Appendix 12

Dirac spinors and matrix elements

A12.1 General properties

We discuss here some properties of four-component spinors and the Dirac matrices, which are particularly useful in the computation of helicity amplitudes. We shall not touch on the usual elementary considerations of the Dirac equation and the finding of its free-particle solutions. For that, the reader should consult Bjorken and Drell (1964).

The γ-matrices satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{A12.1}$$

and we define

$$\gamma_5 \equiv \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A12.2}$$

and

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{A12.3}$$

For any 4-vector A^μ we use

$$\not{A} \equiv \mathrm{A}_\mu \gamma^\mu = \mathrm{A}^0\gamma^0 - \mathrm{A}^1\gamma^1 - \mathrm{A}^2\gamma^2 - \mathrm{A}^3\gamma^3. \tag{A12.4}$$

The particle spinors u and the antiparticle spinors v satisfy the Dirac equations

$$(\not{p} - m)u(p) = 0 \tag{A12.5}$$

$$(\not{p} + m)v(p) = 0. \tag{A12.6}$$

Our normalization is

$$u^\dagger u = 2E \qquad v^\dagger v = 2E \tag{A12.7}$$

which implies

$$\bar{u}u = 2m \qquad \bar{v}v = -2m, \tag{A12.8}$$

the above holding also if $m = 0$.

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With this normalization the cross-section formula (B.1) of Appendix B of Bjorken and Drell (1964) holds for both mesons and fermions, massive or massless.

Of very great importance are the properties of the traces of products of the γ-matrices. The most useful ones are:

$$\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_N}) = 0 \qquad \text{if } N \text{ is odd} \tag{A12.9}$$

$$\mathrm{Tr}\,(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu} \tag{A12.10}$$

$$\mathrm{Tr}\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right) = 4\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right) \tag{A12.11}$$

$$\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{N-1}}\gamma^{\mu_N}) = \mathrm{Tr}\,(\gamma^{\mu_N}\gamma^{\mu_{N-1}}\ldots\gamma^{\mu_2}\gamma^{\mu_1}) \tag{A12.12}$$

$$\mathrm{Tr}\,\gamma_5 = 0 \tag{A12.13}$$

$$\mathrm{Tr}\,(\gamma_5\gamma^{\alpha}) = \mathrm{Tr}\left(\gamma_5\gamma^{\alpha}\gamma^{\beta}\right) = \mathrm{Tr}\left(\gamma_5\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\right) = 0 \tag{A12.14}$$

$$\mathrm{Tr}\left(\gamma_5\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right) = -4i\epsilon^{\alpha\beta\mu\nu}, \tag{A12.15}$$

where $\epsilon_{\alpha\beta\mu\nu}$, the totally antisymmetric tensor in Minkowski space, is defined with $\epsilon_{0123} = +1$.

From the γ-matrices one can construct a set of 16 linearly independent matrices,

$$\begin{gathered}\Gamma_S \equiv I \qquad \Gamma_V^{\mu} \equiv \gamma^{\mu} \qquad \Gamma_T^{\mu\nu} \equiv \sigma^{\mu\nu} \\ \Gamma_A^{\mu} = \gamma^{\mu}\gamma_5 \qquad \Gamma_P = i\gamma_5,\end{gathered} \tag{A12.16}$$

chosen so that for each of them

$$\gamma^0\Gamma^{\dagger}\gamma^0 = \Gamma. \tag{A12.17}$$

Moreover, from the above,

$$\mathrm{Tr}\,\Gamma_j = 0 \qquad \text{for all } j \neq S \tag{A12.18}$$

and

$$\mathrm{Tr}\,\Gamma_S = 4.$$

An arbitrary 4×4 matrix can thus be written

$$M = SI + V_{\mu}\gamma^{\mu} + \tfrac{1}{2}T_{\mu\nu}\sigma^{\mu\nu} + A_{\mu}\gamma^{\mu}\gamma_5 + Pi\gamma_5 \tag{A12.19}$$

where $T_{\mu\nu} = -T_{\nu\mu}$.

