S. Lu and H. Wu
Nagoya Math. J.
Vol. 173 (2004), 205–223

ON THE COMMUTATORS OF SINGULAR INTEGRALS RELATED TO BLOCK SPACES

SHANZHEN LU*AND HUOXIONG WU

Abstract. In this paper, the commutators of singular integrals with rough kernels are considered. By the method of block decomposition for kernel function and Fourier transform estimates, some new results about the $L^p(\mathbb{R}^n)$ boundedness for these commutators are obtained.

§1. Introduction

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)$ be a homogeneous function of degree zero and have mean value zero on S^{n-1}. Suppose that $h(t) \in L^\infty(0,\infty)$. Define the singular integral operator T by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y)dy. \tag{1.1}$$

For a positive integer k and $a(x) \in BMO(\mathbb{R}^n)$, define the k-th order commutator $T_{a,k}$ generated by T and a

$$T_{a,k}f(x) = T((a(x)-a(\cdot))^k f)(x), \ f \in C_0^\infty(\mathbb{R}^n). \tag{1.2}$$

It was proved by Coifman, Rochberg and Weiss [4] that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})\,(0 < \alpha \leq 1)$ and $h \equiv 1$, then $T_{a,1}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|a\|_{BMO(\mathbb{R}^n)}$ for $1 < p < \infty$. Afterwards, by a well-known result of Duoandikoetxea [6] and the boundedness criterion of Alvarez-Bagby-Kurtz-Pérez for the commutators of linear operator (see [2]), we have obtained the following theorem (see also [10]):

Received June 12, 2002.
Revised January 7, 2003, February 25, 2003.
2000 Mathematics Subject Classification: 42B20, 47B20, 42B35, 47A30.
*The first author is supported by the National 973 project and the SEDF of China.

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

THEOREM A. ([6, 2, 10]) *Let* Ω, a, k *be as above and* $h \equiv 1$, $1 < p < \infty$. *If* $\Omega \in \cup_{q>1} L^q(S^{n-1})$, *then* $T_{a,k}$ *is bounded on* $L^p(\mathbb{R}^n)$.

Recently, to weaken the condition imposed on Ω, Hu Guoen et al. employed the method of Littlewood-Paley theory and Fourier transform estimates from [7] to obtain the following results.

THEOREM B. *Let* Ω, a, k *be as above. Then* $T_{a,k}$ *is bounded on* $L^p(\mathbb{R}^n)$ *with bound* $C\|a\|^k_{BMO(\mathbb{R}^n)}$, *if one of the following conditions holds.*

(i) (see [12]). $p = 2$, $h \equiv 1$, $\Omega \in L(\log^+ L)^{k+1}(S^{n-1})$.

(ii) (see [9]). $p = 2$, $h \equiv 1$ *and for some* $\alpha > k+1$, Ω *satisfies*

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \left(\log \frac{1}{|\theta \cdot \xi|}\right)^{\alpha} d\theta. \tag{1.3}$$

(iii) (see [13] or [9]). *For some* $\alpha > k+1$, $\Omega \in L(\log^+ L)^{\alpha}(S^{n-1})$ *and for some* $s > 1$, h *satisfies* $\sup_{R>0} \int_R^{2R} |h(r)|^s r^{-1} dr < \infty$, $2\alpha/(2\alpha - (k+1)) < p < 2\alpha/(k+1)$ *or* $p = 2$.

Theorem B certainly improve Theorem A since both the condition $\Omega \in L(\log^+ L)^{\alpha}(S^{n-1})$ $(\alpha > k+1)$ and the size condition (1.3) are properly weaker than the condition $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$. Unfortunately, the condition on Ω in Theorem B greatly depends on the order k of $T_{a,k}$. It is natural to ask whether there exists a weaker size condition on Ω, which is independent of k, such that $T_{a,k}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. The main purpose of this paper is to give a positive answer to this problem. Inspired by [1], we shall show that $T_{a,k}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if $\Omega \in B_q^{0,0}(S^{n-1})$ for some $q > 1$. Here $B_q^{0,0}(S^{n-1})$ denotes certain block spaces introduced by Jiang and Lu(see [15]). We remark that some ideas in the proof of our main results are taken from [7] and [11]. Before stating the main results, we briefly review some pertinent concepts.

DEFINITION 1. ([15]) A q-block on S^{n-1} is an $L^q(1 < q \le \infty)$ function $b(\cdot)$ that satisfies

$$\text{(i)}\ \ \operatorname{supp}(b) \subseteq Q, \qquad \text{(ii)}\ \ \|b\|_{L^q(S^{n-1})} \le |Q|^{\frac{1}{q}-1},$$

where $Q = S^{n-1} \cap \{y \in \mathbb{R}^n : \ |y - \varsigma| < \rho \ \ for \ \ some \ \ \varsigma \in S^{n-1} \ \ and \ \ \rho \in (0, 1]\}$.

