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ON THE DEFINITION OF C^*-ALGEBRAS II

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0. Introduction. The theory of noncommutative involutive Banach algebras (briefly Banach *-algebras) owes its origin to Gelfand and Naimark, who proved in 1943 the fundamental representation theorem that a Banach *-algebra $\mathscr{B}$ with C^*-condition

$$(C^*) \quad \|a^*a\| = \|a\|^2 \quad \forall a \in \mathscr{B}$$

is *-isomorphic and isometric to a norm-closed self-adjoint subalgebra of all bounded operators on a suitable Hilbert space.

At the same time they conjectured that the C^*-condition can be replaced by the B^*-condition.

$$(B^*) \quad \|a^*a\| = \|a^*\|\,\|a\| \quad \forall a \in \mathscr{B}.$$

In other words any B^*-algebra is actually a C^*-algebra. This was shown by Glimm and Kadison [**5**] in 1960.

Further weakening of the axioms appeared in a paper [**2**] by Araki and Elliott in 1973 by proving that the C^*-condition and the B^*-condition also, if continuity on involution assumed, imply the submultiplicativity of a linear and complete norm on a *-algebra. They asked if it is enough to assume (C^*) and (B^*) only for normal elements and the continuity of $*$ in the second case. A recent survey of some developments is presented by Doran and Wichmann in [**4**].

The second named author proved in [**9**] that

$$(SC^*) \quad \|a^*a\| \leqq \|a\|^2 \quad \forall a \in \mathscr{B}$$

together with (C^*) for normal elements imply (C^*); in [**11**, **12**] that every C^*-seminorm is automatically submultiplicative. For further weakening ([**11**]) see Theorem 5. It was also claimed to prove ([**10**]) that continuity of the involution can be dropped with respect to the (B^*) assumption and that

$$(SB^*) \quad \|a^*a\| \leqq \|a^*\|\,\|a\| \quad \forall a \in \mathscr{B}$$

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together with (B^*) for normal elements are enough as well. However, G. A. Elliott has pointed out in his review an error in the proof in [**10**], namely on line 4 of page 212.

Our purpose is to give a complete proof of these statements in a rarely detailed manner so that this paper serves as a continuation of [**10**] without any reference to that. The ground of our treatment is [**11**] where further localization of these properties appeared, namely to commutative selfadjoint *-subalgebras, which are generated by one selfadjoint element, say $h = h^* \in B$, denoted by $\langle h \rangle$. Denote by $\mathcal{P}$ the complex polynomials in one variable and without constant term, thus

$$(LC^*) \quad \|a^*a\| = \|a\|^2 \quad \forall a = P(h), h = h^* \in \mathcal{B}, P \in \mathcal{P};$$

$$(LB^*) \quad \|a^*a\| = \|a^*\| \, \|a\| \quad \forall a = P(h), h = h^* \in \mathcal{B}, P \in \mathcal{P}$$

are the corresponding local (C^*) and local (B^*) properties of a norm (or seminorm) on a *-algebra $\mathcal{B}$. Note that a norm (or seminorm) on a *-algebra denotes always a linear norm (or seminorm) except its submultiplicativity is assumed separately, for example in case of a Banach (or C^*)-algebra. Moreover, we use [**3**] without any reference.

The remainder of this paper consists of five distinct sections. Section 1 is due to the first named author and contains a detailed analysis on the spectrum of a selfadjoint element h, actually that it is purely real, in a Banach *-algebra provided such a norm p exists for which

$$p(a^*)p(a) = r(a^*a)$$

holds for any a in $\langle h \rangle$, where r denotes the spectral radius (Theorem 1). Section 2 is a simple reformulation of results in [**2**] with some simplification in its proof (in Theorem 2).

Theorem 3 of Section 3 is taken from [**11**] and is a strengthened version of a statement included in [**2**] which serves as a ground for our main result obtained in Theorem 4 of Section 4. Section 5 is an application of Theorem 4 to the seminorm case and contains a simple counterexample for B^*-seminorms.

1. Hermiticity in a Banach *-algebra.

THEOREM 1. *Let h be a selfadjoint element in a Banach *-algebra $\mathcal{B}$ with spectral radius r. Assume there is a norm p on $\langle h \rangle$, the *-subalgebra generated by h in $\mathcal{B}$, such that*

(i) $\quad p(a^*)p(a) = r(a^*a) \quad \forall a \in \langle h \rangle.$

Then h has purely real spectrum, that is

$$\mathrm{Sp}(h) \subseteq \mathbf{R}.$$

The proof will consist of two different parts. Part I contains independent propositions, while in Part II we shall prove the statement utilizing the result of Part I.

In what follows we shall say that a set K in $\mathbf{C}$, the complex plane, is a *cross* if there is a real number s so that

$$K \subseteq \mathbf{R} \cup \{s + it : t \in \mathbf{R}\}.$$

A set K of $\mathbf{C}$ is said to be *symmetric* if it is stable under conjugation, that is $\bar{z} \in K \ \forall z \in K$.