It is *not* implied that the coefficients are Lorentz vectors or tensors etc. for a general M.

The expansion coefficients can be found from M simply by taking appropriate traces:

$$\begin{gathered}S = \tfrac{1}{4}\,\mathrm{Tr}\,M \qquad V_{\mu} = \tfrac{1}{4}\,\mathrm{Tr}\,\gamma_{\mu}M \qquad T_{\mu\nu} = \tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M \\ A_{\mu} = \tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_{\mu}M \qquad iP = \tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M\end{gathered} \tag{A12.20}$$

A12.2 Helicity spinors and Lorentz transformations

Corresponding to the definition of helicity states given in subsection 1.2.1, the particle (or antiparticle) spinors for momentum p^μ are related to those at rest (see eqn (2.4.14)) by

$$u_n(\mathbf{p},\lambda) = D_{nm}[h(\mathbf{p})]u_m(\overset{\circ}{p},\lambda). \tag{A12.21}$$

For Dirac spinors, the representation matrices are given by

(i)

$$D[r_j(\theta)] = e^{-i\theta\Sigma_j/2} \tag{A12.22}$$

where $r_j(\theta)$ is a rotation through angle θ about the j-axis and

$$\Sigma_j \equiv \tfrac{1}{2}\epsilon_{jkl}\sigma^{kl}; \tag{A12.23}$$

(ii)

$$D\left[l_j(v)\right] = e^{-\omega\gamma^j\gamma^0/2}$$

where $l_j(v)$ is a boost along the j-axis and $\tanh\omega = v/c$.

Thus for $\mathbf{p} = (p,\theta,\phi)$

$$D\left[h(\mathbf{p})\right] = e^{-i\phi\Sigma_3/2}e^{-i\theta\Sigma_2/2}e^{-\omega\gamma^3\gamma^0/2} \tag{A12.24}$$

where now

$$\cosh\omega/2 = \sqrt{(E+mc^2)/(2mc^2)}$$

and

$$\tanh\omega/2 = pc/(E+mc^2).$$

The actual form of the spinors depends upon the representation used for the γ-matrices. Two useful choices will be discussed later.

The helicity spinors of course represent eigenstates of helicity. They are thus eigenspinors of the matrix that represents the helicity operator. The most convenient form is in terms of the covariant helicity spin vectors $\mathscr{S}^\mu(\mathbf{p},\lambda)$ introduced in Section 3.4. One finds that

$$\begin{aligned}\gamma_5 \not{\mathscr{S}}(\mathbf{p},\lambda)u(\mathbf{p},\lambda) &= mu(\mathbf{p},\lambda)\\ \gamma_5 \not{\mathscr{S}}(\mathbf{p},\lambda)v(\mathbf{p},\lambda) &= mv(\mathbf{p},\lambda)\end{aligned} \tag{A12.25}$$

where

$$\mathscr{S}^\mu(\mathbf{p},\lambda) = 2\lambda(p, E\hat{\mathbf{p}}) \tag{A12.26}$$

and

$$p_\mu\mathscr{S}^\mu(\mathbf{p},\lambda) = 0. \tag{A12.27}$$

Sometimes it is convenient to label the spinors by p^μ and $\mathscr{S}^\mu$, but it should be remembered that the explicit form for the spinors depends upon the

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representation used for the γ-matrices and so cannot be written as a covariant combination of p^μ and $\mathscr{S}^\mu$.

