Adv. Appl. Prob. **24**, 1010–1011 (1992)
Printed in N. Ireland

A SIMPLE COUPLING OF RENEWAL PROCESSES

T. LINDVALL,* *University of Göteborg*

Abstract

We use a simple coupling to prove the classical result that the renewal function U of a zero-delayed renewal process satisfies $U(t)-\lambda\,.\,t\to\lambda^2\mu_2/2$ as $t\to\infty$ if the life-length distribution is of non-lattice type and has finite first and second moments μ and μ_2 respectively; λ is the renewal intensity, and is equal to $1/\mu$.

COUPLING; RENEWAL FUNCTION

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60K05

Let $Y_1, Y_2, \cdots$ be i.i.d. non-negative random variables with non-lattice distribution F, having finite first and second moments μ and μ_2 respectively. To the zero-delayed renewal process $S=(S_n)_0^\infty$, where $S_n=\sum_1^n Y_i$, $S_0=0$, we associate the point process N: $N(B)=\#\{n; S_n\in B\}$ for $B\in\mathscr{R}_+$, and the counting process $(N_t)_0^\infty$: $N_t=N[0,t]$ for $t\geqq 0$. Let $U(B)=\boldsymbol{E}[N(B)]$ and $U(t)=\boldsymbol{E}[N_t]$. Notice that $U(0)\geqq 1$ always, and $U(0)=1$ if $F(0)=0$. It is rather well known how to use a coupling to prove Blackwell's renewal theorem, which states that

$$U(t, t+A)\to\lambda\,.\,A \text{ as } t\to\infty \tag{1}$$

for all $A>0$, cf. Lindvall (1992), p. 73ff.; here $\lambda=1/\mu$. The classical result

$$U(t)-\lambda\,.\,t\to\lambda^2\mu_2/2 \text{ as } t\to\infty \tag{2}$$

has, however, not yet been given a coupling proof; the established route is to apply the so-called key renewal theorem to a renewal equation which is solved by $U(t)-\lambda t$, cf. Feller (1966), p. 357. The purpose of this letter is to show how a simple coupling works to prove (2).

Let Y_0' be independent of S and have density $\lambda\,.\,(1-F(y))$, $y\geqq 0$. We know that if a renewal process with lifelength distribution F has Y_0' as delay, it is stationary. Let $S'=(S_n')_0^\infty$ be defined by

$$S_n'=Y_0'+S_n, \qquad n\geqq 0,$$

and let N', U' have obvious meanings. We have

$$U(t)-\lambda\,.\,t=U(t)-U'(t)=\boldsymbol{E}[N_t]-\boldsymbol{E}[N_t']=\boldsymbol{E}[N(t-Y_0', t]]. \tag{3}$$

But

$$\boldsymbol{E}[N(t-Y_0', t]]=\boldsymbol{E}[\boldsymbol{E}[N(t-Y_0', t]\mid Y_0']]=\boldsymbol{E}[U(t-Y_0', t]]. \tag{4}$$

Now

$$U(t-Y_0', t]\to\lambda\,.\,Y_0' \text{ as } t\to\infty \tag{5}$$

due to (1). We are allowed to transpose expectation and limit in (4) because of dominated convergence. Indeed,

$$U(t-Y_0', t]\leqq U(Y_0')$$

Received 8 July 1992; revision received 24 August 1992.

* Postal address: Department of Mathematics, University of Göteborg, S-41296 Göteborg, Sweden.

for all $t \geqq 0$, and $E[U(Y_0')] < \infty$ since $U(s) \leqq C.(1+s)$ for some $C > 0$ and $E[Y_0'] < \infty$. From (3)–(5) we may now deduce (2) since $E[Y_0'] = \lambda\mu_2/2$.

The idea of constructing a stationary parallel process by introducing a suitable delay phase as above is of value for the study of regenerative processes in general. The consequences of such coupling will be presented at a later opportunity.

References

FELLER, W. (1966) *An Introduction to Probability Theory and its Applications,* Vol. II. Wiley, New York.

LINDVALL, T. (1992) *Lectures on the Coupling Method.* Wiley, New York.