oplus\overline{\mathcal{Y}}$. Besides, by (10.16), $\omega_\mathbb{R}$ written as a matrix $\mathcal{Y}\oplus\overline{\mathcal{Y}}\to\mathcal{Y}^\#\oplus\overline{\mathcal{Y}}^\#$ is

$$\omega_\mathbb{R} = \frac{1}{2}\begin{bmatrix}0 & \overline{\omega}\\ \omega & 0\end{bmatrix}.$$

□

10.2 Quantum quadratic Hamiltonians

As in the previous section, $(\mathcal{Y},\omega)$ is a finite-dimensional symplectic space. We also fix an irreducible regular CCR representation $\mathcal{Y}\ni y\mapsto W(y)\in U(\mathcal{H})$. Recall that, for $b\in\mathcal{S}'(\mathcal{Y}^\#)$, $\mathrm{Op}(b)$ denotes the Weyl–Wigner quantization of b.

Recall also that $\mathbb{C}\mathrm{Pol}_\mathrm{s}^{\leq 2}(\mathcal{Y}^\#)$ denotes the set of polynomials on $\mathcal{Y}^\#$ of degree ≤ 2. $\mathrm{CCR}_{\leq 2}^{\mathrm{pol}}(\mathcal{Y})$ will denote the set of operators on $\mathcal{H}$ obtained as the Weyl–Wigner quantization of elements of $\mathbb{C}\mathrm{Pol}_\mathrm{s}^{\leq 2}(\mathcal{Y}^\#)$. These operators will be called (quantum) quadratic Hamiltonians. (Obviously, in the above definition we can replace the Weyl–Wigner quantization with $x,D-$, $D,x-$, Wick or anti-Wick quantizations.)

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This section is devoted to the study of quantum quadratic Hamiltonians. We will see in particular that they behave to a large extent in a classical way.

10.2.1 Commutation properties of quadratic Hamiltonians

Recall that $\nabla^{(2)}b$ denotes the second derivative of $b \in \mathcal{S}'(\mathcal{Y}^{\#})$. We treat it as a distribution on $\mathcal{Y}^{\#}$ with values in $L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$.

The following theorem is one of the most striking expressions of the correspondence principle between classical and quantum mechanics.

Theorem 10.13 (1) *For* $\chi \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$, $b \in \mathcal{S}'(\mathcal{Y}^{\#})$ *we have*

$$\mathrm{i}[\mathrm{Op}(\chi), \mathrm{Op}(b)] = \mathrm{Op}(\{\chi, b\}), \tag{10.18}$$
$$\frac{1}{2}\big(\mathrm{Op}(\chi)\mathrm{Op}(b) + \mathrm{Op}(b)\mathrm{Op}(\chi)\big) = \mathrm{Op}\left(\chi b + \frac{1}{8}\mathrm{Tr}\,\omega(\nabla^{(2)}\chi)\omega\nabla^{(2)}b\right).$$

(2) *The map*

$$\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#}) \ni \chi \mapsto \mathrm{Op}(\chi) \in \mathrm{CCR}_{\leq 2}^{\mathrm{pol}}(\mathcal{Y})$$

is a $*$*-isomorphism of Lie algebras, where* $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$ *is equipped with the Poisson bracket* $\{\cdot,\cdot\}$ *and* $\mathrm{CCR}_{\leq 2}^{\mathrm{pol}}(\mathcal{Y})$ *is equipped with* $\mathrm{i}[\cdot,\cdot]$.

Proof (1) follows from (8.41) by expanding the exponential. (2) follows immediately from (10.18). □

In the following definition one cannot replace the Weyl–Wigner quantization by the other four basic quantizations.

Definition 10.14 *We denote by* $\mathrm{CCR}_2^{\mathrm{pol}}(\mathcal{Y})$ *the set of operators obtained by the Weyl–Wigner quantization of polynomials in* $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{2}(\mathcal{Y}^{\#})$. *Elements of this space will be called* purely quadratic (quantum) Hamiltonians.

It will be convenient to introduce the following notation for purely quadratic Hamiltonians:

Definition 10.15 *If* $\zeta \in L_{\mathrm{s}}(\mathbb{C}\mathcal{Y}^{\#}, \mathbb{C}\mathcal{Y})$, *then* $\mathrm{Op}(\zeta)$ *will denote the Weyl–Wigner quantization of* $\mathcal{Y}^{\#} \ni v \mapsto v{\cdot}\zeta v$.

Note that if $\chi(v) = v{\cdot}\zeta v$, then $\nabla\chi(v) = 2\zeta v$ and $\nabla^{(2)}\chi = 2\zeta$.

Proposition 10.16 (1) *For* $\zeta_1, \zeta_2 \in L_{\mathrm{s}}(\mathbb{C}\mathcal{Y}^{\#}, \mathbb{C}\mathcal{Y})$,

$$[\mathrm{Op}(\zeta_1), \mathrm{Op}(\zeta_2)] = 2\mathrm{i}\mathrm{Op}(\zeta_2{\cdot}\omega\zeta_1 - \zeta_1{\cdot}\omega\zeta_2).$$

Hence

$$sp(\mathcal{Y}) \ni a \mapsto \frac{\mathrm{i}}{2}\mathrm{Op}(a\omega^{-1}) \in \mathrm{CCR}_2^{\mathrm{pol}}(\mathcal{Y})$$

is an isomorphism of Lie algebras.

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(2) *For* $\zeta \in L_{\mathrm{s}}(\mathbb{C}\mathcal{Y}^{\#}, \mathbb{C}\mathcal{Y})$, $y \in \mathcal{Y}$,

$$W(y)\mathrm{Op}(\zeta)W(y)^{*} = \mathrm{Op}(\zeta) + 2\phi\,(\zeta\omega y) - (y\cdot\omega\zeta\omega y)\mathbb{1}.$$

Proof (1) immediately follows from (10.18). To prove (2), we use $W(y)\phi(y_1)W(y)^{*} = \phi(y_1) + y_1\cdot\omega y\mathbb{1}$. □

10.2.2 Infimum of positive quadratic Hamiltonians

Quantizations of positive quadratic Hamiltonians are positive. One can give a formula for their infimum, which in quantum physics is responsible for the so-called *Casimir effect*.

Theorem 10.17 *Let* $\zeta \in L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$, $\zeta \geq 0$. *Then* $\mathrm{Op}(\zeta)$ *extends to a bounded below self-adjoint Hamiltonian and*

$$\inf \mathrm{Op}(\zeta) = \frac{1}{2}\mathrm{Tr}\sqrt{-(\zeta\omega)^{2}}. \tag{10.19}$$

Remark 10.18 *By Prop. 10.8,* $-(\omega\zeta)^{2}$ *is a diagonalizable operator with non-negative eigenvalues, hence* $\sqrt{-(\omega\zeta)^{2}}$ *is well defined.*

Proof of Thm. 10.17. Let $(e_1, \dots, e_{2d}, e^1, \dots, e^{2d})$ be as in the proof of Prop. 10.8. Writing ϕ_i for $\phi(e_i)$, we obtain

$$\mathrm{Op}(\zeta) = \sum_{j=1}^{p} \lambda_j(\phi_{2j-1}^{2} + \phi_{2j}^{2}) + \sum_{k=p+1}^{m} \phi_{2k-1}^{2}.$$

Clearly, $\inf \phi_k^2 = 0$. By the well-known properties of the harmonic oscillator, $\inf(\phi_{2j+1}^{2} + \phi_{2j+2}^{2}) = 1$. Thus

$$\inf \mathrm{Op}(\zeta) = \sum_{j=1}^{p} \lambda_j.$$

Now,

$$\begin{aligned} -(\zeta\omega)^{2}e_j &= \lambda_j^{2}e_j, \quad && 1 \leq j \leq 2p, \\ -(\zeta\omega)^{2}e_j &= 0, && 2p+1 \leq k \leq 2d. \end{aligned}$$

Thus, $\mathrm{Tr}\sqrt{-(\omega\zeta)^{2}} = 2\sum_{j=1}^{p} \lambda_j$. □

10.2.3 Scale of oscillator spaces

In the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$, a distinguished role is played by the number operator N. It allows us to define a family of weighted Hilbert spaces $(N+\mathbb{1})^{t}\Gamma_{\mathrm{s}}(\mathcal{Z})$, which is often used in applications.

Recall that in this section we consider a regular CCR representation over a finite-dimensional symplectic space $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$. In this framework, in general we do not have a single distinguished operator similar to N. However, a similar role is played by the whole family of positive definite quadratic Hamiltonians. They define a family of equivalent norms, as shown by the following proposition.

