Canad. Math. Bull. Vol. 19 (2), 1976

ON CRITICAL r, λ-SYSTEMS

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SUMMARY. Critical r, λ-systems are introduced such that any nontrivial r, λ-system must be an extension of some critical system. It is shown that parametric values for which critical systems can exist are restricted to $\lambda(v-1) \leq r(r-1)$ and, further, to $\lambda(v-1) < r(r-1)$ if the critical system is extendible.

1. **Introduction.** An r, λ-system is a collection of v objects (or varieties) arranged into b subsets (blocks) such that each variety appears in exactly r blocks and each pair of varieties appears in exactly λ blocks. For $\lambda = 0$, each block must contain a single variety and, for $\lambda = r$, each block must contain all varieties. We call such designs trivial and avoid them by assuming $1 \leq \lambda < r$.

A theorem by Ryser [1], proves that if $b = v$ in an r, λ-system, then $\lambda(v-1) = r(r-1)$ and the system is a symmetric balanced incomplete block design. The r, λ-system,

$$(abcdeg)(ag)(bd)(ce)(abcf)(adef)(befg)(cdfg)$$

has $v=7$, $b=8$, $r=4$ and $\lambda=2$. Since $(v-1)=12=r(r-1)$ but $b \neq v$ the converse of Ryser's theorem does not hold. We partition r, λ-systems into three classes by defining $D = \lambda(v-1) - r(r-1)$ and calling a system elliptic, parabolic or hyperbolic according to whether D is negative, zero or positive respectively.

2. **Reducible r, λ-systems.** Consider any r, λ-system S. Adding a complete block (that is, a block consisting of all v varieties) will result in a new r, λ-system with the values of r and λ each increased by one. Adding a complete singles set (that is each variety appears as a block consisting of a single element) will result in a new r, λ-system with the value of r increased by one. These simple additions can be used to construct an infinite family of systems from any given system. Any r, λ-system containing a complete block or a complete singles set will be called a reducible system. We will concern ourselves primarily with r, λ-systems that are not reducible and which we call irreducible.

Stanton and Mullin [2] proved that for $\lambda = 1$, all irreducible systems are elliptic or parabolic. It is conjectured that this is true for any value of λ and the following results support this conjecture.

Received by the editors May 5, 1975.

https://doi.org/10.4153/CMB-1976-034-2 Published online by Cambridge University Press

3. **Embedding** r, λ**-systems.** Let S be an r, λ-system with $v>1$ varieties. Deleting all occurrences of some variety, say x, will leave an r, λ-system with $v-1$ varieties and the same values for r and λ as S. The reverse process, of adding a new variety, say y, to r blocks to generate an r, λ-system S' with $v+1$ varieties is not so simple and not always possible. When it is possible we say S is embedded in S'. If S is embedded in S' and S' is embedded in S'' we say "S is embedded in S''".

LEMMA 1. *An irreducible system cannot be embedded in a reducible system.*

Proof. A reducible system must contain a complete block or a complete singles set. Deleting any variety still leaves a complete block or a complete singles set. Hence any system embedded in a reducible system would be itself reducible.

4. **Critical** r, λ**-systems.** If S is some irreducible r, λ-system such that the complete removal of any variety from S yields a reducible system we call S a critical irreducible system or, more simply, we say "S is critical".

THEOREM 1. *All critical* r, λ*-systems are elliptic or parabolic.*

Proof. Assume S is a critical r, λ-system on v varieties with $v>3$. Let X be the set of varieties in S such that the complete removal of any $x \in X$ leaves a system with a complete block. For each $x \in X$, S must contain a block consisting of all varieties of S except x. Let Y be the set of varieties in S that are not in X. Since S is critical, the complete removal of any $y \in X$ must leave a system with a complete singles set. Let T be the set of varieties in S that do not occur as single elements. Since S is irreducible, T is not empty. Then for each $y \in Y$ and each $t \in T$ $(y \neq t)$, S must contain a block consisting of precisely the pair (y, t).