Part I. Let K be throughout this part a symmetric non-void compact subset of the complex plain. Denote the customary sup-norm in $C(K)$ the complex valued continuous functions on K, by r. Define an involution $(*)$ on $C(K)$ by setting

$$f^*(z) = \overline{f(\bar{z})} \quad \forall z \in K$$

which is correct because of the symmetry of K and norm-preserving as well. Let

$$A = \{p_{|K} : p \in \mathscr{P}\}$$

be the *-subalgebra in $C(K)$ of the complex polynomials on K without constant term. Suppose further that a seminorm p is given on A with

$$(P1) \quad p(f^*)p(f) = r(f^*f) \quad \forall f \in A.$$

We shall prove that the existence of such a seminorm implies that the shape of K is very special.

PROPOSITION 1.1. *Let B be the norm-closure of A in $C(K)$ then p has a unique continuous extension to B, denoted by p too such that* $(P1)$ *remains valid and*

$$(P2) \quad p(h) = r(h) \quad \forall h = h^* \in B$$

$$(P3) \quad p(a) \leqq 2r(a) \quad \forall a \in B$$

will also hold.

The easy proof is omitted.

PROPOSITION 1.2. *K is a cross.*

Proof. Suppose the contrary. We shall show

$$p(f) + p(g) < p(f + g) \quad \text{for some } f, g \text{ in } B$$

contradicting the subadditivity of p. Denote by C (resp. β) the maximum of K of $|z|$ (resp. $\operatorname{Im} z$). Note that $C, \beta > 0$ because K is symmetric and is not a cross. Let $\alpha \in \mathbf{R}$ be such that $\alpha + i\beta \in K$ and denote $w_1 = \alpha + i\beta$, $w_2 = \bar{w}_1 \in K$, $m = |w_1|$.

LEMMA 1.3. *For any $n \in \mathbf{R}$ there are a, b in B such that*

(1) $r(a^*a),\ r(b^*b) \leqq C^2$

(2) $r(a) = r(b) > n$

(3) $|b(w_1)| = |b(w_2)| = m$

(4) $|a(w_1)| \geqq \frac{m}{C} r(a)$

(5) $|a(w_2)| < \frac{m}{2}.$

Proof. Put

$$a_t(z) = z \cdot \exp(-it(z - \alpha)),$$
$$b_t(z) = z \cdot \exp(-it(z - \alpha)^2),$$

where t is real. Then $a_{t|K}$, $b_{t|K}$ are in B for any t. Since K is not a cross there are real γ and $\delta \neq 0$ such that

$$\alpha \neq \gamma,\ 0 \neq u_1 = \gamma + i\delta \in K,$$
$$|b_t(u_1)| = |u_1| \exp(2t(\gamma - \alpha)\delta)$$

while

$$|b_t(\bar{u}_1)| = |u_1| \exp(-2t(\gamma - \alpha)\delta)$$

where $(\gamma - \alpha)\delta \neq 0$ and $\bar{u}_1 \in K$. Hence there is a $t \in \mathbf{R}$ with $r(b_{t|K}) > n$ and let $b = b_{t|K}$ with such a t. Since

$$|a_t(w_1)| = m \cdot \exp(t\beta),$$
$$|a_t(w_2)| = m \cdot \exp(-t\beta)$$

there is a real t with

$$|a_t(w_2)| < \frac{m}{2}, \quad r(a_{t|K}) > r(b).$$

With such a t, let

$$a = \frac{r(b)}{r(a_{t|K})} a_{t|K}.$$

It is easy to prove (1) - (5) for these a, b because of

$$r(a_{t|K}) \leqq C \exp(t\beta).$$

LEMMA 1.4. *Assume for an $a \in B$ that*

(6) $r(a^*a)^{1/2} \leqq C \leqq \frac{r(a)}{2}$

holds. Then we have

$$\min(p(a), p(a^*)) \leqq \frac{4C^2}{r(a)}. \tag{7}$$

Proof. Since p is seminorm $(P2)$ implies

$$p(a) + p(a^*) \geqq p(a + a^*) = r(a + a^*).$$

Choosing z in K with $r(a) = |a(z)|$ we have by (6)

$$|a^*(z)|r(a) = |a^*(z)||a(z)| = |(a^*a)(z)| \leqq C^2,$$

$$|a^*(z)| \leqq \frac{C^2}{r(a)} \leqq \frac{C}{2} \leqq \frac{r(a)}{4},$$

and thus

$$r(a + a^*) \geqq |a(z) + a^*(z)| \geqq |a(z)| - |a^*(z)|$$

$$\geqq r(a) - \frac{r(a)}{4} \geqq \frac{r(a)}{2}.$$

We have then

$$p(a) + p(a^*) \geqq \frac{r(a)}{2}, p(a^*)p(a) = r(a^*a) \leqq C^2$$

by using $(P1)$ and (6) too. Hence

$$\min(p(a), p(a^*)) \leqq \frac{C^2}{\max(p(a), p(a^*))} \leqq \frac{4C^2}{r(a)}$$

since

$$\max(p(a), p(a^*)) \geqq \frac{r(a)}{4}$$

follows from the first inequality.