Proposition 10.19 *Let $\zeta, \zeta_1 \in L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$, where $\zeta, \zeta_1 > 0$. Then for any $t > 0$ there exist $0 < C_t$ such that*

$$C_t^{-1} \|\mathrm{Op}(\zeta_1)^t \Psi\| \leq \|\mathrm{Op}(\zeta)^t \Psi\| \leq C_t \|\mathrm{Op}(\zeta_1)^t \Psi\|, \quad \Psi \in \mathcal{H}. \tag{10.20}$$

Proof Choose a basis, as in Prop. 10.8. Using this basis, we can identify $\mathcal{H}$ with $\Gamma_{\mathrm{s}}(\mathbb{C}^d)$ and $\mathrm{Op}(\zeta)$ with $\mathrm{d}\Gamma(h) + \frac{\mathrm{Tr}h}{2}\mathbb{1}$, where the operator h is diagonal and has positive eigenvalues. Using the natural o.n. basis of $\Gamma_{\mathrm{s}}(\mathbb{C}^d)$ we easily check that for any $n = 1, 2, \dots$ there exists C_n such that

$$\|\mathrm{Op}(\zeta_1)^n \Psi\|^2 \leq C_n \|\mathrm{Op}(\zeta)^n \Psi\|^2.$$

By interpolation, this implies the first inequality in (10.20). Reversing the role of ζ and ζ_1 we obtain the second inequality. □

Definition 10.20 *For any $t \geq 0$, the t-th* oscillator space *$\mathcal{H}^t$ is defined as $\mathrm{Dom}\,\mathrm{Op}(\zeta)^t$, where $\zeta \in L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$, $\zeta > 0$. By Thm. 10.19, $\mathcal{H}^t$ does not depend on the choice of ζ and has the structure of a Hilbertizable space. We set $\mathcal{H}^{-t} := (\mathcal{H}^t)^*$.*

Recall that in Def. 8.50 we defined $\mathcal{H}^\infty$ and in Def. 8.51 we defined $\mathcal{H}^{-\infty}$. They are related to spaces $\mathcal{H}^t$ as follows:

$$\mathcal{H}^\infty = \bigcap_{t>0} \mathcal{H}^t, \qquad \mathcal{H}^{-\infty} := \bigcup_{t<0} \mathcal{H}^t. \tag{10.21}$$

10.2.4 Quadratic Hamiltonians as closed operators

Prop. 10.19 shows that all $\mathrm{Op}(\zeta)$ with $\zeta > 0$ have the same domain and in particular are essentially self-adjoint on $\mathcal{H}^\infty$. The following theorem describes more general classes of quadratic Hamiltonians.

Theorem 10.21 (1) *Let $\chi \in \mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$ (χ is a real quadratic polynomial). Then $\mathrm{Op}(\chi)$ is essentially self-adjoint on $\mathcal{H}^\infty$.*

(2) *Let $\chi \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$ (χ is a complex quadratic polynomial). Assume that the purely quadratic part of χ is positive definite. (In other words, $\chi(v) = \mu + y{\cdot}v + \frac{1}{2} v{\cdot}\zeta v$ with $\mu \in \mathbb{C}$, $y \in \mathbb{C}\mathcal{Y}$, $\zeta \in \mathbb{C}L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$ and $\mathrm{Re}\,\zeta > 0$.) Then $\mathrm{Op}(\chi)$ is closed on $\mathcal{H}^1$ and maximal accretive.*

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Proof Fix $\zeta_0 \in L_{\rm s}(\mathcal{Y}, \mathcal{Y}^{\#})$ such that $\zeta_0 > 0$, and set $N = \mathrm{Op}(\zeta_0)$. $\mathrm{Op}(\chi)$ is Hermitian on $\mathcal{H}^\infty$. By (10.20), we have

$$\|\mathrm{Op}(\chi)\Phi\| \leq C\|N\Phi\|, \quad \Phi \in \mathcal{H}^1.$$

Next we have $[\mathrm{Op}(\chi), \mathrm{i}N] = \mathrm{Op}\{\chi, \zeta\}$. Since $\{\chi, \zeta_0\} \in \mathrm{Pol}^{\leq 2}(\mathcal{Y}^{\#})$, we have $\{\chi, \zeta_0\} \leq C\zeta_0$ for some C, and by Thm. 10.17 we get $[\mathrm{Op}(\chi), \mathrm{i}N] \leq C(N + \mathbb{1})$. Applying Nelson's commutator theorem, Thm. 2.74 (1), we obtain that $\mathrm{Op}(\chi)$ is essentially self-adjoint on $\mathrm{Dom}\, N$, hence also on $\mathcal{H}^\infty$, since N is essentially self-adjoint on $\mathcal{H}^\infty$. This proves (1).

To prove (2), we set $\chi_1 = \mathrm{Re}\,\chi$, $\chi_2 = \mathrm{Im}\,\chi$, $B_i = \mathrm{Op}(\chi_i)$, $B = \mathrm{Op}(\chi)$. We note that

$$\pm[B_1, \mathrm{i}B_2] = \mathrm{Op}(\{\chi_1, \chi_2\}) \leq C(B_1 + \mathbb{1}), \tag{10.22}$$

by Thm. 10.17. We write

$$\begin{aligned} B^*B &= B_1^2 + B_2^2 + [B_1, \mathrm{i}B_2] \\ &\geq B_1^2 - C_1(B_1 + \mathbb{1}) \geq \tfrac{1}{2}B_1^2 - C_2\mathbb{1}, \end{aligned} \tag{10.23}$$

using (10.22), which shows that B is closed on $\mathrm{Dom}\, B_1$. Next

$$\mathrm{Re}(\Psi|B\Psi) = (\Psi|B_1\Psi) \geq 0, \tag{10.24}$$

by Thm. 10.17. It remains to prove that $B + \lambda$ is invertible for large enough λ. Inequalities (10.24) and (10.23) for B replaced by $B + \lambda\mathbb{1}$ show that $\mathrm{Ker}(B + \lambda\mathbb{1}) = \{0\}$ and that $\mathrm{Ran}(B + \lambda\mathbb{1})$ is closed. Next we have

$$\begin{aligned} &\tfrac{1}{2}(B + \lambda\mathbb{1})(B_1 + c\mathbb{1})^{-1} + \tfrac{1}{2}(B_1 + c\mathbb{1})^{-1}(B^* + \lambda\mathbb{1}) \\ &= (B_1 + c\mathbb{1})^{-\frac{1}{2}}(B_1 + \lambda\mathbb{1})(B_1 + c\mathbb{1})^{-\frac{1}{2}} + \tfrac{1}{2}(B_1 + c\mathbb{1})^{-1}[B_1, \mathrm{i}B_2](B_1 + c\mathbb{1})^{-1} \\ &\geq (\lambda\mathbb{1} - C)(B_1 + c\mathbb{1})^{-1}, \end{aligned}$$

again using (10.22). If $\Psi \in \mathrm{Ran}(B + \lambda\mathbb{1})^\perp$, then

$$\mathrm{Re}\,(\Psi|(B + \lambda\mathbb{1})(B_1 + c\mathbb{1})^{-1}\Psi) = 0,$$

and hence $\Psi = 0$ if λ is large. This completes the proof of (2). □

10.2.5 One-parameter groups of Bogoliubov $*$-automorphisms

Classical quadratic Hamiltonians generate one-parameter groups of linear symplectic transformations. On the quantum level one can assign two roles to a quadratic Hamiltonian H: $\mathrm{i}[H, \cdot]$ generates a one-parameter group of $*$-automorphisms $\mathrm{e}^{\mathrm{i}tH} \cdot \mathrm{e}^{-\mathrm{i}tH}$, and $\mathrm{i}H$ generates the one-parameter unitary group $\mathrm{e}^{\mathrm{i}tH}$. The following theorem describes the former group. The latter group, which is somewhat more difficult, is discussed in the following section.

Theorem 10.22 *Let $\chi \in \mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$, i.e. χ is a real quadratic polynomial. Let $b \in \mathcal{S}'(\mathcal{Y}^{\#})$ and $b_t(v) = b(\mathrm{e}^{t\omega\nabla\chi}v)$. Then*

$$\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)}\mathrm{Op}(b)\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)} = \mathrm{Op}(b_t). \tag{10.25}$$

In particular, for $y \in \mathcal{Y}$,

$$\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)}W(y)\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)} = W(\mathrm{e}^{-t\nabla\chi\omega}y). \tag{10.26}$$

Proof Let $\Phi, \Psi \in \mathcal{H}^{\infty}$, $b \in \mathcal{S}(\mathcal{Y}^{\#})$. By (10.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}b_t(v) = \{\chi, b_t\}(v).$$

Set

$$\Phi_t := \mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)}\Phi, \quad \Psi_t := \mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)}\Psi.$$

We know that $\mathrm{Op}(\chi)$ is self-adjoint and $\Phi, \Psi \in \mathrm{Dom}\,\mathrm{Op}(\chi)$. Hence, by Thm. 10.13 (1),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t|\mathrm{Op}(b_t)\Psi_t) = -\mathrm{i}(\Phi_t|[\mathrm{Op}(\chi), \mathrm{Op}(b_t)]\Psi_t)& \\ +(\Phi_t|\mathrm{Op}(\{\chi, b_t\})\Psi_t) \;=\; 0.&\end{aligned}$$

Hence,

$$\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}\mathrm{Op}(b_t)\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)} = \mathrm{Op}(b).$$

This proves (10.25) for $b \in \mathcal{S}(\mathcal{Y}^{\#})$. We extend (10.25) to $\mathcal{S}'(\mathcal{Y}^{\#})$ by duality. □

For further reference, let us restate Thm. 10.22 for purely quadratic Hamiltonians.