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Definition 2. ([15]) The block spaces $B_q^{0,0}$ on S^{n-1} are defined by

$$B_q^{0,0}(S^{n-1}) = \{\Omega \in L^1(S^{n-1}) : \Omega(y') = \sum_s C_s b_s(y'),\ M_q^{0,0}(\{C_s\}) < \infty\},$$

where each C_s is a complex number, each b_s is a q-block supported in Q_s, and

$$M_q^{0,0}(\{C_s\}) = \sum_s |C_s| \left\{1 + \log^+ \frac{1}{|Q_s|}\right\}.$$

It should be pointed out that the method of block decomposition for functions was invented by Taibleson and Weiss [17] in the study of the convergence of the Fourier series. Later on, many application of the block decomposition to harmonic analysis were discovered (see [1], [14]–[16] etc.). For further background and information about the theory of spaces generated by blocks and its applications to harmonic analysis, one can consult the book [15]. In [14], Keitoku and Sato showed that for any $q > 1$,

$$\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,0}(S^{n-1}),$$

which is a proper inclusion. And from [14], we easily see that $B_q^{0,0}(S^{n-1})$ is not contained in $L(\log^+ L)^{1+\varepsilon}(S^{n-1})$ for any $\varepsilon > 0$ although the relationship between $B_q^{0,0}(S^{n-1})$ and $L \log^+ L(S^{n-1})$ remains open.

Definition 3. ([3]) A locally integrable function $a(x)$ will be said to belong to $BLO(\mathbb{R}^n)$, if there is a constant C such that for any cube Q

$$m_Q(a) - \inf_{x \in Q} a(x) \le C,$$

where $m_Q(a) = |Q|^{-1} \int_Q a(x)dx$.

If $a \in BLO(\mathbb{R}^n)$, then we denote $\|a\|_{BLO(\mathbb{R}^n)} = \sup_Q \{m_Q(a) - \inf_{x\in Q} a(x)\}$.

Obviously, $L^\infty(\mathbb{R}^n) \subset BLO(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and if $a \in BLO(\mathbb{R}^n)$, then

$$\|a\|_{BMO(\mathbb{R}^n)} \le 2\|a\|_{BLO(\mathbb{R}^n)}. \tag{1.4}$$

Now let us formulate our main results.

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

THEOREM 1. *Let Ω be homogeneous of degree zero and have mean value zero, k be a positive integer and $a \in BMO(\mathbb{R}^n)$. If $h(t) \in L^\infty(0,\infty)$ and $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$, then the commutator $T_{a,k}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|a\|^k_{BMO(\mathbb{R}^n)}$.*

For the case of $p \neq 2$, $1 < p < \infty$, we need to impose some restrictions on BMO functions $a(x)$ as follows.

THEOREM 2. *Let Ω, h, k be as in Theorem 1, $1 < p < \infty$. If $a \in BLO(\mathbb{R}^n)$ and $a(x)$ is subharmonic, then $T_{a,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|a\|^k_{BLO(\mathbb{R}^n)}$.*

Remark 1. It is worth pointing out that a BMO function $a(x)$ satisfying the restrictive conditions in Theorem 2 exists. A typical example is $\log|x|$.

Remark 2. $\bigcup_{r>1} L^r(S^{n-1})$ is properly contained in $B_q^{0,0}(S^{n-1})$ for any $q > 1$, and $B_q^{0,0}(S^{n-1})$ is independent of the order of $T_{a,k}$ and is not contained in $L(\log^+ L)^\alpha(S^{n-1})$ $(\alpha > 1)$. Therefore our theorems are an essential improvement on Theorem A and an great extension of the result in Theorem B.

In proving Theorem 2, we shall use the following L^p-boundedness of $M^\Omega_{a,k}$, a maximal operator related to higher order commutators, defined by

$$M^\Omega_{a,k} f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |a(x) - a(y)|^k |h(|x-y|)\Omega(x-y) f(y)| dy.$$

THEOREM 3. *Under the same hypothesis as in Theorem 2, the operator $M^\Omega_{a,k}$ satisfies*

$$\|M^\Omega_{a,k} f\|_p \le C\|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p.$$

Throughout this paper, C always denotes positive constants that are independent of the essential variables but whose value may vary at each occurrence.

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

§2. Proof of Theorem 1

Let us begin with some preliminary lemmas.

LEMMA 1. ([11]) *Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}\phi \subset \{1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1, \ |\xi| \ne 0.$$

Denote by S_l the multiplier operator

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi),$$

and $S_l^2 f(x) = S_l(S_l f)(x)$. For any positive integer k and $a \in BMO(\mathbb{R}^n)$, consider the k-th order commutator of S_l and S_l^2, respectively, defined by

$$S_{l;a,k}f(x) = S_l((a(x)-a(\cdot))^k f)(x), \qquad f \in C_0^\infty(\mathbb{R}^n)$$

and

$$S^2_{l;a,k}f(x) = S^2_l((a(x)-a(\cdot))^k f)(x), \qquad f \in C_0^\infty(\mathbb{R}^n).$$

Then for all $1 < p < \infty$,

(a) $$\left\| \left(\sum_{l\in\mathbb{Z}} |S_{l;a,k}f|^2 \right)^{\frac{1}{2}} \right\|_p \le C\|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_p;$$

(b) $$\left\| \left(\sum_{l\in\mathbb{Z}} |S^2_{l;a,k}f|^2 \right)^{\frac{1}{2}} \right\|_p \le C\|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_p;$$

(c) $$\left\| \sum_{l\in\mathbb{Z}} S_{l;a,k}f_l \right\|_p \le C(n,k,p)\|a\|^k_{BMO(\mathbb{R}^n)} \left\| \left(\sum_{l\in\mathbb{Z}} |f_l|^2 \right)^{\frac{1}{2}} \right\|_p,$$
$$f_l \in C_0^\infty(\mathbb{R}^n)(l \in \mathbb{Z}).$$