To prove Proposition 1.2 let $a, b \in B$ be such that (1) - (5) hold with

$$n = 2 \cdot C + (10C)^3 \cdot m^{-2}.$$

Let further f (resp. g) be that from a and a^* (resp. b and b^*) for which p is less. Lemma 1.4 implies then

$$p(f) + p(g) \leqq 2 \cdot \frac{4C^2}{n} \leqq \frac{8m^2}{1000C} < \frac{m^2}{100C}.$$

On the other hand $(P1)$ and (2) - (5) give us

$$p(f + g)p(f^* + g^*) = r((f^* + g^*)(f + g))$$

$$\geqq |(f^* + g^*)(f + g)(w_1)|$$

$$\geqq \left(\frac{m}{C}r(f) - m\right)\left(m - \frac{m}{2}\right) \geqq \frac{m^2}{4C}r(f)$$

while $(P3)$ and (2) imply

$$p(f^* + g^*) \leqq 2r(f^* + g^*) \leqq 4r(f)$$

and thus

$$p(f + g) \geqq \frac{m^2}{16C} > \frac{m^2}{100C} > p(f) + p(g),$$

the desired contradiction follows. The proof of Proposition 1.2 is complete.

PROPOSITION 1.5. *If* $\mathrm{card}(K \setminus \mathbf{R}) = 2$ *then*

$$K \cap \mathbf{R} \subseteq \{0\}.$$

Proof. Suppose $K \setminus \mathbf{R} = \{w, \bar{w}\}$. Since $\mathbf{C} \setminus K$ is connected, by Runge's theorem there are polynomials P_k converging in $C(K)$ to $\frac{1}{w} \cdot 1_{\{w\}}$, where $1_{\{w\}}$ denotes the characteristic function of the one point set $\{w\}$. Hence $zP_k(z)$ converges in $C(K)$ to $1_{\{w\}}$, $1_{\{w\}}$ is in B.

Then by $(P1)$

$$0 = r(0) = r(1_{\{w\}}1^*_{\{w\}}) = p(1_{\{w\}})p(1^*_{\{w\}})$$

and hence one of the functions $1_{\{w\}}$ and $1^*_{\{w\}}$, say f, is such that $p(f) = 0$. This implies

$$p(f + g) = p(g) \quad \forall g \in \mathcal{B}.$$

Applying this to $g = f^*$ we infer by $(P2)$ that $p(f^*) = 1$. Let $h(z) \equiv z$ on K and

$$k = \frac{1}{3r(h)}(h - w1_{\{w\}} - \bar{w}1^*_{\{w\}}).$$

Then k is self-adjoint and $r(k) \leqq 1/3$. Further,

$$k \cdot f = k \cdot f^* = 0.$$

Thus

$$r((k + f)^*(k + f)) = r(k^2) = r(k)^2 \leqq \frac{1}{3} \cdot r(k).$$

On the other hand by the above observation

$$p(k + f) = p(k) = r(k)$$

while

$$p(k + f^*) \geqq p(f^*) - p(k) = 1 - r(k) > \frac{1}{3}.$$

Thus we infer by $(P1)$ that $r(k) = 0$. But it is equivalent to $K \cap \mathbf{R} \subseteq \{0\}$ (by the definition of K).

COROLLARY. 1.6. *If p is a norm and*

$$\operatorname{card}(K \setminus \mathbf{R}) \leqq 2$$

then $K \subseteq \mathbf{R}$.

Proof. Suppose to the contrary that

$$K \setminus \mathbf{R} = \{w, \bar{w}\}.$$

Then by Proposition 1.5, K is finite and therefore there is an f in A (namely $f = 1_{\{w\}}$) such that $f \neq 0, f^* \neq 0$ but $ff^* = 0$ contradicting $(P1)$ in case of a norm.

Part II. Observe that if $g \in \langle h \rangle$, $g^* = g$ then $\langle g \rangle \subseteq \langle h \rangle$ and therefore the conditions assumed for h (in the theorem) also hold for g. Thus the consequences of these conditions (formulated with h) remain true for g, too.

If $P \in \mathcal{P}$, we write

(8) $\quad P^*(z) = \overline{P(\bar{z})}.$

In other words if

$$P(z) = \sum_1^n a_i z^i$$

then

$$P^*(z) = \sum_1^n \bar{a}_i z^i.$$

Hence it is clear that $P(h)^* = P^*(h)$.

In each *-algebra $\mathrm{Sp}(a^*) = \overline{\mathrm{Sp}(a)}$ for any a; hence $\mathrm{Sp}(h)$ is symmetric. (i) easily implies that $p(g) = r(g)$ if $g^* = g \in \langle h \rangle$ and $p \leqq 2r$ on $\langle h \rangle$ because (*) is isometric with respect to r. Hence r is a norm on $\langle h \rangle$. Let

$$\phi : \langle h \rangle \mapsto C(\mathrm{Sp}(h)),\ \phi(P(h)) = P_{|\mathrm{Sp}(h)}.$$

This definition is correct, moreover ϕ is norm-preserving with respect to $(\langle h \rangle, r)$. Indeed, it follows from the well-known fact

$$\mathrm{Sp}(P(h)) = P(\mathrm{Sp}(h)).$$

Furthermore ϕ is a *-homomorphism onto

$$A = \{P_{|\mathrm{Sp}(h)}; P \in \mathscr{P}\}$$

(endowed with involution (8)).

Thus $p \circ \phi^{-1}$ is a norm on A satisfying ($P1$). Therefore Proposition 1.2 and Corollary 1.6 are available and we have

(9) $\mathrm{Sp}(h)$ is a cross

(10) if $\mathrm{card}(\mathrm{Sp}(h) \setminus \mathbf{R}) \leqq 2$ then $\mathrm{Sp}(h) \subseteq \mathbf{R}$.