Corollary 10.23 *Let $\zeta \in L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$. Then for $b \in \mathcal{S}'(\mathcal{Y})^{\#}$, $b_t(v) = b(\mathrm{e}^{t\omega\nabla\zeta}v)$,*

$$\mathrm{e}^{\frac{\mathrm{i}t}{2}\mathrm{Op}(\zeta)}\mathrm{Op}(b)\mathrm{e}^{-\frac{\mathrm{i}t}{2}\mathrm{Op}(\zeta)} = \mathrm{Op}(b_t).$$

In particular, for $y \in \mathcal{Y}$,

$$\mathrm{e}^{\frac{\mathrm{i}t}{2}\mathrm{Op}(\zeta)}W(y)\mathrm{e}^{-\frac{\mathrm{i}t}{2}\mathrm{Op}(\zeta)} = W(\mathrm{e}^{-t\zeta\omega}y).$$

10.3 Metaplectic group

In this section, as in the previous one, $(\mathcal{Y}, \omega)$ is a finite-dimensional symplectic space and $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ is an irreducible regular CCR representation. In this section we study unitary operators of the form $\mathrm{e}^{\mathrm{i}H}$, where H is a purely quadratic Hamiltonian. We show that they form a group, called the metaplectic group, isomorphic to the double cover of the symplectic group.

10.3.1 Implementation of Bogoliubov transformations

It follows from the Stone–von Neumann theorem that, for a finite-dimensional symplectic space, all Bogoliubov rotations can be implemented by unitary operators. The set of such unitary implementers forms a group.

Definition 10.24 *We define $Mp^{\rm c}(\mathcal{Y})$ to be the set of $U \in U(\mathcal{H})$ such that*

$$\big\{UW(y)U^* \ : \ y \in \mathcal{Y}\big\} = \big\{W(y) \ : \ y \in \mathcal{Y}\big\}.$$

Proposition 10.25 *Let $U \in Mp^{\rm c}(\mathcal{Y})$. Then there exists a unique $r \in Sp(\mathcal{Y})$ satisfying*

$$UW(y)U^* = W(ry), \quad y \in \mathcal{Y}. \tag{10.27}$$

The map $Mp^{\rm c}(\mathcal{Y}) \to Sp(\mathcal{Y})$ obtained this way is a group homomorphism.

Definition 10.26 *If (10.27) is satisfied, we say that U* implements *r.*

Note that (10.27) is equivalent to

$$U\mathrm{Op}(a)U^* = \mathrm{Op}(a \circ r^{\#}), \quad a \in \mathcal{S}'(\mathcal{Y}^{\#}). \tag{10.28}$$

There also exists a smaller group that is sufficient to implement all linear symplectic transformations. Its definition is more involved. As a preparation for this definition, with every $r \in Sp(\mathcal{Y})$ we associate a pair of unitaries $\pm U_r$ differing by a sign:

Definition 10.27 (1) *Let $r \in Sp(\mathcal{Y})$ be regular (see Def. 10.6). Let $\gamma \in sp(\mathcal{Y})$ be the Cayley transform of r, that is, $\gamma = \frac{1\!\!1 - r}{1\!\!1 + r}$ (see Subsect. 1.4.6). Set*

$$\begin{aligned} &\pm U_r := \pm \mathrm{Op}(f), \\ \textit{where} \quad &f(v) \ = \ \det(1\!\!1 + \gamma)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}v \cdot \gamma\omega^{-1}v}. \end{aligned} \tag{10.29}$$

(2) *Let $r \in Sp(\mathcal{Y})$ be arbitrary. Let $r = r_0\kappa$ be the canonical decomposition of r into a regular $r_0 \in Sp(\mathcal{Y})$ and an involution $\kappa \in Sp(\mathcal{Y})$ given by (10.4). Let $\mathcal{Y} = \mathcal{Y}_{\rm sg} \oplus \mathcal{Y}_{\rm reg}$ be the decomposition of the symplectic space such that $\kappa = (-1\!\!1) \oplus 1\!\!1$. Let $m = \dim \mathcal{Y}_{\rm sg}$. Then we set*

$$\pm U_r := \pm U_\kappa U_{r_0}$$

for

$$U_\kappa := \pm \mathrm{Op}((\mathrm{i}\pi)^{m/2}\delta_{\rm sg}),$$

where $\delta_{\rm sg}$ is the Dirac delta function at zero on $\mathcal{Y}_{\rm sg}^{\#}$ times 1 *on $\mathcal{Y}_{\rm reg}^{\#}$.*

Note that under the assumptions of Def. 10.27 (2), our CCR representation can be decomposed as the tensor product of a representation over $\mathcal{Y}_{\rm sg}$ and over $\mathcal{Y}_{\rm reg}$, and then

$$\pm U_\kappa = \pm \mathrm{i}^{m/2} I_{\rm sg} \otimes 1\!\!1_{\rm reg},$$

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where $I_{\rm sg}$ is the parity operator corresponding to $\mathcal{Y}_{\rm sg}$, defined as in Subsect. 8.4.4.

Definition 10.28 *$Mp(\mathcal{Y})$ is the set of operators of the form $\pm U_r$ for some $r \in Sp(\mathcal{Y})$. It is called the* metaplectic group of $\mathcal{Y}$.

Theorem 10.29 *Let $r \in Sp(\mathcal{Y})$.*

(1) *The set of elements of $Mp(\mathcal{Y})$ implementing r consists of a pair operators differing by the sign $\pm U_r = \{U_r, -U_r\}$.*
(2) *The set of elements of $Mp^{\rm c}(\mathcal{Y})$ implementing r consists of operators of the form μU_r with $|\mu| = 1$.*
(3) *If $r_1, r_2 \in Sp(\mathcal{Y})$, then $U_{r_1} U_{r_2} = \pm U_{r_1 r_2}$.*

The above statements can be summarized by the following commuting diagram consisting of exact horizontal and vertical sequences:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & U(1) & \rightarrow & U(1) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Mp(\mathcal{Y}) & \rightarrow & Mp^{\rm c}(\mathcal{Y}) & \rightarrow & U(1) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Sp(\mathcal{Y}) & \rightarrow & Sp(\mathcal{Y}) & \rightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{10.30}
$$

The meaning of all the arrows in the above diagram should be obvious. In particular, the horizontal arrow $U(1) \to U(1)$ is just $\mu \to \mu^2$.

It remains to prove Thm. 10.29. We start by considering the case of regular symplectic maps. Recall that the formula for $\pm U_r$ is then given in (10.29).

Lemma 10.30 *Let $r \in Sp(\mathcal{Y})$ be regular. Then*

(1) *U_r intertwines r, i.e.*

$$U_r \phi(y) = \phi(ry) U_r, \quad y \in \mathcal{Y}.$$

(2) *U_r is unitary.*
(3) *If $r_1, r_2, r \in Sp(\mathcal{Y})$ are regular and $r_1 r_2 = r$, then $U_{r_1} U_{r_2} = \pm U_r$.*

Proof Let $y \in \mathcal{Y}$. Set $b(v) = v{\cdot}\gamma\omega^{-1}v$. Using Thm. 10.13 (1), we obtain

$$\mathrm{Op}(\mathrm{e}^{\mathrm{i}b})\phi(y) = \mathrm{Op}(\mathrm{e}^{\mathrm{i}b})\mathrm{Op}(y) = \mathrm{Op}\left(\mathrm{e}^{\mathrm{i}b} y - \frac{\mathrm{i}}{2}\{y, \mathrm{e}^{\mathrm{i}b}\}\right).$$

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Now,

$$\mathrm{e}^{\mathrm{i}b}y - \frac{\mathrm{i}}{2}\left\{y, \mathrm{e}^{\mathrm{i}b}\right\} = \mathrm{e}^{\mathrm{i}b}(\mathbb{1}-\gamma)y$$
$$= \mathrm{e}^{\mathrm{i}b}(\mathbb{1}+\gamma)ry = \mathrm{e}^{\mathrm{i}b}ry + \frac{\mathrm{i}}{2}\left\{ry, \mathrm{e}^{\mathrm{i}b}\right\}.$$

Hence,

$$\mathrm{Op}(\mathrm{e}^{\mathrm{i}b})\phi(y) = \mathrm{Op}\left(\mathrm{e}^{\mathrm{i}b}(ry) + \frac{\mathrm{i}}{2}\left\{ry, \mathrm{e}^{\mathrm{i}b}\right\}\right) = \phi(ry)\mathrm{Op}(\mathrm{e}^{\mathrm{i}b}).$$

Thus $\mathrm{Op}(\mathrm{e}^{\mathrm{i}b})$ intertwines r, and hence (1) is true.