The following results for the 4×4 matrices formed from the spinors are important in calculating physical cross-sections:

$$\sum_\lambda \left[u_\alpha(\mathbf{p},\lambda)\bar{u}_\beta(\mathbf{p},\lambda) - v_\alpha(\mathbf{p},\lambda)\bar{v}_\beta(\mathbf{p},\lambda)\right] = \delta_{\alpha\beta}2m \tag{A12.28}$$

$$\begin{aligned}\sum_\lambda u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) &= \not{p} + m \\ \sum_\lambda v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda) &= \not{p} - m\end{aligned} \tag{A12.29}$$

and, using (A12.25),

$$\begin{aligned}u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) &= \frac{\not{p}+m}{2}\left[1 + \frac{\gamma_5\,\not{\mathscr{S}}(\mathbf{p},\lambda)}{m}\right] \\ v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda) &= \frac{\not{p}-m}{2}\left[1 + \frac{\gamma_5\,\not{\mathscr{S}}(\mathbf{p},\lambda)}{m}\right].\end{aligned} \tag{A12.30}$$

Now note that, irrespective of whether we have particle or antiparticle spinors, *any* matrix element of the form $\bar{u}(1)\Gamma u(2)$ can be written as a trace:

$$\begin{aligned}\bar{u}(1)\Gamma u(2) &= \bar{u}_\alpha(1)\Gamma_{\alpha\beta}u_\beta(2) = \Gamma_{\alpha\beta}u_\beta(2)\bar{u}_\alpha(1) \\ &= \text{Tr}\ [\Gamma u(2)\bar{u}(1)].\end{aligned} \tag{A12.31}$$

Generally this trick is not useful because of the complexity of $u(2)\bar{u}(1)$ when $m \neq 0$. However, when $p_1^\mu = p_2^\mu$ we can use (A12.30) to derive the following helpful results:

$$\begin{aligned}\bar{u}(\mathbf{p},\mathscr{S})\gamma^\mu u(\mathbf{p},\mathscr{S}) &= 2p^\mu \\ \bar{u}(\mathbf{p},\mathscr{S})\sigma^{\mu\nu} u(\mathbf{p},\mathscr{S}) &= \frac{2}{m}\epsilon^{\alpha\beta\mu\nu}p_\alpha \mathscr{S}_\beta \\ \bar{u}(\mathbf{p},\mathscr{S})\gamma^\mu\gamma_5 u(\mathbf{p},\mathscr{S}) &= 2\mathscr{S}^\mu \\ \bar{u}(\mathbf{p},\mathscr{S})\gamma_5 u(\mathbf{p},\mathscr{S}) &= 0\end{aligned} \tag{A12.32}$$

We consider now the specific form of the spinors in particular representations of the γ-matrices.

A12.3 The Dirac–Pauli representation

One takes

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad k = 1, 2, 3$$
$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{A12.33}$$
$$\Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \gamma^j\gamma^0 = -\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad j = 1, 2, 3$$

where, as usual,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A12.34}$$

For the transpose (T) of the matrices one has

$$\begin{aligned} {\gamma^j}^T &= \gamma^j \qquad j = 0, 2, 5 \\ {\gamma^j}^T &= -\gamma^j \qquad j = 1, 3 \end{aligned} \tag{A12.35}$$

and for the hermitian conjugate (†)

$${\gamma^0}^\dagger = \gamma^0 \qquad {\gamma_5}^\dagger = \gamma_5 \qquad {\gamma^j}^\dagger = -\gamma^j \qquad j = 1, 2, 3 \tag{A12.36}$$

For the rest-frame spinors one usually takes

$$u(\mathring{p}, \lambda) = \sqrt{2m}\begin{pmatrix} \chi_\lambda \\ 0 \end{pmatrix} \qquad v(\mathring{p}, \lambda) = \sqrt{2m}\begin{pmatrix} 0 \\ \chi_{-\lambda} \end{pmatrix} \tag{A12.37}$$

where χ_λ is a two-component spinor and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A12.38}$$