Suppose that $\mathrm{Sp}(h) \subsetneqq \mathbf{R}$. Then by (9) and (10) there are w_1, $w_2 \in \mathrm{Sp}(h) \setminus \mathbf{R}$ so that $w_2 \neq w_1$, $w_2 \neq \bar{w}_1$, and $\mathrm{Re}\, w_1 = \mathrm{Re}\, w_2$. Then $\forall s \in \mathbf{R} \setminus \{0\}$

$$\mathrm{Re}(sw_1^2) \neq \mathrm{Re}(sw_2^2)$$

and if s is small enough then

$$w_1 + sw_1^2, \quad w_2 + sw_2^2 \in \mathbf{C} \setminus \mathbf{R}.$$

Thus $\mathrm{Sp}(h + sh^2)$ is not a cross with suitable real s, contradicting (9) (which is available to $g = h + sh^2$). Thus $\mathrm{Sp}(h) \subseteq \mathbf{R}$, and the proof of Theorem 1 is complete.

2. The commutative case.

THEOREM 2. *Let $(\mathscr{A}, r)$ be a commutative C^*-algebra, and let p be a seminorm on it satisfying*

($P1$) $\quad p(a^*)p(a) = r(a^*a) \quad \forall a \in \mathscr{A}.$

Then $p = r$.

Proof. It is easy to infer from ($P1$) that

($P2$) $\quad p(h) = r(h) \quad \forall h = h^* \in \mathscr{A}$

($P3$) $\quad p(a) \leqq 2r(a) \quad \forall a \in \mathscr{A}.$

We treat first the finite dimensional case. Consider $\mathbf{C}^n$ as a C^*-algebra $C(T)$, where $T = \{1, \dots, n\}$ is a discrete space. In this special case we write q (resp. s) instead of p (resp. r) and one bracket instead of double bracket, e.g.,

$$s(x_1, \dots, x_n) = r((x_1, \dots, x_n)) = \max\{|x_i|; i = 1, \dots, n\}.$$

Case 1. $\mathscr{A} = \mathbf{C}^2$. Let

$$D = \{z \in \mathbf{C}; |z| \leqq 1\} \quad \text{and}$$

$$f: D \mapsto \mathbf{R}^+, \quad f(z) = q(z, 1).$$

It is enough to prove that $f(z) = 1 \quad \forall z \in D$, because that implies

$$q(x, y) = s(x, y) \quad \text{for } |x| \leqq |y|$$

(since q is a seminorm) and it is enough by the symmetry. Let $0 \leqq \lambda \leqq 1$, $\mu = 1 - \lambda$, then

$$f(\lambda z_1 + \mu z_2) = q(\lambda z_1 + \mu z_2, \lambda + \mu)$$
$$\leqq \lambda f(z_1) + \mu f(z_2) \quad (z_1, z_2 \in D).$$

By $(P2)$ $q(1, 0) = s(1, 0) = 1$ and hence

$$q(z, 0) = |z| \quad \text{for every } z \in \mathbf{C};$$

thus

$$f(z_1) + f(z_2) = q(z_1, 1) + q(-z_2, -1)$$
$$\geqq q(z_1 - z_2, 0) = |z_1 - z_2|.$$

From $(P1)$ we have $f(z) \cdot f(\bar{z}) = 1$. Thus the following situation stands:

(1) f is a non-negative convex function on D
(2) $f(z_1) + f(z_2) \geqq |z_1 - z_2|$
(3) $f(\bar{z}) \cdot f(z) = 1$.

We will show that (1), (2), (3) imply $f \equiv 1$.

Step 1. $f(z) = 1$ if z is real. This is clear from (3) and the non-negativity of f.

Step 2. $f(z) \leqq 1 + 2(\text{Im } z)^2$ if $|\text{Im } z|$ is small (e.g. $|\text{Im } z| \leqq 1/2$ is enough). Let

$$\text{Im } z = b \quad \left(|b| \leqq \frac{1}{2}\right),$$

$$z_1 = -\sqrt{1 - b^2} + ib,$$

$$z_2 = \sqrt{1 - b^2} + ib.$$

Let $d = |1 - \bar{z}_1|$. Then by (2) and step 1,

$$f(\bar{z}_1) \geqq d - 1,$$

as well as

$$f(\bar{z}_2) \geqq |\bar{z}_2 + 1| - f(-1) = d - 1,$$

and hence by (3)

$$f(z_1), f(z_2) \leqq \frac{1}{d - 1}.$$

But z is a convex combination of z_1 and z_2, hence

$$f(z) \leqq \frac{1}{d-1}.$$

Thus our statement follows from

$$\frac{1}{d-1} \leqq 1 + 2b^2$$

which is true for small b.

Step 3. $f \geqq 1$. Suppose to the contrary that $f(w) < 1$ where $w = a + ib$. By step 1, $b \neq 0$. Then by the convexity and $f(a) = 1$ (step 1) we infer

$$f(a - i\lambda b) \geqq 1 + \lambda(1 - f(w)) \quad (\lambda \in [0, 1])$$

and this contradicts step 2 for small positive λ.

(3) and step 3 clearly imply $f = 1$.