Let b_1, b_2, b, $\gamma_1, \gamma_2, \gamma$ be related to r_1, r_2, r as in (10.29). We know that $\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_1})\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_2})$ intertwines r. Likewise, we know that $\mathrm{Op}(\mathrm{e}^{\mathrm{i}b})$ intertwines r. Hence, for some c,

$$\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_1})\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_2}) = c\,\mathrm{Op}(\mathrm{e}^{\mathrm{i}b}).$$

Next using Thm. 8.70 and formula (4.12), we obtain that $\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_1})\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_2})$ has the symbol

$$\begin{aligned}&\int \exp\big(\mathrm{i}v_1\cdot\gamma_1\omega^{-1}v_1 + \mathrm{i}v_2\cdot\gamma_2\omega^{-1}v_2 - 2\mathrm{i}v_1\cdot\omega^{-1}v - 2\mathrm{i}v\cdot\omega^{-1}v_2 + 2\mathrm{i}v_1\cdot\omega^{-1}v_2\big)\frac{\mathrm{d}v_1\mathrm{d}v_2}{\pi^{2d}}\\ &= \pi^{-2d}\int \exp\big(\mathrm{i}(v_1,v_2)\cdot\sigma(v_1,v_2) + 2\mathrm{i}\theta\cdot(v_1,v_2)\big)\frac{\mathrm{d}v_1\mathrm{d}v_2}{\pi^{2d}},\end{aligned} \tag{10.31}$$

where

$$\theta := (\omega^{-1}v, -\omega^{-1}v), \quad \sigma := \begin{bmatrix}\gamma_1\omega^{-1} & \omega^{-1}\\ -\omega^{-1} & \gamma_2\omega^{-1}\end{bmatrix}.$$

(10.31) equals

$$\det(-\mathrm{i}\sigma)^{-\frac{1}{2}}\exp(-\mathrm{i}\theta\cdot\sigma^{-1}\theta).$$

Setting $v = 0$ and using Subsect. 1.1.2, we obtain

$$c = \det(-\mathrm{i}\sigma)^{-\frac{1}{2}} = \pm\det(\mathbb{1}+\gamma_1\gamma_2)^{-\frac{1}{2}}.$$

Next, by (1.49),

$$\mathbb{1}+\gamma = (\mathbb{1}+\gamma_2)(\mathbb{1}+\gamma_1\gamma_2)^{-1}(\mathbb{1}+\gamma_1).$$

Hence,

$$\det(\mathbb{1}+\gamma)^{\frac{1}{2}} = \pm\det(\mathbb{1}+\gamma_2)^{\frac{1}{2}}(\mathbb{1}+\gamma_1\gamma_2)^{\frac{1}{2}}(\mathbb{1}+\gamma_1)^{\frac{1}{2}}.$$

Therefore,

$$\det(\mathbb{1}+\gamma)^{\frac{1}{2}}\mathrm{Op}(\mathrm{e}^{\mathrm{i}b}) = \pm\det(\mathbb{1}+\gamma)^{\frac{1}{2}}\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_1})(\mathbb{1}+\gamma_1)^{\frac{1}{2}}\mathrm{Op}(\mathrm{e}^{\mathrm{i}b_2}).$$

This proves (3).

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It remains to prove that U_r is unitary. We have $U_r^* = \lambda U_{r^{-1}}$, for

$$\lambda = \pm \det \overline{(\mathbb{1}+\gamma)}^{\frac{1}{2}} \det(\mathbb{1}-\gamma)^{-\frac{1}{2}}.$$

Since by (3) $U_{r^{-1}} U_r = \pm U_{\mathbb{1}} = \pm\mathbb{1}$, it suffices to verify that $|\lambda| = 1$. But using that $\det a = \det a^{\#}$, we get

$$\det(\mathbb{1}+\gamma) = \det\big(\omega(\mathbb{1}-\gamma)\omega^{-1}\big) = \det(\mathbb{1}-\gamma).$$

This implies that $|\det(\mathbb{1}+\gamma)^{\frac{1}{2}}| = |\det(\mathbb{1}-\gamma)^{\frac{1}{2}}|$, which completes the proof of (2). □

To treat the general case we will need more lemmas.

Lemma 10.31 *Let r_1, r_2, r_3 be regular. Then we can write r_2 as $r_2 = \tilde{r}_2\hat{r}_2$ with $\tilde{r}_2, \hat{r}_2, r_1\tilde{r}_2$ and $\hat{r}_2 r_3$ regular.*

Proof Let $D = \big\{r \in Sp(\mathcal{Y}) \ : \ r \text{ and } r_1 r \text{ are regular}\big\}$. Clearly, D is an open dense subset of $Sp(\mathcal{Y})$ containing $\mathbb{1}$. Hence, we can write r_2 as $r_2 = \tilde{r}_2\hat{r}_2$, where $\tilde{r}_2$, $r_1\tilde{r}_2$ are regular and $\mathbb{1} - \hat{r}_2$ can be made as small as we wish. Then if $\mathbb{1} - \hat{r}_2$ is sufficiently small, $\hat{r}_2$ and $\hat{r}_2 r_3$ are regular. □

Lemma 10.32 *Let $r_i, \tilde{r}_i \in Sp(\mathcal{Y})$ be regular for $1 \le i \le p$. Assume that $r_1 \cdots r_p = \tilde{r}_1 \cdots \tilde{r}_p$. Then*

$$U_{r_1} \cdots U_{r_p} = \pm U_{\tilde{r}_1} \cdots U_{\tilde{r}_p}.$$

Proof If r is regular, so is r^{-1}, and hence, by Lemma 10.30, $U_{r^{-1}} = \pm U_r^{-1}$. Therefore, we are reduced to proving that

$$r_1 \cdots r_p = \mathbb{1} \ \Rightarrow U_{r_1} \cdots U_{r_p} = \pm\mathbb{1}. \tag{10.32}$$

Using Lemma 10.31, we write $r_1 r_2 r_3$ as $r_1\tilde{r}_2\hat{r}_2 r_3$. Then, by Lemma 10.30, we get

$$U_{r_2} = \pm U_{\tilde{r}_2} U_{\hat{r}_2}, \quad U_{r_1} U_{r_2} U_{r_3} = \pm U_{r_1} U_{\tilde{r}_2} U_{\hat{r}_2} U_{r_3} = \pm U_{r_1\tilde{r}_2} U_{\hat{r}_2 r_3}.$$

Relabeling the r_i, we are reduced to showing (10.32) with p replaced by $p-1$. Continuing in this way we end up with

$$r_1 r_2 = \mathbb{1} \ \Rightarrow U_{r_1} U_{r_2} = \pm\mathbb{1},$$

which holds since r_1, r_2 and $\mathbb{1}$ are regular. □

Lemma 10.33 *Let κ be a symplectic involution, so that there exists a decomposition $\mathcal{Y} = \mathcal{Y}_{\rm reg} \oplus \mathcal{Y}_{\rm sg}$ into mutually ω-orthogonal subspaces and $\kappa = (-\mathbb{1}) \oplus \mathbb{1}$. Decompose $\mathcal{Y}_{\rm sg}$ further as $\mathcal{Y}_{\rm sg} = \mathcal{X}_{\rm sg} \oplus \mathcal{X}_{\rm sg}$, where $\mathcal{X}_{\rm sg}$ is a Euclidean space, with the standard symplectic form on $\mathcal{Y}_{\rm sg}$. Set*

$$u := \begin{bmatrix} 0 & -\mathbb{1} & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{bmatrix} \in Sp(\mathcal{Y}).$$

Then $u \in Sp(\mathcal{Y})$ is regular, $u^2 = \kappa$ and $\pm U_\kappa = \pm U_u^2$.

Proof The lemma follows by the properties of the evolution generated by the harmonic oscillator; see Subsect. 10.5.1. □

Proof of Thm. 10.29. Let us show that $Mp(\mathcal{Y})$ is a group. Let $r_1, r_2, r_3 \in Sp(\mathcal{Y})$, $r_1 r_2 = r_3$. Let $r_i = \kappa_i r_{0i}$ be the decomposition described in (10.4). Applying Lemma 10.33, we can write $\kappa_i = u_i^2$, where $u_i \in Sp(\mathcal{Y})$ are regular. Thus, we have

$$r_1 r_2 = r_{01} u_1^2 r_{02} u_2^2 = r_{03} u_3^2 = r_3.$$

By definition and then Lemma 10.33,

$$\pm U_{r_1} U_{r_2} = \pm U_{r_{01}} U_{u_1}^2 U_{r_{02}} U_{u_2}^2, \qquad (10.33)$$

$$\pm U_{r_3} = \pm U_{r_{03}} U_{u_3}^2. \qquad (10.34)$$

By Lemma 10.32, (10.33) equals (10.34). This proves also that $Mp(\mathcal{Y}) \to Sp(\mathcal{Y})$ is a homomorphism with the kernel consisting of $\{1\!\!1, -1\!\!1\} \simeq \mathbb{Z}_2$.