The explicit form of the helicity spinors is, then, from (A12.24),

$$\begin{aligned} u(\mathbf{p}, \lambda) &= \frac{1}{\sqrt{E+m}}\begin{pmatrix} E+m \\ 2p\lambda \end{pmatrix}\chi_\lambda(\hat{\mathbf{p}}) \\ v(\mathbf{p}, \lambda) &= \frac{1}{\sqrt{E+m}}\begin{pmatrix} -2p\lambda \\ E+m \end{pmatrix}\chi_{-\lambda}(\hat{\mathbf{p}}) \end{aligned} \tag{A12.39}$$

where

$$\chi_\lambda(\hat{\mathbf{p}}) \equiv e^{-i\phi\sigma_3/2}e^{-i\theta\sigma_2/2}\chi_\lambda. \tag{A12.40}$$

One finds

$$\begin{aligned} \chi_+(\mathbf{p}) &= \begin{pmatrix} e^{-i\phi/2} & \cos\theta/2 \\ e^{i\phi/2} & \sin\theta/2 \end{pmatrix} \\ \chi_-(\mathbf{p}) &= \begin{pmatrix} -e^{-i\phi/2} & \sin\theta/2 \\ e^{i\phi/2} & \cos\theta/2 \end{pmatrix} \end{aligned} \tag{A12.41}$$

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Note that in this representation we have

$$v(\mathbf{p}, \lambda) = i\gamma^2 u^*(\mathbf{p}, \lambda) \tag{A12.42}$$

A12.4 The Weyl representation

This is particularly useful in the relativistic limit and in the massless case. One takes

$$\begin{gathered}
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad j = 1, 2, 3 \\
\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \gamma^j \gamma^0 = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}
\end{gathered} \tag{A12.43}$$

Equations (A12.35) and (A12.36) for transpose and hermitian conjugate continue to hold.

The explicit form of the helicity spinors can be taken, via (A12.24), as

$$\begin{aligned}
u(\mathbf{p}, \lambda) &= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} E + m + 2p\lambda \\ E + m - 2p\lambda \end{pmatrix} \chi_\lambda(\hat{\mathbf{p}}) \\
v(\mathbf{p}, \lambda) &= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} p - 2\lambda(E+m) \\ p + 2\lambda(E+m) \end{pmatrix} \chi_{-\lambda}(\hat{\mathbf{p}})
\end{aligned} \tag{A12.44}$$

in the Weyl representation. Note that if $m \neq 0$ this corresponds to the choice

$$\begin{aligned}
u(\overset{\circ}{p}, \lambda) &= \sqrt{m} \begin{pmatrix} \chi_\lambda \\ \chi_\lambda \end{pmatrix} \\
v(\overset{\circ}{p}, \lambda) &= 2\lambda\sqrt{m} \begin{pmatrix} -\chi_{-\lambda} \\ \chi_{-\lambda} \end{pmatrix}
\end{aligned} \tag{A12.45}$$

and in this representation

$$v(\mathbf{p}, \lambda) = -i\gamma^2 u^*(\mathbf{p}, \lambda). \tag{A12.46}$$

A12.5 Massless fermions

When $m = 0$, remarkable simplifications occur in the Weyl representation. For example, if $m \ll E$ or $m = 0$ then (A12.44) becomes

$$\begin{aligned}
u(\mathbf{p}, +) &= \sqrt{2E} \begin{pmatrix} \chi_+(\mathbf{p}) \\ 0 \end{pmatrix} = v(\mathbf{p}, -) \\
u(\mathbf{p}, -) &= \sqrt{2E} \begin{pmatrix} 0 \\ \chi_-(\mathbf{p}) \end{pmatrix} = v(\mathbf{p}, +)
\end{aligned} \tag{A12.47}$$

with

$$\not{p}u(\mathbf{p}) = \not{p}v(\mathbf{p}) = 0$$
$$\bar{u}_\lambda(\mathbf{p})u_\lambda(\mathbf{p}') = \bar{v}_\lambda(\mathbf{p})v_\lambda(\mathbf{p}') = 0. \tag{A12.48}$$