Case 2. $\mathscr{A} = \mathbf{C}^n$. If

$$t = (t_1, \ldots, t_{n-1}) \in \mathbf{R}^{n-1} \quad \text{and}$$
$$r = (r_1, \ldots, r_{n-1}) \in [0, 1]^{n-1}$$

then we write

$$f(t, r) = q(r_1 \exp(it_1), \ldots, r_{n-1} \exp(it_{n-1}), 1).$$

It is enough to prove that $f = 1$. Since q is a seminorm f is convex in r. By $(P1)$ we have

$$f(t, r)f(-t, r) = 1.$$

It follows from these facts that the set

$$H_t = \{r \in [0, 1]^{n-1}; f(t, r) = 1\}$$

is convex. Indeed, if

$$f(t, u) = 1 = f(t, v),$$

then

$$f(-t, u) = f(-t, v) = 1$$

and hence if $w = \lambda t + \mu v$ then

$$f(-t, w) \leqq 1, \quad f(t, w) \leqq 1,$$

thus $f(t, w) = 1$.

We will show that if $r \in [0, 1]^{n-1}$ and $\lambda r \in H_t$ $(0 < \lambda \leqq 1)$ then $r \in H_t$. Suppose the contrary. Then one of $f(t, r)$ and $f(-t, r)$ is less than 1. On the other hand $f(t, 0) = 1$ for every t (by $(P2)$). Thus, by convexity, $f(t, \lambda r)$ or $f(-t, \lambda r)$ is less than 1 contradicting $\lambda r \in H_t$.

Let

$$a^j_k = \delta_{jk}, \quad a^j = (a^j_1, \ldots, a^j_{n-1}).$$

We know from Case 1 that $a^j \in H_t$ for every t, j. Now if $r = (r_1, \ldots, r_{n-1}) \in [0, 1]^{n-1}$ then

$$r = (1 - \sum r_j)0 + \sum_1^{n-1} r_j a^j$$

shows $r \in H_t$ if $\sum r_j \leqq 1$, and

$$\frac{1}{\sum r_j} r = \sum_1^{n-1} \frac{r_j}{\sum r_j} a^j$$

shows it if $\sum r_j > 1$.

General case. By the commutative Gelfand-Naimark theorem we consider $\mathscr{A}$ as $C_0(T)$, where T is locally compact T_2 space. Let $y_1, \ldots, y_n \in \mathscr{A}$ so that

$$y_j \geqq 0, \quad r(y_j) = 1, \quad r(\sum y_j) = 1.$$

Fixing them let

$$q(x_1, \ldots, x_n) = p(\sum x_j y_j)$$

where $x_j \in \mathbf{C}$. This q is a seminorm on $\mathbf{C}^n$. We assert that

$$s(x_1, \ldots, x_n) = r(\sum x_j y_j).$$

From the conditions about y_j's it follows that

$$\sum x_j y_j(t) \in \mathrm{co}(0, x_1, \ldots, x_n)$$

in $\mathbf{C}$ for every t and that there is t_k such that $y_j(t_k) = \delta_{jk}$ and hence

$$\sum x_j y_j(t_k) = x_k.$$

Thus

$$\begin{aligned} r(\sum x_j y_j) &= \max\{\, |x_j|, j = 1, \ldots, n\} \\ &= s(x_1, \ldots, x_n). \end{aligned}$$

Since y_j is self-adjoint,

$$\sum \bar{x}_j y_j = (\sum x_j y_j)^*.$$

Thus

$$\begin{aligned} q(x)q(x^*) &= p(\sum x_j y_j)p(\,(\sum x_j y_j)^*) \\ &= r(\sum x_j y_j(\sum x_j y_j)^*) \\ &= r(\sum x_j y_j)^2 = s(x)^2 = s(x^*x). \end{aligned}$$

Therefore by case 2, $q(x) = s(x)$, that is

$$p(\sum x_j y_j) = q(x) = s(x) = r(\sum x_j y_j).$$

Because of the continuity of p it is enough to show that $\forall \epsilon$ and $\forall w \in C_0(T)\ \exists x_j, y_j$ (with properties above) for which

$$r(w - \sum x_j y_j) \leqq \epsilon.$$

Let $G_0, \ldots, G_n$ be an open covering of the compact set $\mathrm{Sp}(w)$ in $\mathbf{C}$ so that

(a) $0 \in G_0,\ 0 \notin G_k$ for $k > 0$
(b) $\exists x_i \in G_i,\ x_0 = 0$ such that if $z \in G_i$ then $|z - x_i| \leqq \epsilon$
(c) $\forall k > 0\ \exists z_k \in \mathrm{Sp}(w)\quad z_k \notin \cup \{G_i;\ i \neq k\}$

(that is, each G_k is "necessary"). Let $f_0, \ldots, f_n$ be a partition of unity under the covering $G_0, \ldots, G_n$ on $\mathrm{Sp}(w)$, and

$$y_k = f_k \circ w \quad (k = 1, \ldots, n).$$

Then it is clear $y_k \in \mathscr{A}$ for $k > 0$ (from (a)) $y_k \geqq 0$,

$$\sum_1^n y_k \leqq 1,$$

and, for $k > 0$, $r(y_k) = 1$ (from (c)),

$$w - \sum_1^n x_k y_k = \left(z - \sum_1^n x_k f_k\right) \circ w$$

and

$$\left|z - \sum_1^n x_k f_k\right| = \left|z\left(\sum_0^n f_k\right) - \sum_0^n x_k f_k\right|$$

$$= \left|\sum_0^n f_k(z - x_k)\right| \leqq \epsilon$$

(from (b)).