It is obvious that $Mp^{\rm c}(\mathcal{Y})$ is a group. It clearly contains $Mp(\mathcal{Y})$, and hence the homomorphism $Mp^{\rm c}(\mathcal{Y}) \to Sp(\mathcal{Y})$ is onto. By the irreducibility of the CCR representation, the kernel of this homomorphism is $U(1)$. □

10.3.2 Semi-groups generated by quadratic Hamiltonians

In the next two theorems, we will compute the Weyl–Wigner symbol of the semi-group generated by a maximal accretive quadratic Hamiltonian and of the unitary group generated by a self-adjoint quadratic Hamiltonian. We start with the case of a maximal accretive Hamiltonian.

Theorem 10.34 *Let $y \in \mathbb{C}\mathcal{Y}$, $\zeta \in \mathbb{C}L_{\rm s}(\mathcal{Y}^{\#}, \mathcal{Y})$. Assume that $\mathrm{Re}\,\zeta > 0$. Consider the complex quadratic polynomial*

$$\chi(v) = y{\cdot}v + v{\cdot}\zeta v. \qquad (10.35)$$

Then for $t \geq 0$, the bounded operator $\mathrm{e}^{-t\mathrm{Op}(\chi)}$ has the Weyl–Wigner symbol

$$f_t(v) = (\det\cos t\omega\zeta)^{-\frac{1}{2}} \exp\Big(-v{\cdot}\omega^{-1}\mathrm{tg}(t\omega\zeta)v \qquad (10.36)$$
$$-y{\cdot}(\omega\zeta)^{-1}\mathrm{tg}(t\omega\zeta)v + \frac{1}{4}y{\cdot}\left(t1\!\!1 - (\omega\zeta)^{-1}\mathrm{tg}(t\omega\zeta)\right)\zeta^{-1}y\Big).$$

The next theorem describes the case of a quadratic self-adjoint Hamiltonian.

Theorem 10.35 *Let $y \in \mathcal{Y}$, $\zeta \in L_{\rm s}(\mathcal{Y}^{\#}, \mathcal{Y})$. Consider the real quadratic polynomial χ defined as in (10.35). For $t \in \mathbb{R}$, let $g_t(v)$ be the Weyl–Wigner symbol of the unitary operator $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}$.*

(1) *If $\pi\mathbb{Z} \cap \mathrm{spec}\, t\omega\zeta = \emptyset$, then*

$$g_t(v) = (\det\cosh t\omega\zeta)^{-\frac{1}{2}} \exp\Big(\mathrm{i}v{\cdot}\omega^{-1}\tanh(t\omega\zeta)v \qquad (10.37)$$
$$+\mathrm{i}y{\cdot}(\omega\zeta)^{-1}\tanh(t\omega\zeta)v + \frac{\mathrm{i}}{4}y{\cdot}\left((\omega\zeta)^{-1}\tanh(t\omega\zeta) - t1\!\!1\right)\zeta^{-1}y\Big).$$

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(2) *In the general case, to find the Weyl–Wigner symbol of* $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}$ *we can do as follows. We choose* $\zeta_1 \in L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$ *with* $\zeta_1 > 0$*. We set* $\zeta_\epsilon := \zeta + \mathrm{i}\epsilon\zeta_1$*, and let* $g_{\epsilon,t}$ *be defined by (10.37), where* ζ *is replaced with* ζ_ϵ*. Then*

$$g_t(v) = \begin{cases} \lim\limits_{\epsilon\searrow 0} g_{\epsilon,t}(v), & t \geq 0; \\ \lim\limits_{\epsilon\nearrow 0} g_{\epsilon,t}(v), & t \leq 0. \end{cases}$$

Proof of Thm. 10.34. We first note that $\mathrm{e}^{-t\mathrm{Op}(\chi)}$ is well defined as a strongly continuous semi-group, since $\mathrm{Op}(\chi)$ is maximal accretive. Note also from Lemma 10.11 that $\omega\zeta$ and $\zeta\omega$ have no real eigenvalues, so the operator $\mathrm{tg}(t\omega\zeta)$ is well defined by the holomorphic functional calculus and $\cos(t\omega\zeta)^{\frac{1}{2}} \neq 0$.

Since

$$\partial_t \mathrm{e}^{-t\mathrm{Op}(\chi)} = -\frac{1}{2}\big(\mathrm{Op}(\chi)\mathrm{e}^{-t\mathrm{Op}(\chi)} + \mathrm{e}^{-t\mathrm{Op}(\chi)}\mathrm{Op}(\chi)\big),$$

it suffices, using Thm. 10.13, to verify that

$$\begin{cases} \partial_t f_t(v) = -\chi(v) f_t(v) + \frac{1}{8}\mathrm{Tr}(\nabla^{(2)}\chi)\omega\nabla^{(2)} f_t(v)\omega, \\ \quad f_0(v) = 1. \end{cases} \tag{10.38}$$

We have

$$\begin{aligned} \partial_t f_t(v) &= f_t(v)\Big(-v{\cdot}\zeta\cos^{-2}(t\omega\zeta)v - y{\cdot}\cos^{-2}(t\omega\zeta)v \\ &\qquad + \frac{1}{4}y{\cdot}\left(\mathbb{1} - \cos^{-2}(t\omega\zeta)\right)\zeta^{-1}y - \frac{1}{2}\partial_t \log\det\cos(t\omega\zeta)\Big) \\ &= f_t(v)\Big(-v{\cdot}\zeta v - y{\cdot}v - v{\cdot}\zeta\mathrm{tg}^2(t\omega\zeta)v - y{\cdot}\mathrm{tg}^2(t\omega\zeta)v \\ &\qquad + \frac{1}{4}y{\cdot}\mathrm{tg}^2(t\omega\zeta)\zeta^{-1}y + \frac{1}{2}\mathrm{Tr}\omega\zeta\mathrm{tg}(t\omega\zeta)\Big). \end{aligned} \tag{10.39}$$

Now,

$$\begin{aligned} &\nabla^{(2)}\chi(v) = 2\zeta, \\ &\nabla^{(2)} f_t(v) = -f_t(v)\Bigg(2\omega^{-1}\mathrm{tg}(t\omega\zeta) \\ &\qquad - \Big|\mathrm{tg}(t\omega\zeta)\big(2\omega^{-1}v + (\omega\zeta)^{-1}y\big)\Big\rangle\Big\langle\mathrm{tg}(t\omega\zeta)\big(2\omega^{-1}v + (\omega\zeta)^{-1}\big)y\Big|\Bigg). \end{aligned}$$

Using that $\mathrm{Tr}|y_1\rangle\langle y_2| = \langle y_2|y_1\rangle$, we get

$$\begin{aligned} \mathrm{Tr}(\nabla^{(2)}\chi)\omega\nabla^{(2)} f_t(v)\omega = f_t(v)\Big(&4\mathrm{Tr}\omega\zeta\mathrm{tg}(t\omega\zeta) \\ &+8v{\cdot}\omega\zeta\mathrm{tg}^2(t\omega\zeta)v + 8v{\cdot}\mathrm{tg}^2(t\omega\zeta)y + 2y{\cdot}\zeta^{-1}y\Big). \end{aligned} \tag{10.40}$$

Comparing (10.39) and (10.40) we see that (10.38) is true. □

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Proof of Thm. 10.35. We may assume that $t \geq 0$. For $\epsilon > 0$, set $\chi_\epsilon(v) = \chi(v) - \mathrm{i}\epsilon v{\cdot}\zeta_1 v$. Then we have

$$\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)} = \mathrm{s} - \lim_{\epsilon \searrow 0} \mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi_\epsilon)}.$$

This implies that $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi_\epsilon)}$ converges to $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}$ in $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{Y})$. This implies that the Weyl–Wigner symbol of $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi_\epsilon)}$ converges to the Weyl–Wigner symbol of $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}$ in $\mathcal{S}'(\mathcal{Y}^\#)$. Hence Thm. 10.35 follows from Thm. 10.34. □

The following theorem provides an alternative definition of the metaplectic group:

Theorem 10.36 (1) *Let* $\zeta \in L_{\mathrm{s}}(\mathcal{Y}^\#, \mathcal{Y})$. *Then* $\mathrm{e}^{\mathrm{i}\mathrm{Op}(\zeta)} \in Mp(\mathcal{Y})$.
(2) *Conversely,* $Mp(\mathcal{Y})$ *is generated by operators of the form* $\mathrm{e}^{\mathrm{i}\mathrm{Op}(\zeta)}$ *with* $\zeta \in L_{\mathrm{s}}(\mathcal{Y}^\#, \mathcal{Y})$.

Proof By Thm. 10.35, $\mathrm{e}^{\mathrm{i}\mathrm{Op}(\zeta)} = \mathrm{Op}(g)$, where

$$g(v) = (\det\cosh t\omega\zeta)^{-\frac{1}{2}} \exp\left(\mathrm{i}v{\cdot}\omega^{-1}\tanh(t\omega\zeta)v\right).$$

Set $r = \mathrm{e}^{-\zeta\omega}$. Then

$$\gamma = \frac{\mathrm{e}^{\zeta\omega} - \mathrm{e}^{-\zeta\omega}}{\mathrm{e}^{\zeta\omega} + \mathrm{e}^{-\zeta\omega}} = \tanh\zeta\omega,$$

$$\mathbb{1} + \gamma = 2(\mathbb{1} + r)^{-1} = \frac{\mathrm{e}^{\zeta\omega}}{\cosh\zeta\omega}.$$

Taking into account that $\det\mathrm{e}^{\zeta\omega} = 1$, we obtain that

$$g(v) = \det(\mathbb{1} + \gamma\omega)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}v\cdot\gamma v}.$$

This proves (1).