LEMMA 2. ([11]) *Let $0 < \delta < \infty$, and take a function $m_\delta \in C_0^\infty(\mathbb{R}^n)$ with support contained in $\{\xi \in \mathbb{R}^n : |\xi| \le \delta\}$. Suppose that for some positive constant α,*

$$\|m_\delta\|_\infty \le C\min\{\delta^\alpha,\, \delta^{-\alpha}\}, \qquad \|\nabla m_\delta\|_\infty \le C.$$

Let T_δ be the multiplier operator defined by

$$\widehat{T_\delta f}(\xi) = m_\delta(\xi)\hat{f}(\xi).$$

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

For a positive integer k and $a \in BMO(\mathbb{R}^n)$, let $T_{\delta;a,k}$ be the k-th order commutator of T_δ. Then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n,k,v)$ such that

$$\|T_{\delta;a,k}f\|_2 \le C\min\{\delta^{\alpha v}, \delta^{-\alpha v}\}\|a\|^k_{BMO(\mathbb{R}^n)}\|f\|_2.$$

LEMMA 3. *Let $\Omega(x') = \sum_s C_s b_s(x')$, $h(t)$ be as in Theorem 1. For $j \in \mathbb{Z}$, set*

$$K_j(x) = \frac{\Omega(x)}{|x|^n}h(|x|)\chi_{\{2^j\le|x|<2^{j+1}\}}(x),$$

$$K_{j,s}(x) = \frac{b_s(x)}{|x|^n}h(|x|)\chi_{\{2^j\le|x|<2^{j+1}\}}(x),$$

and $m_j(\xi) = \widehat{K_j}(\xi)$, $m_{j,s}(\xi) = \widehat{K_{j,s}}(\xi)$. Then we have

(i) $|m_j(\xi)| \le C|2^j\xi|$;

(ii) $|m_{j,s}(\xi)| \le |2^j\xi|^{\frac{1}{2\log|Q_s|}}$, *if* $|Q_s| < e^{\frac{q}{1-q}}$;

(iii) $|m_{j,s}(\xi)| \le C|2^j\xi|^{-\omega}$, *if* $|Q_s| \ge e^{\frac{q}{1-q}}$.

Here C and ω are positive constants independent of j, s, ξ and b_s.

Proof. By the mean zero property and the integrability of Ω on S^{n-1}, we have

$$\begin{aligned} |m_j(\xi)| &= \left|\int_{2^j\le|y|<2^{j+1}} h(|y|)|y|^{-n}\Omega(y')e^{-2\pi iy\cdot\xi}dy\right| \\ &= \left|\int_{2^j}^{2^{j+1}} h(t)t^{-1}\int_{S^{n-1}}\Omega(y')e^{-2\pi ity'\cdot\xi}d\sigma(y')dt\right| \\ &\le C\int_{2^j}^{2^{j+1}} t^{-1}\left|\int_{S^{n-1}}\Omega(y')(e^{-2\pi ity'\cdot\xi}-1)d\sigma(y')\right|dt \\ &\le C\int_{2^j}^{2^{j+1}} t^{-1}\int_{S^{n-1}}|\Omega(y')||2\pi ty'\cdot\xi|d\sigma(y')dt \\ &\le C\|\Omega\|_{L^1(S^{n-1})}|\xi|\int_{2^j}^{2^{j+1}} dt \le C|2^j\xi|. \end{aligned}$$

Thus, (i) is proved. (ii) and (iii) are the special cases of (ii) and (iii) Lemma 2.2 in [1]. The proof of Lemma 3 is complete. ∎

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Proof of Theorem 1. For $j \in \mathbb{Z}$, let $K_j(\xi)$, $m_j(\xi)$ be as in Lemma 3 and ϕ be as in Lemma 1. Define the multiplier operator S_l by

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi).$$

Set $m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi)$ and $\widehat{T_j^l f}(\xi) = m_j^l(\xi)\hat{f}(\xi)$. Let

$$U_l f(x) = \sum_{j\in\mathbb{Z}} \left(\left(S_{l-j}T_j^l S_{l-j}\right)_{a,k} f\right)(x).$$

We know from [11] that for f, $g \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x)T_{a,k}f(x)dx = \int_{\mathbb{R}^n} g(x)\sum_{l\in\mathbb{Z}} U_l f(x)dx.$$

Hence

$$\|T_{a,k}f\|_2 \le \sum_{l\in\mathbb{Z}} \|U_l f\|_2. \tag{2.1}$$

With the aid of the formula

$$(a(x)-a(y))^k = \sum_{m=0}^{k} C_k^m \left(a(x)-a(z)\right)^m \left(a(z)-a(y)\right)^{k-m},\ x,\ y,\ z \in \mathbb{R}^n,$$

we get

$$\begin{aligned}&\int_{\mathbb{R}^n} g(x)U_l f(x)dx\\ &\qquad = \sum_{m=0}^{k} C_k^m \int_{\mathbb{R}^n} g(x)\sum_{j\in\mathbb{Z}} S_{l-j;a,k-m}\left(\left(T_j^l S_{j-l}\right)_{a,m} f\right)(x)dx,\end{aligned}$$

for f, $g \in C_0^\infty(\mathbb{R}^n)$ by a straightforward computation.

By Lemma 1(c), we get

$$\begin{aligned}\|U_l f\|_2 &\le C\sum_{m=0}^{k}\left\|\sum_{j\in\mathbb{Z}} S_{j-l;a,k-m}\left(\left(T_j^l S_{l-j}\right)_{a,m} f\right)\right\|_2\\ &\le C\sum_{m=0}^{k}\|a\|_{BOM(\mathbb{R}^n)}^{k-m}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^l S_{l-j}\right)_{a,m} f\right|^2\right)^{\frac{1}{2}}\right\|_2.\end{aligned} \tag{2.2}$$

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Case 1. We first consider the L^2-boundedness of U_l for $l \le 0$.