Choose some $t \in T$. If $t \in X$, then t occurs in at least $|X|-1$ blocks of length $v-1$, one for each of the other elements in X. Also t occurs in $|Y|$ blocks of length two. Since $v>3$, these occurrences are distinct and t must occur in at least $|Y|+|X|-1$ blocks. If $t \in Y$, then t occurs in $|X|$ blocks of length $v-1$ and $|Y|-1$ blocks of length two. Again t occurs in at least $|X|+|Y|-1=v-1$ blocks. Since t must belong to X or Y we conclude $r=$(occurrences of $t) \geq v-1$. Substituting this result into the hyperbolic condition $D=\lambda(v-1)-r(r-1)>0$, we obtain $\lambda(v-1)>r(r-1) \geq (v-1)(r-1)$ or $\lambda>r-1$. This contradicts $\lambda<r$ so S was not hyperbolic.

Now suppose $v \leq 3$. Any r, λ-system with $v=1$ or $v=2$ is reducible so it couldn't be critical. For $v=3$, the hyperbolic property becomes $\lambda \cdot 2>r(r-1)$. Using $\lambda \leq r-1$ we obtain $2 \cdot \lambda>(\lambda+1)\lambda$ or $\lambda<1$. This says that S was trivial so it couldn't be critical. We conclude that there are no critical hyperbolic r, λ-systems.

A system obtained by taking as blocks all possible subsets of order $v-1$ from v varieties is called a full combinatorial design. Counting arguments show that there will be $b=v$ blocks, $r=v-1$ occurrences of each variety and $\lambda=v-2$ occurrences of each pair of varieties. The design is parabolic since $\lambda(v-1)=(v-2)(v-1)=r(r-1)$ and critical since the complete removal of any variety leaves a complete block.

THEOREM 2. *A critical parabolic system with v varieties must be a full combinatorial design with $r=v-1$ and $\lambda=v-2$.*

Proof. Suppose S is a critical design with $v>3$. Then

(1) $$r \geq (v-1) \quad \text{as shown in Theorem 1.}$$

Assume that S is also parabolic. That is, $\lambda(v-1)=r(r-1)$ or $r^2-r-\lambda(v-1)=0$. Solving as a quadratic equation in r, $r=(1\pm(1+4\lambda v-4\lambda)^{1/2})/2$. Since r is a positive integer we omit the negative sign to obtain,

(2) $$r=(1+(1+4\lambda v-4\lambda)^{1/2})/2.$$

Combining (1) and (2), $1+(1+4\lambda v-4\lambda)^{1/2} \geq 2(v-1)$. Simplifying, $1+4\lambda v-4\lambda \geq 4v^2-12v+9$, or

(3) $$0 \geq v^2-(3+\lambda)v+2+\lambda=[v-(2+\lambda)][v-1].$$

Since the roots of the quadratic equation $v^2-(3+\lambda)v+(2+\lambda)=0$ are $v=1$ and $v=\lambda+2$ we obtain from (3) that $1 \leq v \leq \lambda+2$. Suppose $v \leq \lambda+1$. Then $\lambda^2 \geq \lambda(v-1)$ and, using the parabolic property and $\lambda \leq r-1$, we obtain $\lambda(v-1)=r(r-1) \geq (\lambda+1)\lambda=\lambda^2+\lambda$. Hence $\lambda^2 \geq \lambda^2+\lambda$ and no such non-trivial systems exist. Therefore $v=\lambda+2$. From $\lambda(v-1)=r(r-1)$ we now get $r=\lambda+1$. S contains $\binom{v}{2}$ distinct pairs, each occurring $\lambda=v-2$ times for a total of $v(v-1)(v-2)/2$ pairs. The maximum number of varieties per block is $v-1$ and blocks of this size contain $(v-1)(v-2)/2$ pairs. We have $v \cdot r=v(v-1)$ entries, so we can construct v blocks of length $v-1$ for a total of $v(v-1)(v-2)/2$ pairs. Since this is precisely the number of pairs required and any other arrangement of varieties will produce strictly fewer pairs, then S must take this form and is the full combinatorial design. For $v \leq 3$ we argue as in Theorem 1 and find the only critical parabolic system to be $(ab)(ac)(bc)$ which is also a full combinatorial design.

Theorem 4 from Stanton and Mullin [2] tells us that these full combinatorial designs can be embedded in other r, λ-systems only if k (the constant block size) divides $r-\lambda$. In this case that would imply that $k=1$ and the system would be reducible. Hence no critical parabolic r, λ-system can be embedded in some other r, λ-system.

https://doi.org/10.4153/CMB-1976-034-2 Published online by Cambridge University Press

Since all irreducible r, λ-systems are critical or have some critical system embedded in them, we are now left with the fact that any irreducible hyperbolic r, λ-systems must be extensions of critical elliptical systems.

References

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