All elements of $Sp(\mathcal{Y})$ in a neighborhood of $\mathbb{1}$ are of the form $r = \mathrm{e}^a$ for $a \in sp(\mathcal{Y})$. By (1), the corresponding $\pm U_r$ are of the form $\mathrm{e}^{\mathrm{i}\mathrm{Op}(\zeta)}$ for $\zeta \in L_{\mathrm{s}}(\mathcal{Y}^\#, \mathcal{Y})$. But the whole group $Sp(\mathcal{Y})$ is generated by a neighborhood of $\mathbb{1}$. This proves (2). □

10.3.3 $Mp(\mathcal{Y})$ as the two-fold covering of $Sp(\mathcal{Y})$

Definition 10.37 *Let G be a path-connected topological group. A* covering group *of G is a path-connected topological group $\tilde{G}$ with a surjective homomorphism $\pi : \tilde{G} \to G$. If for each $g \in G$ the set $\pi^{-1}(g)$ has n elements, then $\tilde{G}$ is called an* n-fold covering of G.

Introducing an arbitrary Kähler structure on $\mathcal{Y}$ and considering the polar decomposition, we easily see that $Sp(\mathcal{Y})$ is path-connected. The same argument shows that its fundamental group, that is, $\pi_1(Sp(\mathcal{Y}))$, equals $\mathbb{Z}$. Hence, for any $n \in \{1, 2, \dots, \aleph_0\}$ the n-fold covering of $Sp(\mathcal{Y})$ exists and is unique up to an isomorphism.

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The group $Mp(\mathcal{Y})$ is clearly path-connected, since $\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\zeta)}$, $t \in [0,1]$, is a continuous path joining $\mathbb{1}$ and $\mathrm{e}^{\mathrm{i}\mathrm{Op}(\zeta)}$. For $U \in Mp(\mathcal{Y})$, let $\pi(U) \in Sp(\mathcal{Y})$ denote the symplectic transformation r implemented by U. By Thm. 10.29, $\pi^{-1}(r) = \{U_r, -U_r\}$. Hence, $Mp(\mathcal{Y})$ is the double covering of $Sp(\mathcal{Y})$.

10.4 Symplectic group on a space with conjugation

Throughout this section we fix a finite-dimensional space $\mathcal{X}$ and consider the space $\mathcal{X}^{\#} \oplus \mathcal{X}$ equipped with the symplectic form ω and the conjugation τ given by

$$\omega = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}.$$

Recall that its dual is isomorphic to $\mathcal{X} \oplus \mathcal{X}^{\#}$ with the symplectic form ω^{-1} and conjugation $\tau^{\#}$:

$$\omega^{-1} = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \quad \tau^{\#} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}.$$

The Poisson bracket on $\mathcal{X} \oplus \mathcal{X}^{\#}$ takes the familiar form

$$\{b_1, b_2\} = \nabla_\xi b_1 \cdot \nabla_x b_2 - \nabla_x b_1 \cdot \nabla_\xi b_2, \quad b_1, b_2 \in C^1(\mathcal{X} \oplus \mathcal{X}^{\#}).$$

Recall from Thm. 1.47 that every finite-dimensional symplectic space can be equipped with a conjugation and is isomorphic to $\mathcal{X}^{\#} \oplus \mathcal{X}$. This section is devoted to a discussion of symplectic and infinitesimally symplectic transformations in a symplectic space with conjugation. It is a preparation for the next section, where we consider the Schrödinger CCR representation on $L^2(\mathcal{X})$.

As already discussed in Remark 10.1, we actually have two symplectic spaces with conjugation at our disposal: $\mathcal{Y} = \mathcal{X}^{\#} \oplus \mathcal{X}$ and $\mathcal{Y}^{\#} = \mathcal{X} \oplus \mathcal{X}^{\#}$. They are dual to one another and, as we know, both are relevant, as seen e.g. from the relations (10.27) and (10.28). We will explicitly describe $Sp(\mathcal{Y}^{\#})$ and $sp(\mathcal{Y}^{\#})$, since they appear more naturally in the quantization of classical symbols (but, obviously, it is easy to pass to $Sp(\mathcal{Y})$ and $sp(\mathcal{Y})$, to which they are naturally isomorphic).

10.4.1 Symplectic transformations on a space with conjugation

Let $a^{\#} \in L(\mathcal{X} \oplus \mathcal{X}^{\#})$. $a^{\#}$ belongs to $sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ iff

$$a^{\#} = \begin{bmatrix} c & \beta \\ -\alpha & -c^{\#} \end{bmatrix},$$

where $\alpha \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#})$, $c \in L(\mathcal{X})$, $\beta \in L_{\mathrm{s}}(\mathcal{X}^{\#}, \mathcal{X})$.

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Let $(q,\eta)\in\mathcal{X}\oplus\mathcal{X}^{\#}$ and $a^{\#}\in sp(\mathcal{X}\oplus\mathcal{X}^{\#})$. Clearly, $\big((q,\eta),a^{\#}\big)\in asp(\mathcal{X}\oplus\mathcal{X}^{\#})$. Its Hamiltonian is

$$\mathcal{X}\oplus\mathcal{X}^{\#}\ni(x,\xi)\mapsto\chi(x,\xi)=-\eta\cdot x+q\cdot\xi+\frac{1}{2}x\cdot\alpha x+\xi\cdot cx+\frac{1}{2}\xi\cdot\beta\xi.$$

Let $r^{\#}\in L(\mathcal{X}\oplus\mathcal{X}^{\#})$. Write $r^{\#}$ as

$$r^{\#}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}. \tag{10.41}$$

$r^{\#}\in Sp(\mathcal{X}\oplus\mathcal{X}^{\#})$ iff

$$a^{\#}d-c^{\#}b=\mathbb{1},\quad c^{\#}a=a^{\#}c,\quad d^{\#}b=b^{\#}d, \tag{10.42}$$

or, equivalently,

$$ad^{\#}-bc^{\#}=\mathbb{1},\quad ab^{\#}=ba^{\#},\quad cd^{\#}=dc^{\#}. \tag{10.43}$$

In fact, (10.42) is equivalent to (10.1) and (10.43) is equivalent to (10.2). We have

$$r^{\#\,-1}=\begin{bmatrix}d^{\#} & -b^{\#}\\ -c^{\#} & a^{\#}\end{bmatrix}.$$

10.4.2 Generating function of a symplectic transformation

In the next theorem we prove a factorization result for symplectic transformations, similar to the one discussed in Subsect. 1.1.2. It will be used to define its generating function.

Theorem 10.38 *Let $r^{\#}\in Sp(\mathcal{X}\oplus\mathcal{X}^{\#})$ be as in (10.41) with b invertible.*

(1) *We have the factorization*

$$r^{\#}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}\mathbb{1} & 0\\ e & \mathbb{1}\end{bmatrix}\begin{bmatrix}0 & b\\ -b^{\#\,-1} & 0\end{bmatrix}\begin{bmatrix}\mathbb{1} & 0\\ f & \mathbb{1}\end{bmatrix}, \tag{10.44}$$

where

$$e=db^{-1}=b^{\#\,-1}d^{\#}\in L_{\mathrm{s}}(\mathcal{X},\mathcal{X}^{\#}),$$
$$f=b^{-1}a=a^{\#}b^{\#\,-1}\in L_{\mathrm{s}}(\mathcal{X},\mathcal{X}^{\#}).$$

(2) *Define $S\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{X}\oplus\mathcal{X})$ by setting*

$$\mathcal{X}\times\mathcal{X}\ni(x_1,x_2)\mapsto S(x_1,x_2):=(b^{-1}q)\cdot x_1+(-eq+\eta)\cdot x_2$$
$$+\frac{1}{2}x_1\cdot fx_1-x_1\cdot b^{-1}x_2+\frac{1}{2}x_2\cdot ex_2.$$

Then

$$\begin{bmatrix}q\\ \eta\end{bmatrix}+\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x_1\\ \xi_1\end{bmatrix}=\begin{bmatrix}x_2\\ \xi_2\end{bmatrix} \tag{10.45}$$

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iff

$$\nabla_{x_1} S(x_1, x_2) = -\xi_1, \quad \nabla_{x_2} S(x_1, x_2) = \xi_2. \tag{10.46}$$

Proof The proofs are direct computations, using (10.42) and (10.43). □

Definition 10.39 *The function $S(x_1, x_2)$ is called a* generating function of the affine symplectic transformation (10.45)*.*

10.4.3 Point transformations

Definition 10.40 *Elements of $sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ that commute with the conjugation $\tau^{\#}$ are called* infinitesimal point transformations*.*

Their set is the image of the following injective homomorphism:

$$gl(\mathcal{X}) \ni c \mapsto \begin{bmatrix} c & 0 \\ 0 & -c^{\#} \end{bmatrix} \in sp(\mathcal{X} \oplus \mathcal{X}^{\#}). \tag{10.47}$$

$\mathcal{X} \oplus \mathcal{X}^{\#} \ni (x, \xi) \mapsto \xi \cdot cx = x \cdot c^{\#} \xi$ is the Hamiltonian of (10.47).