Let $\tilde{T}_j^l$ be the operator defined by

$$\widehat{\tilde{T}_j^l f}(\xi) = m_j^l(2^{-j}\xi)\hat{f}(\xi).$$

By the vanishing moment and the integrability of Ω, we have

$$|\widehat{K_j}(\xi)| \le C|2^j\xi|, \qquad \|\nabla\widehat{K_j}\|_\infty \le C2^j.$$

Thus

$$\|m_j^l(2^{-j}\cdot)\|_\infty \le C2^l, \qquad \|\nabla m_j^l(2^{-j}\cdot)\|_\infty \le C.$$

Using this and Lemma 2, we obtain that for any fixed $0 < v < 1$ and positive integer i,

$$\|\tilde{T}_{j;a,i}^l f\|_2 \le C2^{vl}\|a\|_{BMO(\mathbb{R}^n)}^i\|f\|_2,$$

which by dilation-invariance implies

$$\|T_{j;a,i}^l f\|_2 \le C2^{vl}\|a\|_{BMO(\mathbb{R}^n)}^i\|f\|_2. \tag{2.3}$$

On the other hand, the Plancherel theorem tells us that

$$\|T_j^l f\|_2 \le C2^l\|f\|_2. \tag{2.4}$$

Observe that for $f,\, g \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x)\left(T_j^l S_{l-j}\right)_{a,m} f(x)dx = \sum_{i=0}^m C_m^i \int_{\mathbb{R}^n} g(x)T_{j;,a,i}^l\left(S_{l-j;a,m-i}f\right)(x)dx.$$

It follows from (2.3), (2.4) and Lemma 1(a) that

$$\begin{aligned}
&\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^l S_{l-j}\right)_{a,m} f\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\
&\le C\sum_{i=0}^m\left\|\left(\sum_{j\in\mathbb{Z}}\left|T_{j;a,i}^l\left(S_{l-j;a,m-i}f\right)\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\
&\le C2^{2vl}\sum_{i=0}^m\|a\|_{BMO(\mathbb{R}^n)}^{2i}\sum_{j\in\mathbb{Z}}\|S_{l-j;a,m-i}f\|_2^2 \\
&\le C2^{2vl}\|a\|_{BMO(\mathbb{R}^n)}^{2m}\|f\|_2^2, \qquad f \in C_0^\infty(\mathbb{R}^n).
\end{aligned} \tag{2.5}$$

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Therefore

$$\|U_l f\|_2 \le C2^{vl}\|a\|^k_{BMO(\mathbb{R}^n)}\|f\|_2. \tag{2.6}$$

Case 2. Next we consider the L^2-estimate of U_l for $l > 0$.

Let $K_{j,s}$, $m_{j,s}$ be as in Lemma 3. Then $K_j(\xi) = \sum_s C_s K_{j,s}(\xi)$. Define the operator $T_j^{l,s}$ by

$$\widehat{T_j^{l,s}f}(\xi) = \widehat{K_{j,s}}(\xi)\phi(2^{j-l}\xi)\hat{f}(\xi).$$

Then

$$T_j^l f(\xi) = \sum_s C_s T_j^{l,s} f(\xi),$$

$$\left(T_j^l S_{l-j}\right)_{a,m} f(x) = \sum_s C_s \left(T_j^{l,s} S_{l-j}\right)_{a,m} f(x).$$

And

$$U_l f(x) = \sum_s C_s U_l^s f(x),$$

where

$$U_l^s f(x) = \sum_{j\in\mathbb{Z}} \left(S_{l-j} T_j^{l,s} S_{l-j}\right)_{a,k} f(x).$$

So

$$\|U_l f\|_2 \le \sum_s |C_s|\|U_l^s f\|_2. \tag{2.7}$$

Similarly to (2.2), we have

$$\|U_l^s f\|_2 \le C\sum_{m=0}^{k}\|a\|^{k-m}_{BMO(\mathbb{R}^n)}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^{l,s}S_{l-j}\right)_{a,m} f\right|^2\right)^{\frac{1}{2}}\right\|_2. \tag{2.8}$$

In what follows, we estimate $\|U_l^s f\|_2$ for each s. Set

$$m_j^{l,s}(\xi) = \widehat{K_{j,s}}(\xi)\phi(2^{j-l}\xi) = m_{j,s}(\xi)\phi(2^{j-l}\xi).$$

And let $\bar{T}_j^{l,s}$ be the operator defined by

$$\widehat{\bar{T}_j^{l,s}f}(\xi) = m_j^{l,s}(2^{-j}\xi)\hat{f}(\xi).$$

By (ii) and (iii) of Lemma 3, we may assume, without loss of generality, that the support Q_s of b_s are uniformly small such that $|Q_s| < e^{\frac{q}{1-q}}$. Thus

$$|m_{j,s}(\xi)| = |\widehat{K_{j,s}}(\xi)| \le C|2^j\xi|^{\frac{1}{2\log|Q_s|}}.$$