If in (A12.26) we write

$$\begin{aligned}\mathscr{S}^\mu(\mathbf{p},\lambda) &= 2\lambda(E,\mathbf{p}) + 2\lambda(p-E,(E-p)\hat{\mathbf{p}}) \\ &= 2\lambda p^\mu + 2\lambda(E-p)(-1,\hat{\mathbf{p}})\end{aligned} \tag{A12.49}$$

where $\hat{\mathbf{p}}$ is the unit vector along $\mathbf{p}$ then, using (A12.5) and (A12.25), we see that for $m \ll E$ or $m = 0$

$$\left.\begin{aligned}\gamma_5 u(\mathbf{p},\lambda) &= 2\lambda u(\mathbf{p},\lambda) \\ \gamma_5 v(\mathbf{p},\lambda) &= -2\lambda v(\mathbf{p},\lambda)\end{aligned}\right\} \qquad (m=0) \tag{A12.50}$$

from which follow

$$\begin{aligned}\bar{u}(\mathbf{p},\lambda)\gamma_5 &= -2\lambda\bar{u}(\mathbf{p},\lambda) \\ \bar{v}(\mathbf{p},\lambda)\gamma_5 &= 2\lambda\bar{v}(\mathbf{p},\lambda).\end{aligned} \tag{A12.51}$$

The consequences of these and the connection with chirality are discussed in subsection 4.6.3.

The relations (A12.29) become

$$u_+(\mathbf{p})\bar{u}_+(\mathbf{p}) + u_-(\mathbf{p})\bar{u}_-(\mathbf{p}) = \not{p} = v_+(\mathbf{p})\bar{v}_+(\mathbf{p}) + v_-(\mathbf{p})\bar{v}_-(\mathbf{p}). \tag{A12.52}$$

Multiplying by $(1 \pm \gamma_5)/2$ we obtain

$$\begin{aligned}u_\pm(\mathbf{p})\bar{u}_\pm(\mathbf{p}) &= \tfrac{1}{2}(1 \pm \gamma_5)\not{p} \\ v_\pm(\mathbf{p})\bar{v}_\pm(\mathbf{p}) &= \tfrac{1}{2}(1 \mp \gamma_5)\not{p}\end{aligned} \tag{A12.53}$$

Consider now the very important matrix $M = u_+(\mathbf{p}_1)\bar{u}_+(\mathbf{p}_2)$. It can be expanded as in (A12.19). Using the fact that

$$\gamma_5 M = M = -M\gamma_5 \tag{A12.54}$$

the coefficients in the expansion must satisfy

$$\mathrm{Tr}\,(\Gamma M) = -\,\mathrm{Tr}\,(\Gamma\gamma_5 M\gamma_5) = -\,\mathrm{Tr}\,(\gamma_5\Gamma\gamma_5 M).$$

But $\gamma_5\,\{I,\sigma^{\mu\nu},\gamma_5\}\,\gamma_5 = \{I,\sigma^{\mu\nu},\gamma_5\}$, so that $S = T_{\mu\nu} = P = 0$. Also $A_\mu = -V_\mu$.

Thus for the massless case the only relevant coefficients are

$$V_\mu = \tfrac{1}{4}\,\mathrm{Tr}\,\left(\gamma_\mu u_+(\mathbf{p}_1)\bar{u}_+(\mathbf{p}_2)\right) \tag{A12.55}$$

However, using (A12.31) we can rewrite (A12.55) as

$$V_\mu = \tfrac{1}{4}\bar{u}_+(\mathbf{p}_2)\gamma_\mu u_+(\mathbf{p}_1)$$

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so that finally

$$u_+(\mathbf{p}_1)\bar{u}_+(\mathbf{p}_2) = \tfrac{1}{4}\left[\bar{u}_+(\mathbf{p}_2)\gamma_\mu u_+(\mathbf{p}_1)\right]\gamma^\mu(1-\gamma_5). \tag{A12.56}$$

In similar fashion one finds

$$u_-(\mathbf{p}_1)\bar{u}_-(\mathbf{p}_2) = \tfrac{1}{4}\left[\bar{u}_-(\mathbf{p}_2)\gamma_\mu u_-(\mathbf{p}_1)\right]\gamma^\mu(1+\gamma_5). \tag{A12.57}$$