Definition 10.41 *Elements of $Sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ that commute with the conjugation $\tau^{\#}$ are called* point transformations*.*

Their set is the image of the following injective homomorphism:

$$GL(\mathcal{X}) \ni m \mapsto \begin{bmatrix} m & 0 \\ 0 & m^{\# -1} \end{bmatrix} \in Sp(\mathcal{X} \oplus \mathcal{X}^{\#}). \tag{10.48}$$

We have

$$\exp \begin{bmatrix} c & 0 \\ 0 & -c^{\#} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{c} & 0 \\ 0 & (\mathrm{e}^{c})^{\# -1} \end{bmatrix}.$$

10.4.4 Transformations fixing $\mathcal{X}^{\#}$

The set of elements of $sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ that send $\mathcal{X}^{\#}$ to zero is the image of the following injective homomorphism of Lie algebras with the trivial bracket:

$$L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#}) \ni \alpha \mapsto \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \in sp(\mathcal{X} \oplus \mathcal{X}^{\#}). \tag{10.49}$$

The Hamiltonian of (10.49) is $-\frac{1}{2} x \cdot \alpha x$.

The set of elements of $Sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ that fix elements of $\mathcal{X}^{\#}$ is the image of the following injective homomorphism of groups (where $L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#})$ is equipped with the addition):

$$L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#}) \ni \alpha \mapsto \begin{bmatrix} 1\!\!1 & 0 \\ \alpha & 1\!\!1 \end{bmatrix} \in Sp(\mathcal{X} \oplus \mathcal{X}^{\#}). \tag{10.50}$$

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We have

$$\exp \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 0 \\ \alpha & \mathbb{1} \end{bmatrix}.$$

10.4.5 Transformations fixing $\mathcal{X}$

The set of elements of $sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ that send $\mathcal{X}$ to zero is the image of the following injective homomorphism of Lie algebras with the trivial bracket:

$$L_{\mathrm{s}}(\mathcal{X}^{\#}, \mathcal{X}) \ni \beta \mapsto \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \in sp(\mathcal{X} \oplus \mathcal{X}^{\#}). \tag{10.51}$$

The Hamiltonian of (10.51) is $-\frac{1}{2}\xi{\cdot}\beta\xi$.

The set of elements of $Sp(\mathcal{X} \oplus \mathcal{X}^{\#})$ that fix elements of $\mathcal{X}$ is the image of the following injective homomorphism of groups (where $L_{\mathrm{s}}(\mathcal{X}^{\#}, \mathcal{X})$ is equipped with the addition):

$$L_{\mathrm{s}}(\mathcal{X}^{\#}, \mathcal{X}) \ni \beta \mapsto \begin{bmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{bmatrix} \in Sp(\mathcal{X} \oplus \mathcal{X}^{\#}). \tag{10.52}$$

We have

$$\exp \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{bmatrix}.$$

The generating function for the transformation (10.52) is

$$S(x_1, x_2) = -\frac{1}{2}(x_1 - x_2){\cdot}\beta^{-1}(x_1 - x_2).$$

10.4.6 Harmonic oscillator

We choose a scalar product on $\mathcal{X}$ and use it to identify $\mathcal{X}^{\#}$ with $\mathcal{X}$.

Consider the Hamiltonian $\chi(x, \xi) = \frac{1}{2}x^2 + \frac{1}{2}\xi^2$. It generates the flow

$$\mathrm{e}^{t\omega\nabla\chi} \begin{bmatrix} x_0 \\ \xi_0 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ \xi_0 \end{bmatrix} = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}.$$

Its generating function is $S(x_0, x_t) = \frac{(x_0^2 + x_t^2)\cos t - 2x_0{\cdot}x_t}{2\sin t}$.

10.4.7 Transformations swapping $\mathcal{X}$ and $\mathcal{X}^{\#}$

Let $b \in L(\mathcal{X}^{\#}, \mathcal{X})$. Then the following transformation is symplectic:

$$\begin{bmatrix} 0 & b \\ -b^{\#\,-1} & 0 \end{bmatrix}.$$

Its generating function is $S(x_1, x_2) = -x_1{\cdot}b^{-1}x_2$.

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10.5 Metaplectic group in the Schrödinger representation

As in the previous section, $\mathcal{X}$ is a finite-dimensional real vector space. In this section we describe the metaplectic group $Mp(\mathcal{X}^{\#} \oplus \mathcal{X})$ in the Schrödinger CCR representation on $L^2(\mathcal{X})$.

10.5.1 Metaplectic group in $L^2(\mathbb{R})$

We start with the one-dimensional case. Let us consider the Schrödinger representation in $L^2(\mathbb{R})$ over $\mathbb{R} \oplus \mathbb{R}$. We will describe some examples of subgroups of the metaplectic group $Mp(\mathbb{R} \oplus \mathbb{R}) \subset U\big(L^2(\mathbb{R})\big)$.

Example 10.42 *Let* $\chi(x, \xi) = x{\cdot}\xi$. *Then* $\mathrm{Op}(\chi) = \frac{1}{2}(x{\cdot}D + D{\cdot}x)$ *and* $\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}$ *belongs to the metaplectic group. We have*

$$\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}\Psi(x) = \mathrm{e}^{-\frac{1}{2}t}\Psi(\mathrm{e}^{-t}x), \quad \Psi \in L^2(\mathcal{X}).$$

Example 10.43 *The multiplication operator* $\mathrm{e}^{-\frac{\mathrm{i}}{2}tx^2}$ *belongs to the metaplectic group.*

Example 10.44 *The operator* $\mathrm{e}^{-\frac{\mathrm{i}}{2}tD^2}$ *belongs to the metaplectic group. Its integral kernel equals*

$$(2\pi\mathrm{i}t)^{-\frac{1}{2}}\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{(x-y)^2}{t}}.$$

10.5.2 Harmonic oscillator

We still consider the one-dimensional case. Let $\chi(x, \xi) := \frac{1}{2}\xi^2 + \frac{1}{2}x^2$. Then $\mathrm{Op}(\chi) = \frac{1}{2}D^2 + \frac{1}{2}x^2$. The Weyl–Wigner symbol of $\mathrm{e}^{-t\mathrm{Op}(\chi)}$ is

$$w(t, x, \xi) = (\mathrm{ch}\tfrac{t}{2})^{-1}\exp(-(x^2 + \xi^2)\mathrm{th}\tfrac{t}{2}). \tag{10.53}$$

Its integral kernel is given by the so-called *Mehler's formula*

$$W(t, x, y) = \pi^{-\frac{1}{2}}(\mathrm{sh}t)^{-\frac{1}{2}}\exp\left(\frac{-(x^2 + y^2)\mathrm{ch}t + 2xy}{2\mathrm{sh}t}\right).$$

$\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)}$ has the Weyl–Wigner symbol

$$w(\mathrm{i}t, x, \xi) = (\cos\tfrac{t}{2})^{-1}\exp\big(-\mathrm{i}\,(x^2 + \xi^2)\mathrm{tg}\tfrac{t}{2}\big) \tag{10.54}$$

and the integral kernel

$$W(\mathrm{i}t, x, y) = \pi^{-\frac{1}{2}}|\sin t|^{-\frac{1}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}[\frac{t}{\pi}]}\exp\left(\frac{-(x^2 + y^2)\cos t + 2xy}{2\mathrm{i}\sin t}\right).$$

Above, $[c]$ denotes the integral part of c.

It is easy to see that (10.53), resp. (10.54) are special cases of (10.36), resp. (10.37).

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We have $W(\mathrm{i}t+2\mathrm{i}\pi,x,y)=-W(\mathrm{i}t,x,y)$. Note the special cases

$$W(0,x,y)=\delta(x-y),$$
$$W(\tfrac{\mathrm{i}\pi}{2},x,y)=(2\pi)^{-\frac12}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\mathrm{e}^{-\mathrm{i}xy},$$
$$W(\mathrm{i}\pi,x,y)=\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}\delta(x+y),$$
$$W(\tfrac{\mathrm{i}3\pi}{2},x,y)=(2\pi)^{-\frac12}\mathrm{e}^{-\frac{\mathrm{i}3\pi}{4}}\mathrm{e}^{\mathrm{i}xy}.$$

Corollary 10.45 (1) *The operator with kernel* $\pm(2\pi\mathrm{i})^{-\frac12}\mathrm{e}^{-\mathrm{i}xy}$ *belongs to the metaplectic group and implements* $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

(2) *The operator with kernel* $\pm\mathrm{i}\delta(x+y)$ *belongs to the metaplectic group and implements* $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$.