By a straightforward computation, we get

$$|\nabla m_{j,s}(\xi)| = |\nabla \widehat{K_{j,s}}(\xi)| \le C2^j.$$

So

$$|m_j^{l,s}(2^{-j}\xi)| = |m_{j,s}(2^{-j}\xi)\phi(2^{-l}\xi)| \le C2^{\frac{l}{2\log|Q_s|}}$$

and

$$|\nabla m_j^{l,s}(2^{-j}\xi)| = |\nabla(m_{j,s}(2^{-j}\xi)\phi(2^{-l}\xi))| \le C.$$

By Lemma 2 again, there exists some constant $0 < \theta < 1$ such that

$$\left\|\bar{T}_{j;a,m}^{l,s}f\right\|_2 \le C2^{\frac{\theta l}{2\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^m\|f\|_2,$$

which by dilation-invariance implies

$$\left\|T_{j;a,m}^{l,s}f\right\|_2 \le C2^{\frac{\theta l}{2\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^m\|f\|_2.$$

From this and Lemma 1(a), we obtain

$$\begin{aligned}
&\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^{l,s}S_{l-j}\right)_{a,m}f\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\
&\qquad \le C\sum_{i=0}^m\left\|\left(\sum_{j\in\mathbb{Z}}\left|T_{j;a,i}^{l,s}\left(S_{l-j;a,m-i}f\right)\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\
&\qquad \le C\sum_{i=0}^m\|a\|_{BMO(\mathbb{R}^n)}^{2i}2^{\frac{\theta l}{\log|Q_s|}}\sum_j\|S_{l-j;a,m-i}f\|_2^2 \\
&\qquad \le C2^{\frac{\theta l}{\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^{2m}\|f\|_2^2.
\end{aligned}$$

Thus

$$\|U_l^sf\|_2 \le C2^{\frac{\theta l}{2\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^k\|f\|_2. \tag{2.9}$$

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

This shows that

$$\begin{aligned}\sum_{l>0}\|U_l f\|_2 &\le \sum_s |C_s| \sum_{l>0}\|U_l^s f\|_2 \\ &\le C\sum_s |C_s| \sum_{l>0} 2^{\frac{\theta l}{2\log|Q_s|}}\|a\|^k_{BMO(\mathbb{R}^n)}\|f\|_2 \\ &\le C\sum_s |C_s| \left(\log\frac{1}{|Q_s|}\right)\|a\|^k_{BMO(\mathbb{R}^n)}\|f\|_2.\end{aligned} \tag{2.10}$$

Therefore, it follows from (2.6) and (2.10) that

$$\|T_{a,k}f\|_2 \le \sum_{l\le 0}\|U_l f\|_2 + \sum_{l>0}\|U_l f\|_2 \le C\|a\|^k_{BMO(\mathbb{R}^n)}\|f\|_2.$$

This completes the proof of Theorem 1. □

§3. Proof of Theorem 3

The proof of Theorem 3 is based on the following two lemmas.

LEMMA 4. *Let m be a positive number, $1 < p < \infty$. If $a \in BLO(\mathbb{R}^n)$ and $a(x)$ is a subharmonic function, then the operator $M_{a,m}$ defined by*

$$M_{a,m}f(x) = \sup_{r>0}\frac{1}{r^n}\int_{|x-y|\le r}|a(x)-a(y)|^m|f(y)|dy$$

satisfies

$$\|M_{a,m}f\|_p \le C\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p.$$

Note that for any cube Q, $|Q|^{-1}\int_Q |a(x)-a_Q|^m dx \le \|a\|^m_{BLO(\mathbb{R}^n)}$. Since a is a subharmonic function, this lemma follows from the same argument as in the proof of Theorems 2.3 and 2.4 in [8]. We omit the details.

LEMMA 5. *Let Ω_0 be homogeneous of degree zero on $\mathbb{R}^n$, $1 < p < \infty$, a and h be as in Theorem 2 . If $\Omega_0 \in L^\lambda(S^{n-1})$, for $\lambda > 1$, then the operator*

$$M^{\Omega_0}_{a,\widetilde{m}}f(x) = \sup_{r>0}\frac{1}{r^n}\int_{|x-y|\le r}|a(x)-a(y)|^{\widetilde{m}}|h(|x-y|)\Omega_0(x-y)f(y)|dy$$

satisfies

$$\|M^{\Omega_0}_{a,\widetilde{m}}f\|_p \le C\|a\|^{\widetilde{m}}_{BLO(\mathbb{R}^n)}\|\Omega_0\|_{L^\lambda(S^{n-1})}\|f\|_p,$$

for all integer $\widetilde{m} \ge 0$. Here C is independent of λ.

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Proof. For $\widetilde{m} = 0$, Lemma 5 was proved by Calderón and Zygmund [5]. Next, we consider the case, $\widetilde{m} > 0$. For any $\lambda > 1$, write $\lambda' = \frac{\lambda}{\lambda-1}$. Then by a double application of Hölder's inequality, we have

$$\begin{aligned}\|M^{\Omega_0}_{a,\widetilde{m}}f\|_p^p &\le \|h\|_\infty^p \int_{\mathbb{R}^n} \left(M_{a,\lambda'\widetilde{m}}f(x)\right)^{\frac{p}{\lambda'}} \left(M_{\Omega_0^\lambda}f(x)\right)^{\frac{p}{\lambda}} dx \\ &\le C\|M_{a,\lambda'\widetilde{m}}f\|_p^{\frac{p}{\lambda'}} \|M_{\Omega_0^\lambda}f\|_p^{\frac{p}{\lambda}},\end{aligned}$$

where

$$M_{\Omega_0^\lambda}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|\le r} |\Omega_0^\lambda(x-y)f(y)|dy.$$