3. Continuity of seminorms on C^*-algebras.

THEOREM 3. *Let $(\mathscr{B}, \|\cdot\|)$ be a C^*-algebra and let p be a seminorm on it satisfying*

(i) $\quad p(P(h)) \leqq \|P(h)\| \quad \forall h = h^* \in \mathscr{B},\ P \in \mathscr{P}.$

Then p is contractive on $\mathscr{B}$, that is

(ii) $\quad p(a) \leqq \|a\| \quad \forall a \in \mathscr{B}.$

Proof. Consider a new norm on $\mathscr{B}$ defined by

(iii) $\quad q(a) = \max(p(a), \|a\|) \quad \forall a \in \mathscr{B},$

and observe that

(i)′ $\quad q(P(h)) = \|P(h)\| \quad \forall h = h^* \in \mathscr{B}, P \in \mathscr{P}.$

Furthermore we can state

(1) $\quad \|a\| \leqq q(a) \leqq 2\|a\| \quad \forall a \in \mathscr{B},$

in other words that q and $\|\cdot\|$ are equivalent norms on $\mathscr{B}$. Our goal is to prove that q and $\|\cdot\|$ are the same. For this reason fix an element a in $\mathscr{B}$ and denote by $\mathscr{A}$ the C^*-subalgebra in $\mathscr{B}$ generated by a and a^*. Taking $(\mathscr{A}^{**}, \|\cdot\|)$ and $(\mathscr{A}^{**}, q)$ we get isomorphic Banach spaces being the norms $\|\cdot\|$ and q equivalent. Moreover, $(\mathscr{A}^{**}, \|\cdot\|)$ as the second dual of the C^*-algebra $(\mathscr{A}, \|\cdot\|)$ is a W^*-algebra and isomorphic to the weak closure $\overline{U(\mathscr{A})}^w$ of $U(\mathscr{A})$ where U is the universal $*$-representation of $\mathscr{A}$ on a Hilbert space, say $\mathscr{H}$. Here the weak operator topology on $\overline{U(\mathscr{A})}^w$ is identical with the weak* topology of $\mathscr{A}^{**}$ (see **[8]**).

On the other hand we shall consider q as a norm on $(\mathscr{A}^{**}, \|\cdot\|)$ equivalent to the ground norm $\|\cdot\|$. To prove the identity of these norms on $\mathscr{A}^{**}$ (and hence on $\mathscr{A}$) it is enough to show (see **[6]**) this:

(iv) $\quad q(\exp(ih)) \leqq 1 \quad \forall h = h^* \in \mathscr{A}^{**}.$

Assume first $0 \leqq h \in \mathscr{A}$ and consider

$$h_n = h + \frac{a^*a + aa^*}{n}$$

a sequence of strictly positive elements in $\mathscr{A}$ with a being a generating element. Hence U_{h_n} has a dense range in $\mathscr{H}$ **[1]** and $U_{h_n}^{1/m}$ converges as $m \to \infty$ strongly (hence weakly) to the identity operator (as well identity element 1 in $\mathscr{A}^{**}$) of the Hilbert space $\mathscr{H}$. Now $((\overline{\langle h_n \rangle})^{**}, \|\cdot\|)$ is identical with the weak* closure of $\overline{\langle h_n \rangle}$ in $\mathscr{A}^{**}$, whence 1 is in $(\overline{\langle h_n \rangle})^{**}$, the norms $\|\cdot\|$ and q are identical on which such that

$$q(\exp(ih_n)) = 1$$

follows for any $n = 1, 2, \ldots$. Since

$$q(h - h_n) = \frac{1}{n} q(a^*a + aa^*) \to 0 \text{ as } n \to \infty$$

we have

$$q(\exp(ih)) = \lim_{n \to \infty} q(\exp(ih_n)) = 1.$$

If $0 \leqq h \in \mathscr{A}^{**}$ we can choose a net h_j in $\mathscr{A}$ with strong limit h and such that

$$0 \leqq h_j \leqq \|h\| \cdot 1.$$

Thus $\exp(ih_j)$ tends to $\exp(ih)$ strongly, hence weakly and thus finally in the weak* topology.

Since $q(\exp(ih_j)) \leqq 1$ as before, we have

$$q(\exp(ih)) \leqq 1$$

also.

Finally in case $h = h^* \in \mathscr{A}^{**}$, we write

$$k = h + 2n\pi \cdot 1,$$

with such integer n for which

$$-2n\pi \leqq \min(t; t \in \mathrm{Sp}(h)).$$

Then we have

$$0 \leqq k \in \mathscr{A}^{**} \quad \text{and} \quad \exp(ih) = \exp(ik)$$

whence

$$q(\exp(ih)) = q(\exp(ik)) \leqq 1.$$

Proving (iv) we get $q(a) = \|a\|$ and thus the equality of q and $\|\cdot\|$ since a was an arbitrary chosen element in $\mathscr{B}$. The proof is complete.

COROLLARY 3.1. *Let p be a seminorm on the C^*-algebra $(\mathscr{B}, \|\cdot\|)$ satisfying* (i) *with equality for $P(h) \equiv h$ and*

(SC^*) $\quad p(a^*a) \leqq p(a)^2 \quad \forall a \in \mathscr{B}.$

Then $p = \|\cdot\|$.