10.5.3 Quadratic Hamiltonians in the Schrödinger representation

Until the end of the section we consider $\mathcal{X}$ of any finite dimension. Any $\chi\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{Y}^{\#})$ is of the form

$$\mathcal{X}\oplus\mathcal{X}^{\#}\ni(x,\xi)\mapsto\chi(x,\xi)=\alpha(x)+\xi\cdot cx+\beta(\xi),$$

where $\alpha\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{X})$, $\beta\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\leq 2}(\mathcal{X}^{\#})$ and $c\in L(\mathcal{X})$. We have

$$\mathrm{Op}^{x,D}(\chi)=\alpha(x)+x\cdot c^{\#}D+\beta(D),$$
$$\mathrm{Op}(\chi)=\mathrm{Op}^{x,D}(\chi)+\tfrac{\mathrm{i}}{2}\mathrm{Tr}\,c.$$

10.5.4 Integral kernel of elements of the metaplectic group

First we describe various examples of elements of the metaplectic group.

Proposition 10.46 *If* $m\in GL(\mathcal{X})$ *with* $\det m\neq 0$, *then the operator*

$$\pm T_m\Psi(x):=\pm(\det m)^{\frac12}\Psi(mx) \tag{10.55}$$

belongs to $Mp(\mathcal{X}^{\#}\oplus\mathcal{X})$ *and implements* $\begin{bmatrix}m^{\#} & 0\\ 0 & m^{-1}\end{bmatrix}\in Sp(\mathcal{X}^{\#}\oplus\mathcal{X})$.

Proof Assume first that $\det m>0$. Let $c\in gl(\mathcal{X})$ such that $m=\mathrm{e}^{c}$. Recall that if $\chi(x,\xi)=x\cdot c^{\#}\xi$, then

$$\mathrm{Op}(\chi)=x\cdot c^{\#}D+\frac{\mathrm{i}}{2}\mathrm{Tr}\,c,\quad \mathrm{Op}^{x,D}(\chi)=x\cdot c^{\#}D.$$

But $\mathrm{e}^{\frac12\mathrm{Tr}c}=(\det\mathrm{e}^{c})^{\frac12}=(\det m)^{\frac12}$.

Suppose now that $\det m<0$. Fix an arbitrary Euclidean structure in $\mathcal{X}$. We can write m as m_1m_2 where $\det m_1>0$ and $m_2=1\!\!1-2|e\rangle\langle e|$, where $e\in\mathcal{X}$,

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$\|e\| = 1$. We have $\begin{bmatrix} m^{\#} & 0 \\ 0 & m^{-1} \end{bmatrix} = \begin{bmatrix} m_1^{\#} & 0 \\ 0 & m_1^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1} - 2|e\rangle\langle e| & 0 \\ 0 & \mathbb{1} - 2|e\rangle\langle e| \end{bmatrix}$. The first term we implement as above, the second by the exponential of a one dimensional harmonic oscillator; see Corollary 10.45 (2). □

Proposition 10.47 *Let* $\alpha \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#})$. *Then* $\mathrm{e}^{-\frac{\mathrm{i}}{2} x \cdot \alpha x} \in Mp(\mathcal{X}^{\#} \oplus \mathcal{X})$ *and implements* $\begin{bmatrix} \mathbb{1} & \alpha \\ 0 & \mathbb{1} \end{bmatrix}$.

Proposition 10.48 *Let* $b \in L(\mathcal{X}, \mathcal{X}^{\#})$. *Then the operator with the kernel*

$$\pm(2\pi\mathrm{i})^{-\frac{d}{2}} (\det b)^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i} x_1 \cdot b^{-1} x_2} \tag{10.56}$$

belongs to $Mp(\mathcal{X}^{\#} \oplus \mathcal{X})$ *and implements* $\begin{bmatrix} 0 & -b^{-1} \\ b^{\#} & 0 \end{bmatrix}$.

Proof Equip $\mathcal{X}$ with a scalar product. We can identify $\mathcal{X}$ with $\mathcal{X}^{\#}$ and write

$$\begin{bmatrix} 0 & -b^{-1} \\ b^{\#} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \begin{bmatrix} b^{\#} & 0 \\ 0 & b^{-1} \end{bmatrix}.$$

By Corollary 10.45, the operator with integral kernel $\pm(2\pi\mathrm{i})^{-\frac{d}{2}} \mathrm{e}^{-\mathrm{i} x_1 \cdot x_2}$ belongs to $Mp(\mathcal{X} \oplus \mathcal{X})$ and implements $\begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$. By Prop. 10.46, $\begin{bmatrix} b^{\#} & 0 \\ 0 & b^{-1} \end{bmatrix}$ is implemented by (10.55). Then we use the chain rule. □

Let us now describe the case of an (almost) arbitrary $r \in Sp(\mathcal{X}^{\#} \oplus \mathcal{X})$. We can write $r^{\#} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Recall from Thm. 10.38 that, if b is invertible, we can factorize $r^{\#}$ as

$$r^{\#} = \begin{bmatrix} \mathbb{1} & 0 \\ e & \mathbb{1} \end{bmatrix} \begin{bmatrix} 0 & b \\ -b^{\# -1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ f & \mathbb{1} \end{bmatrix},$$

and introduce the generating function of $r^{\#}$:

$$\mathcal{X} \times \mathcal{X} \ni (x_1, x_2) \mapsto S(x_1, x_2) := \frac{1}{2} x_1 \cdot f x_1 - x_1 \cdot b^{-1} x_2 + \frac{1}{2} x_2 \cdot e x_2.$$

The following theorem is one of the most beautiful expressions of the correspondence between classical and quantum mechanics, since the distributional kernel of the (quantum) unitary operator U_r is expressed purely in terms of the generating function for the symplectic transformation $r^{\#}$.

Theorem 10.49 *Let* $r \in Sp(\mathcal{X}^{\#} \oplus \mathcal{X})$ *be such that* b *is invertible. Then the operators* $\pm U_r \in Mp(\mathcal{X}^{\#} \oplus \mathcal{X})$ *implementing* r *have their integral kernels equal to*

$$\pm U_r(x_1, x_2) = \pm(2\pi\mathrm{i})^{-\frac{d}{2}} \sqrt{-\det \nabla_{x_1} \nabla_{x_2} S}\; \mathrm{e}^{-\mathrm{i} S(x_1, x_2)}.$$

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Proof We can write

$$r = \begin{bmatrix} \mathbb{1} & f \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} 0 & -b^{-1} \\ b^{\#} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{1} & e \\ 0 & \mathbb{1} \end{bmatrix} = r_f r_b r_e.$$

r_f and r_e are implemented in $Mp(\mathcal{X}^{\#} \oplus \mathcal{X})$ by $U_e = \mathrm{e}^{-\frac{\mathrm{i}}{2}x\cdot ex}$ and $U_f = \mathrm{e}^{-\frac{\mathrm{i}}{2}x\cdot fx}$. r_b is implemented in $Mp(\mathcal{X}^{\#} \oplus \mathcal{X})$ by U_b, which has the integral kernel (10.56). Hence $r = r_f r_b r_e$ is implemented by $U_f U_b U_e$, which has the integral kernel

$$\pm(2\pi\mathrm{i})^{-\frac{d}{2}}(\det b)^{-\frac{1}{2}}\mathrm{e}^{-\frac{\mathrm{i}}{2}x_1\cdot fx_1}\mathrm{e}^{\mathrm{i}x_1\cdot b^{-1}x_2}\mathrm{e}^{-\frac{\mathrm{i}}{2}x_2\cdot ex_2}.$$

□

10.6 Notes

Normal forms of quadratic Hamiltonians were first established by Williamson (1936). Thus, Prop. 10.8 is a special case of Williamson's theorem.

The fact that Bogoliubov rotations are implemented by a projective unitary representation of the symplectic group was noted by Segal (1959). Its implementation by a representation of the two-fold covering of the symplectic group, the so-called metaplectic representation, is attributed to Weil (1964) and Shale (1962). The metaplectic group plays an important role in the concept of the Maslov index, the semi-classical approximation and microlocal analysis; see Maslov (1972), Leray (1978), Guillemin–Sternberg (1977) and Hörmander (1985). The semi-classical approximation and microlocal analysis are asymptotic theories (where the small parameter is the Planck constant $\hbar$ or the inverse λ^{-1} of the momentum scale). One can obtain for example extensions of Thm. 10.35 or Thm. 10.49 to non-quadratic Hamiltonians or non-linear symplectic maps. In these extensions the expressions of Weyl–Wigner symbols or distributional kernels are given by asymptotic expansions in terms of the small parameter. In the linear case these expansions have only one term and are exact.

The first famous application of the symplectic invariance of CCR seems to be Bogoliubov's theory of the excitation spectrum of the homogeneous Bose gas (Bogoliubov (1947a); see also Fetter–Walecka (1971) and Cornean–Dereziński–Ziń (2009)).

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