It follows from Lemma 4 that

$$\|M_{a,\lambda'\widetilde{m}}f\|_p^{\frac{p}{\lambda'}} \le C\|a\|^{\widetilde{m}p}_{BLO(\mathbb{R}^n)}\|f\|_p^{\frac{p}{\lambda'}}.$$

By the method of rotation of Calderón-Zygmund [5], it yields that

$$\|M_{\Omega_0^\lambda}f\|_p^{\frac{p}{\lambda}} \le C\|\Omega_0\|^p_{L^\lambda(S^{n-1})}\|f\|_p^{\frac{p}{\lambda}}.$$

Combining these estimates above, we complete the proof Lemma 5. □

Proof of Theorem 3. By Definitions 1 and 2, we write $\Omega(y') = \sum_s C_s b_s(y')$, where each b_s is a q-block supported in Q_s. Thus

$$\begin{aligned}&M^{\Omega}_{a,m}f(x) \\ &\quad\le \sum_s |C_s| \sup_{r>0} \int_{|x-y|\le r} |a(x)-a(y)|^m |h(|x-y|)b_s(x-y)f(y)|dy \\ &\quad:= \sum_s |C_s| M^{b_s}_{a,m}f(x).\end{aligned}$$

Consequently,

$$\|M^{\Omega}_{a,m}f\|_p \le \sum_s |C_s| \|M^{b_s}_{a,m}f\|_p.$$

We now estimate $\|M^{b_s}_{a,m}f\|_p$ for each b_s. It follows from Lemma 5 that for any $\lambda > 1$,

$$\|M^{b_s}_{a,m}f\|_p \le C\|b_s\|_{L^\lambda(S^{n-1})}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p.$$

Notice that $\operatorname{supp}(b_s) \subseteq Q_s$ and $\|b_s\|_{L^q(S^{n-1})} \le |Q_s|^{\frac{1}{q}-1}$. If $|Q_s| \ge e^{\frac{q}{1-q}}$, we let $\lambda = q$, to get

$$\begin{aligned}\|M^{b_s}_{a,m}f\|_p &\le C\|b_s\|_{L^q(S^{n-1})}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p \\ &\le C|Q_s|^{\frac{1}{q}-1}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p \le C\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p.\end{aligned}$$

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

If $|Q_s| < e^{\frac{q}{1-q}}$, let $\lambda = \log|Q_s|/(1+\log|Q_s|)$, so that $1 < \lambda < q$ and $\lambda' = -\log|Q_s|$. By Hölder's inequality, we have

$$\begin{aligned}\|M^{b_s}_{a,m}f\|_p &\le C\|b_s\|_{L^q(S^{n-1})}|Q_s|^{\frac{1}{\lambda}-\frac{1}{q}}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p\\ &\le C|Q_s|^{-\frac{1}{\lambda'}}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p \le C\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p.\end{aligned}$$

So, we obtain

$$\|M^{\Omega}_{a,m}f\|_p \le C\sum_s |C_s|\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p \le C\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p$$

and complete the proof of Theorem 3. □

§4. Proof of Theorem 2

To prove Theorem 2, we still need the following auxiliary result.

Let h, a, k and $\Omega(y') = \sum_s C_s b_s(y')$ be as in Theorem 2, $j \in \mathbb{Z}$. Define the following operators:

$$\sigma_{j;a,k}f(x) = \int_{2^j<|x-y|\le 2^{j+1}} [a(x)-a(y)]^k \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|)f(y)dy,$$

$$\sigma^s_{j;a,k}f(x) = \int_{2^j<|x-y|\le 2^{j+1}} [a(x)-a(y)]^k \frac{b_s(x-y)}{|x-y|^n} h(|x-y|)f(y)dy,$$

$$\mu_{j;a,k}f(x) = \int_{2^j<|x-y|\le 2^{j+1}} |a(x)-a(y)|^k \frac{|\Omega(x-y)|}{|x-y|^n} |h(|x-y|)|f(y)dy,$$

$$\mu^s_{j;a,k}f(x) = \int_{2^j<|x-y|\le 2^{j+1}} |a(x)-a(y)|^k \frac{|b_s(x-y)|}{|x-y|^n} |h(|x-y|)|f(y)dy,$$

$$\mu^*_{a,k}f(x) = \sup_{j\in\mathbb{Z}} |\mu_{j;a,k}f(x)| \quad \text{and} \quad \mu^{s*}_{a,k}f(x) = \sup_{j\in\mathbb{Z}} |\mu^s_{j;a,k}f(x)|.$$

Clearly, we have

$$\mu^*_{a,k}f(x) \le CM^{\Omega}_{a,k}f(x) \ \ \textit{and} \ \ \mu^{s*}_{a,k}f(x) \le CM^{b_s}_{a,k}f(x).$$

By Lemma 5 and Theorem 3, it is easy to see that for all $1 < p < \infty$,

$$\|\mu^*_{a,k}f\|_p \le C\|a\|^k_{BLO(\mathbb{R}^n)}\|f\|_p \tag{4.1}$$

and

$$\|\mu^{s*}_{a,k}f\|_p \le C\|b_s\|_{L^\lambda(S^{n-1})}\|a\|^k_{BLO(\mathbb{R}^n)}\|f\|_p, \tag{4.2}$$

and the bounds are independent of b_s.

By applying (4.1) and (4.2), we can obtain the following lemma.