Proof. Using Theorem 3 we have at once

$$p(a)^2 \leqq \|a\|^2 = \|a^*a\| = p(a^*a) \leqq p(a)^2 \quad \forall a \in \mathscr{B}$$

proving the statement.

4. The general case.

THEOREM 4. *Let $\mathscr{A}$ be a *-algebra, and let p be a norm on it satisfying*

(SB^*) $\quad p(a^*a) \leqq p(a^*) \cdot p(a) \quad \forall a \in \mathscr{A}$

(LB^*) $\quad p(a^*a) = p(a^*) \cdot p(a) \quad \forall a = P(h),\ h = h^*,\ P \in \mathscr{P}.$

Then $(\mathscr{A}, p)$ is a pre-C^*-algebra (that is, its completion is a C^*-algebra).

Proof. The following identity holds in a *-algebra:

$$(1) \qquad 4yx = (x + y^*)^*(x + y^*) + i(x + iy^*)^*(x + iy^*) - (x - y^*)^*(x - y^*) - i(x - iy^*)^*(x - iy^*).$$

This and (SB^*) and the subadditivity of p imply

(2) $$4p(yx) \leqq 4(p(x^*) + p(y))(p(x) + p(y^*)).$$

Writing

$$x = \left(p(v^*)^{1/2} + \frac{1}{n}\right)\left(p(v)^{1/2} + \frac{1}{n}\right)u,$$

$$y = \left(p(u^*)^{1/2} + \frac{1}{n}\right)\left(p(u)^{1/2} + \frac{1}{n}\right)v$$

in (2), we get by $n \to \infty$

(3) $$p(vu) \leqq (p(u^*)^{1/2}p(v^*)^{1/2} + p(u)^{1/2}p(v)^{1/2})^2.$$

Define a new norm on $\mathscr{A}$ by setting

(4) $$\|a\|: = 4 \max(p(a), p(a^*)) \quad \forall a \in \mathscr{A},$$

such that we infer

(5) $$\|ab\| \leqq \|a\| \cdot \|b\|, \|a^*\| = \|a\|, p(a) \leqq \|a\| \quad \forall a \in \mathscr{A}.$$

Let $\mathscr{B}$ be the completion of $\mathscr{A}$ with respect to $\|\cdot\|$. There are then unique continuous extensions of $*$ and p to $\mathscr{B}$. Denote these extensions by $*$ and p, too. This p is now a seminorm on $\mathscr{B}$. The multiplication, the involution and p are continuous on $(\mathscr{B}, \|\cdot\|)$. (SB^*), (LB^*) and (4) remain valid on $\mathscr{B}$; furthermore we can sharpen (LB^*) into

$(LB1)$ $$p(a^*a) = p(a^*)p(a) \quad \forall a \in \overline{\langle h \rangle}, h = h^* \in \mathscr{B}.$$

Let r be the spectral radius in $\mathscr{B}$. Since $\mathscr{B}$ is a Banach algebra, we have

(6) $$r(a) = \lim \|a^n\|^{1/n} \quad \forall a \in \mathscr{B}.$$

Consider a self-adjoint h in $\mathscr{B}$. Then by (LB^*)

$$p(h^{2^n}) = p(h^{2^{n-1}})^2 = \ldots = p(h)^{2^n},$$

and hence (by (4))

$$\|h^{2^n}\| = 4 \cdot p(h)^{2^n}$$

so that by (6) we have

(7) $$r(h) = p(h) \quad \forall h = h^* \in \mathscr{B}.$$

This and $(LB1)$ give

(8) $$p(a^*)p(a) = r(a^*a) \quad \text{if } a \in \overline{\langle h \rangle}, h = h^* \in \mathscr{B}.$$

If $h = h^* \in \mathscr{A}$ then p is a norm on $\langle h \rangle$ and hence by (8) we can apply Theorem 1, and infer that $\mathrm{Sp}(h) \subseteq \mathbf{R}$.

Therefore $r(\sin h) \leqq 1$, $r(\cos h - 1) \leqq 2$ via the functional calculus. But $\sin h$, $\cos h - 1$ are selfadjoint since $\mathscr{B}$ is a star-normed algebra and hence by (7), (4) we have

$$\|\sin h\| \leqq 4, \ \|\cos h - 1\| \leqq 8,$$

hence also

(9) $\quad \|e^{ih} - 1\| \leqq 12 \quad \forall h = h^* \in \mathscr{A}.$

Since the selfadjoint part of $\mathscr{A}$ is dense in that of $\mathscr{B}$, (9) remains valid for $\mathscr{B}$ too. This implies that $\|a\|_c = r(a^*a)^{1/2}$ $(a \in \mathscr{B})$ is a C^*-norm on $\mathscr{B}$, equivalent to $\|\cdot\|$ (see [7]). But

$$r(a^*a) = p(a^*a) \quad \forall a \in \mathscr{B}$$

by (7) and hence

(10) $\quad \|a\|_c = p(a^*a)^{1/2} \quad \forall a \in \mathscr{B}.$

If $h = h^* \in \mathscr{B}$, $(\overline{\langle h \rangle}, \|\cdot\|_c)$ is a commutative C^*-algebra and Theorem 2 is available by (8) and hence we have $p = \|\cdot\|_c$ on $\overline{\langle h \rangle}$. Thus Theorem 3 shows that

(11) $\quad p(a) \leqq \|a\|_c \quad \forall a \in \mathscr{B}.$

Then by (10), (11), (SB^*) we have

$$\|a\|_c^2 = p(a^*a) \leqq p(a^*)p(a) \leqq \|a^*\|_c\|a\|_c = \|a\|_c^2$$

that is

(12) $\quad p(a^*)p(a) = \|a^*\|_c\|a\|_c.$

This shows that $p(a) < \|a\|_c$ would imply $\|a^*\|_c = 0$, but

$$\|a^*\|_c = \|a\|_c > p(a) \geqq 0,$$

a contradiction.