One can check explicitly that

$$\bar{u}_-(\mathbf{p}_2)\gamma_\mu u_-(\mathbf{p}_1) = \bar{u}_+(\mathbf{p}_1)\gamma_\mu u_+(\mathbf{p}_2) \tag{A12.58}$$

so that (A12.57) becomes

$$u_-(\mathbf{p}_1)\bar{u}_-(\mathbf{p}_2) = \tfrac{1}{4}\left[\bar{u}_+(\mathbf{p}_1)\gamma_\mu u_+(\mathbf{p}_2)\right]\gamma^\mu(1+\gamma_5). \tag{A12.59}$$

Using similar techniques one can show that

$$u_+(\mathbf{p}_1)\bar{u}_-(\mathbf{p}_2) - u_+(\mathbf{p}_2)\bar{u}_-(\mathbf{p}_1) = \left[\bar{u}_-(\mathbf{p}_2)u_+(\mathbf{p}_1)\right]\tfrac{1}{2}(1+\gamma_5). \tag{A12.60}$$

Equations (A12.56), (A12.59) and (A12.60) are very useful in deriving the rules for Feynman diagrams with massless particles (Chapter 10).

A12.6 The Fierz rearrangement theorem

It sometimes happens, when dealing with the matrix element corresponding to a Feynman diagram involving spin-1/2 particles, that it is convenient to rearrange the order of the spinors in relation to the order they acquire directly from the Feynman diagram.

In general, let $\tilde{\Gamma}^i (i = 1, \ldots, 16)$ stand for any of the independent combinations of unit matrix and γ-matrices $I, \gamma^\mu, \sigma^{\mu\nu}, i\gamma^\mu\gamma_5, \gamma_5$.

Let $\tilde{\Gamma}_i$ stand for the above set of matrices with their Lorentz indices lowered when relevant, i.e. $\tilde{\Gamma}_i$ contains for example γ_μ, whereas $\tilde{\Gamma}^i$ contains γ^μ etc.

As a result of the algebraic properties of the set $\tilde{\Gamma}^i$ it can be shown that

$$\tfrac{1}{4}\sum_i \left(\tilde{\Gamma}_i\right)_{\alpha\beta}\left(\tilde{\Gamma}^i\right)_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{A12.61}$$

If now A and B are any 4×4 matrices, then on multiplying (A12.61) by $A_{\rho\alpha}B_{\nu\gamma}$ we obtain

$$\begin{aligned} \tfrac{1}{4}\sum_i A_{\rho\alpha}\left(\tilde{\Gamma}_i\right)_{\alpha\beta} B_{\nu\gamma}\left(\tilde{\Gamma}^i\right)_{\gamma\delta} &= A_{\rho\delta}B_{\nu\beta} \\ A_{\rho\delta}B_{\nu\beta} &= \tfrac{1}{4}\sum_i \left(A\tilde{\Gamma}_i\right)_{\rho\beta}\left(B\tilde{\Gamma}^i\right)_{\nu\delta}. \end{aligned} \tag{A12.62}$$

Since the 16 $\tilde{\Gamma}^i$ are a complete set of 4×4 matrices, each product $A\tilde{\Gamma}_i$ etc. will reduce to a sum of $\tilde{\Gamma}_i$.

After some labour one can obtain the following relation:

$$[\gamma^\mu(1-\gamma_5)]_{\rho\delta}\ [\gamma_\mu(1-\gamma_5)]_{\nu\beta} = -[\gamma^\mu(1-\gamma_5)]_{\rho\beta}\,[\gamma_\mu(1-\gamma_5)]_{\nu\delta}\,. \qquad \text{(A12.63)}$$

Clearly, analogous relations can be worked out for any product of the $\tilde{\Gamma}$-matrices. Results may be found in Section 2.2B of Marshak, Riazuddin and Ryan (1969).