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

LEMMA 6. *Under the same assumptions as in Theorem 2, for arbitrary functions f_j,*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p \le C\|a\|^k_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p \tag{4.3}$$

and

$$\begin{aligned} &\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma^s_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\qquad \le C\|b_s\|_{L^\lambda(S^{n-1})}\|a\|^k_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p \end{aligned} \tag{4.4}$$

for all $1 < p < \infty$ and for any $\lambda > 1$.

Proof. We prove only (4.3) because the other is essentially similar. The ideas in our proof are taken from those in Lemma of [7] and Lemma 2 of [11]. In fact, it suffices to consider the case $p > 2$ so that $q = (\frac{p}{2})'$, and there exists $g \in L^q_+$ of unit norm such that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|^2_p = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j(x)|^2 g(x)dx.$$

Also, by Hölder's inequality and a simple computation, we have

$$|\sigma_{j;a,k} f(x)|^2 \le C\mu_{j;a,2k}(|f|^2)(x)$$

and

$$\int_{\mathbb{R}^n} \mu_{j;a,k}(|f|^2)(x)g(x)dx = \int_{\mathbb{R}^n} f^2(x)\mu_{j;\widetilde{a},2k}\widetilde{g}(-x)dx,$$

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

where $\widetilde{a}(x) = a(-x)$ and $\widetilde{g}(x) = g(-x)$. Therefore

$$\begin{aligned}\left\|\left(\sum_{j\in\mathbb{Z}}|\sigma_{j;a,k}f_j|^2\right)^{\frac{1}{2}}\right\|_p^2 &\le C\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\mu_{j;a,2k}\left(|f_j|^2\right)(x)g(x)dx\\ &= C\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}f_j^2(x)\mu_{j;\widetilde{a},2k}\widetilde{g}(-x)dx\\ &\le C\int_{\mathbb{R}^n}\sup_{j\in\mathbb{Z}}\left|\mu_{j;\widetilde{a},2k}\widetilde{g}(-x)\right|\sum_{j\in\mathbb{Z}}f_j^2(x)dx\\ &\le C\left\|\mu^*_{\widetilde{a},2k}\widetilde{g}\right\|_q\left\|\sum_{j\in\mathbb{Z}}|f_j|^2\right\|_{\frac{p}{2}}.\end{aligned}$$

By (4.1), we obtain

$$\begin{aligned}\left\|\left(\sum_{j\in\mathbb{Z}}|\sigma_{j;a,k}f_j|^2\right)^{\frac{1}{2}}\right\|_p^2 &\le C\|a\|^{2k}_{BLO(\mathbb{R}^n)}\|g\|_q\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^2\right)^{\frac{1}{2}}\right\|_p^2\\ &= C\|a\|^{2k}_{BLO(\mathbb{R}^n)}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^2\right)^{\frac{1}{2}}\right\|_p^2,\end{aligned}$$

which proves Lemma 6. □

Proof of Theorem 2. Let U_l, T_j^l, S_{l-j} be the same as that in the proof of Theorem 1. Then for $1 < p < \infty$, similarly to (2.1) and (2.2), we have

$$\|T_{a,k}f\|_p \le \sum_{l\in\mathbb{Z}}\|U_lf\|_p \tag{4.5}$$

and

$$\|U_lf\|_p \le C\sum_{m=0}^{k}\|a\|^{k-m}_{BLO(\mathbb{R}^n)}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^lS_{l-j}\right)_{a,m}f\right|^2\right)^{\frac{1}{2}}\right\|_p. \tag{4.6}$$

Now we estimate $\|U_lf\|_p$ in two cases as follows:

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Case 1. First we show the L^p-boundedness of U_l for $l \le 0$.

For $p = 2$, by the same arguments as to (2.6), we obtain

$$\|U_l f\|_2 \le C 2^{\upsilon l} \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_2. \tag{4.7}$$

Next we turn to estimate L^p-boundedness of $U_l f$. Write

$$\left(T^l_j S_{l-j}\right)_{a,m} f(x) = \sum_{i=0}^{m} C^i_m \sigma_{j;a,i}\left(S^2_{l-j;a,m-i} f\right)(x).$$

We know from Lemma 6 and Lemma 1(b) that for all $1 < p < \infty$,

(4.8)
$$\begin{aligned}\|U_l f\|_p &\le C \sum_{m=0}^{k} \|a\|^{k-m}_{BLO(\mathbb{R}^n)} \sum_{i=0}^{m} \left\| \left(\sum_{j\in\mathbb{Z}} \left|\sigma_{j;a,i}\left(S^2_{l-j;a,m-i} f\right)\right|^2 \right)^{\frac12} \right\|_p \\ &\le C \sum_{m=0}^{k} \sum_{i=0}^{m} C^m_i \|a\|^{k-m+i}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left|S^2_{l-j;a,m-i} f\right|^2 \right)^{\frac12} \right\|_p \\ &\le C \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p.\end{aligned}$$

Using interpolation between (4.7) and (4.8), we obtain

$$\sum_{l\le 0} \|U_l f\|_p \le C \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p. \tag{4.9}$$

Case 2. We next consider the L^p-estimate of U_l for $l > 0$.