5. Applications to seminorms.

THEOREM 5. *Let p be a seminorm on the *-algebra $\mathscr{B}$ satisfying*

(SC^*) $\quad p(a^*a) \leqq p(a)^2 \quad \forall a \in \mathscr{B}$

(LC^*) $\quad p(a^*a) = p(a)^2 \quad \forall a \in \langle h \rangle, h = h^* \in \mathscr{B}.$

Then p is a (submultiplicative) C^*-seminorm.

Proof. Define a new seminorm by

(1) $\quad q(a) = \max(p(a), p(a^*)) \quad \forall a \in \mathscr{B}$

we have at once (SC^*), (LC^*) for q because $p = q$ locally (see [**12**]) and moreover

(2) $\quad q(a^*) = q(a) \quad \forall a \in \mathcal{B}.$

The polarization identity (1) in Section 4 gives now

$$4q(yx) \leqq 4(q(y) + q(x^*))^2 = 4(q(x) + q(y))^2$$

such that replacing y (and x) with

$$\frac{a}{q(a) + \frac{1}{n}} \quad \left(\text{and } \frac{b}{q(b) + \frac{1}{n}}\right)$$

and tending with n to infinity, we get

(3) $\quad q(ab) \leqq 4q(a)q(b) \quad \forall a, b \in \mathcal{B}.$

The kernel

$$K_q = \{a \in \mathcal{B}: q(a) = 0\}$$

is now a *-ideal of $\mathcal{B}$ such that the quotient space $\mathcal{B}_q = \mathcal{B}/K_q$ is a *-algebra with norm

(4) $\quad \|a + K_q\| = q(a) \quad \forall a \in \mathcal{B}$

preserving (SC^*), (LC^*) and (2) such that (SB^*) and (LB^*) are trivially satisfied. Theorem 4 says that $\mathcal{B}_q$ is a pre-C^*-algebra and in other words that q has the C^*-property and is submultiplicative. But

$$p(a)^2 \leqq q(a)^2 = q(a^*a) = p(a^*a) \leqq p(a)^2 \quad \forall a \in \mathcal{B}$$

implies that $p = q$ and the theorem is proved.

THEOREM 6. *Let p be a seminorm on the *-algebra $\mathcal{B}$ satisfying*

(SB^*) $\quad p(a^*a) \leqq p(a^*)p(a) \quad \forall a \in \mathcal{B}$

(LB^*) $\quad p(a^*a) = p(a^*)p(a) \quad \forall a \in \langle h \rangle, h = h^* \in \mathcal{B}$

(NI) $\quad p(a) = 0$ *implies* $p(a^*) = 0 \quad \forall a \in \mathcal{B}.$

Then p is a (submultiplicative) C^*-seminorm.

Proof. The polarization identity gives us as before (3) in Section 4

(5) $\quad p(ab)^{1/2} \leqq p(a^*)^{1/2}p(b^*)^{1/2} + p(a)^{1/2}p(b)^{1/2}$

$\quad \forall a, b \in \mathcal{B}.$

It follows that $p(a) = 0$ or $p(b) = 0$ implies $p(ab) = 0$ and thus

$$K_p = \{a \in \mathcal{B}: p(a) = 0\}$$

the kernel of p is a *-ideal in $\mathcal{B}$. The quotient space $\mathcal{B}_p = \mathcal{B}/K_p$ has a norm by defining

(6) $\quad \|a + K_p\| = p(a) \quad \forall a \in \mathcal{B}$

which preserves (SB^*) and (LB^*). Theorem 4 shows that $\mathscr{B}_p$ is a pre-C^*-algebra, that is p is a (submultiplicative) C^*-seminorm.

The following example shows that Theorem 6 does not remain true without (NI).

Example 7. $\mathbf{C}^2$ is $*$-algebra with coordinate-wise multiplication and with involution defined by

$$(w, z)^* = (\bar{z}, \bar{w}) \quad \forall w, z \in \mathbf{C}. \tag{7}$$

Then

$$p(w, z) = |w| \quad \forall w, z \in \mathbf{C} \tag{8}$$

defines a multiplicative B^*-seminorm which is not a C^*-seminorm.

Proof. Since p is trivially multiplicative it is B^*-seminorm too, but

$$p((1, 0)^*(1, 0)) = p((0, 1)(1, 0)) = 0 \neq 1 = p((1, 0))^2$$

$$p((1, 0)^*) = p((0, 1)) = 0 \neq 1 = p((1, 0)).$$

Remark. The above example shows also that Theorem 1 can not be sharpened by writing "p is a seminorm" instead of "p is a norm". Indeed, the above example satisfies these weaker conditions, but $(-i, i)$ is a self-adjoint in it however its spectrum is not real.

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