Let $T^l_{j,s}$, S_{l-j}, U^s_l be as that in the proof of Theorem 1. Similarly to (2.7) and (2.8), we have for $1 < p < \infty$,

$$\|U_l f\|_p \le \sum_s |C_s| \|U^s_l f\|_p, \tag{4.10}$$

and

$$\|U^s_l f\|_p \le C \sum_{m=0}^{k} \|a\|^{k-m}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \left(T^{l,s}_j S_{l-j}\right)_{a,m} f \right|^2 \right)^{\frac12} \right\|_p. \tag{4.11}$$

For each b_s, without loss of generality, we may assume that the support Q_s of b_s are uniformly small such that $|Q_s| < e^{\frac{q}{1-q}}$. Similarly to (2.9), we can get that for some $0 < \theta < 1$,

$$\|U_l^s f\|_2 \le C 2^{\frac{\theta l}{2\log|Q_s|}} \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_2. \tag{4.12}$$

For $1 < p < \infty$, noting that

$$\left(T_j^{l,s} S_{l-j}\right)_{a,m} f(x) = \sum_{i=0}^{m} C_m^i \sigma^s_{j;a,i}\left(S^2_{l-j;a,m-i} f\right)(x)$$

and invoking (4.4) and Lemma 1(b) with $\lambda = \frac{\log|Q_s|}{1+\log|Q_s|}$, we have

(4.13)

$$\begin{aligned}\|U_l^s f\|_p &\le C\sum_{m=0}^{k} \|a\|^{k-m}_{BLO(\mathbb{R}^n)} \sum_{i=0}^{m} \left\| \left(\sum_{j\in\mathbb{Z}} \left|\sigma^s_{j;a,i}\left(S^2_{l-j;a,m-i} f\right)\right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\le C\|b_s\|_{L^\lambda(S^{n-1})} \sum_{m=0}^{k}\sum_{i=0}^{m} \|a\|^{k-m+i}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left|S^2_{l-j;a,m-i} f\right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\le C\|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p.\end{aligned}$$

Using interpolation between (4.12) and (4.13) again, we obtain

$$\|U_l^s f\|_p \le C 2^{\frac{\theta_1 \theta l}{2\log|Q_s|}} \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p, \tag{4.14}$$

for some $0 < \theta_1 \le 1$. This shows that

$$\begin{aligned}\sum_{l>0} \|U_l f\|_p &\le \sum_s |C_s| \sum_{l>0} \|U_l^s f\|_p \\ &\le C\sum_s |C_s| \sum_{l>0} 2^{\frac{\theta_1\theta l}{2\log|Q_s|}} \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p \\ &\le C\sum_s |C_s| \left(\log\frac{1}{|Q_s|}\right) \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p.\end{aligned} \tag{4.15}$$

Therefore, (4.9) and (4.15) now imply

$$\|T_{a,k} f\|_p \le \sum_{l\le 0} \|U_l f\|_p + \sum_{l>0} \|U_l f\|_p \le C\|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p,$$

which completes the proof of Theorem 2. □

https://doi.org/10.1017/S0027763000008771 Published online by Cambridge University Press

Acknowledgements. The authors would like to express their gratitude to the referee for his very valuable comments.

References

[1] A.J. AL-Hasan and D.S. Fan, *A singular integral operator related block spaces*, Hokkaido Math. J., **28** (1999), 285–299.

[2] J. Alvarez, R. Bagby, D. Kurtz and C. Pérez, *Weighted estimates for commutators of linear operators*, Studia Math., **104** (1993), 195–209.

[3] R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc., **79**(2) (1980), 249–254.

[4] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611–636.

[5] A.P. Calderon and A. Zygmund, *On singular integrals*, Amer. J. Math., **78** (1956), 289–309.

[6] J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc., **336** (1993), 869–880.

[7] J. Duoandikoetxea and J.L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., **84** (1996), 541–561.

[8] J. Garcia-Cuerva, E. Harboure, C. Segovia and J.L. Torre, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J., **40** (1991), 1397–1420.

[9] G.E. Hu, *L^2-boundedness for the commutators with convolution operators*, Nagoya Math.J., **163** (2001), 55–70.

[10] ———, *Weighted norm inequalities for commutators of homogeneous singular integral operators*, Acta Math. Sinica, New series, **11** (1995), Special Issue: 77–88.

[11] G.E. Hu, S.Z. Lu and B.L. Ma, *The commutators of convolution operators* (in Chinese), Acta Math. Sinica, **42** (1999), 359–368.

[12] G.E. Hu and B.L. Ma, *$L^2(\mathbb{R}^n)$ boundedness for the commutators of homogeneous singular integral operators*, Acta Math. Sinica (to appear).

[13] G.E. Hu and Q.Y. Sun, *L^p boundedness for the commutators of convolution operators*, Chinese Ann. of Math. (Ser. A), **22**(4) (2001), 509–516.

[14] M. Keitoku and S. Sato, *Block spaces on the unit sphere in $\mathbb{R}^n$*, Proc. Amer. Math. Soc., **119** (1993), 453–455.

[15] S.Z. Lu, M. Taibleson and G. Weiss, Spaces Generated by Blocks, Beijing Normal University Press, Beijing, 1989.

[16] S.Z. Lu and H.X. Wu, *Oscillatory singular integrals and commutators with rough kernels*, Ann. Sci. Math. Québec, **27**(1) (2003), 47–66.

[17] M.H. Taibleson and G. Weiss, *Certain function spaces associated with a.e. convergence of Fourier series*, Univ. of Chicago Conf. in honor of Zygmund, Vol.I, Woodsworth, 1983, 95–113.

Shanzhen Lu
Department of Mathematics
Beijing Normal University
Beijing, 100875
P. R. China
lusz@bnu.edu.cn

Huoxiong Wu
School of Mathematical Sciences
Xiamen University
Xiamen, Fujian 361005
China
huoxwu@xmu.edu.cn