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Towards Regulator Formulae for the K-Theory of Curves over Number Fields*

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Abstract. In this paper we study the group $K_{2n}^{(n+1)}(F)$ where F is the function field of a complete, smooth, geometrically irreducible curve C over a number field, assuming the Beilinson–Soulé conjecture on weights. In particular, we compute the Beilinson regulator on a subgroup of $K_{2n}^{(n+1)}(F)$, using the complexes constructed in *Compositio Math.* **96** (1995), pp. 197–247. We study the boundary map in the localization sequence for $n = 2$ and $n = 3$. We combine our results with results of Goncharov in order to obtain a complete description of the image of the regulator map on $K_4^{(3)}(C)$ and $K_6^{(4)}(C)$ (which have the same images as $K_4(C) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $K_6(C) \otimes_{\mathbb{Z}} \mathbb{Q}$, respectively), independent of any conjectures.

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1. Introduction

Let C be a smooth, proper, geometrically irreducible curve over a number field k. We want to consider $K_m(C)$ for $m \geqslant 0$, and would specifically like a more concrete description of this group rather than the abstract definition. The case $m = 0$ is classical, and we shall assume $m \geqslant 1$ from now on. For C as above, there are regulator maps from $K_m(C)$ to Deligne cohomology groups. More precisely, for C as above, $K_m(C) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{n=1}^{m+1} K_m^{(n)}(C)$, where the $K_m^{(n)}(C)$ are the weight n subspaces of $K_m(C) \otimes_{\mathbb{Z}} \mathbb{Q}$, which are eigenspaces for particular operators, the Adams operations. Let C_{an} be the analytic manifold associated to $C \otimes_{\mathbb{Q}} \mathbb{C}$. C_{an} is a disjoint union of $[k : \mathbb{Q}]$ Riemann surfaces of genus the genus of C. C_{an} has an involution σ coming from complex conjugation on $\mathbb{C}$. There are regulator maps to Deligne cohomology,

$$\mathrm{reg} : K_m^{(n)}(C) \to H_{\mathcal{D}}^{2n-m}(C_{\mathrm{an}}; \mathbb{R}(n)),$$

where $\mathbb{R}(n) = (2\pi i)^n \mathbb{R} \subset \mathbb{C}$. For $m \geqslant 1$, the only nonzero $K_m^{(1)}(C)$ is $K_1^{(1)}(C) \cong k^* \otimes_{\mathbb{Z}} \mathbb{Q}$. For $n \geqslant 2$, $H_{\mathcal{D}}^{2n-m}(C_{\mathrm{an}}; \mathbb{R}(n))$ is isomorphic to

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$H_{\mathrm{dR}}^{2n-m-1}(C_{\mathrm{an}}; \mathbb{R}(n-1))$. The most interesting case for $n \geqslant 2$ for the target is therefore the case $m = 2n-2$, so that we land in H_{dR}^1. Replacing n with $n+1$ to simplify notation somewhat in the rest of the paper, we are interested in the groups $K_{2n}^{(n+1)}(C)$ for $n \geqslant 1$. For $n = 1$, the group $K_2^{(2)}(C)$ can be described in terms of the exact localization sequence

$$\cdots \xrightarrow{\partial} \coprod_{x \in C^{(1)}} K_{2n}^{(n)}(k(x)) \to K_{2n}^{(n+1)}(C) \to K_{2n}^{(n+1)}(F) \xrightarrow{\partial} \coprod_{x \in C^{(1)}} K_{2n-1}^{(n)}(k(x)) \to \cdots,$$

where $C^{(1)}$ is the set of codimension 1 (i.e., closed) points of C. Because $K_{2n}(k(x))$ is torsion for $n > 0$ as $k(x)$ is a number field, the map $K_2^{(2)}(C) \to K_2^{(2)}(F)$ is injective. $K_2^{(2)}(F) = K_2(F) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $K_2(F)$ can be described completely in terms of generators and relations by Matsumoto's theorem. The boundary map ∂ is described in terms of the tame symbol. Similarly, because the map $K_{2n}^{(n+1)}(C) \to K_{2n}^{(n+1)}(F)$ is always injective for $n \geqslant 1$, one can try to describe $K_{2n}^{(n+1)}(C)$ by describing $K_{2n}^{(n+1)}(F)$ and computing the boundary map to the $K_{2n-1}^{(n)}(k(x))$ in the localization sequence. This is the approach taken in this paper.

In order to construct elements in $K_{2n}^{(n+1)}(F)$ we use (cohomological) complexes constructed in previous work by the author for $n+1 \geqslant 2$. Those complexes exist for any field F of characteristic zero. Let us write $F_{\mathbb{Q}}^*$ for $F^* \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a cohomological complex $\widetilde{\mathcal{M}}_{(n+1)}^{\bullet}(F)$ in degrees $1, \ldots, n+1$,

$$\widetilde{M}_{(n+1)}(F) \to \widetilde{M}_{(n)}(F) \otimes F_{\mathbb{Q}}^* \to \cdots \to \widetilde{M}_{(2)}(F) \otimes \bigwedge^{n-1} F_{\mathbb{Q}}^* \to \bigwedge^{n+1} F_{\mathbb{Q}}^*,$$

with $\widetilde{M}_{(k)}(F)$ a $\mathbb{Q}$-vector space generated by symbols $[f]_k$ for f in F^*. The maps are given by

$$\mathrm{d}([f]_l \otimes g_1 \wedge \cdots \wedge g_{n-l+1}) = [f]_{l-1} \otimes f \wedge g_1 \wedge \cdots \wedge g_{n-l+1}$$

for $l \neq 2$, and

$$\mathrm{d}\big([f]_2 \otimes g_1 \wedge \cdots \wedge g_{n-1}\big) = (1-f) \wedge f \wedge g_1 \wedge \cdots \wedge g_{n-1}$$

for $l = 2$, with $\mathrm{d}\big([1]_2 \otimes g_1 \wedge \cdots \wedge g_{n-1}\big) = 0$. There is a map

$$H^p(\widetilde{\mathcal{M}}_{(n+1)}^{\bullet}(F)) \to K_{2(n+1)-p}^{(n+1)}(F)$$

from the cohomology groups to the K-theory of the field, at least assuming a standard conjecture in K-theory that certain weight parts of K-groups vanish, see Section 2 for more details. The map is natural only up to sign, which will result in some statements up to sign below. We shall be mostly interested in the case $p = 2$, so we get a map to $K_{2n}^{(n+1)}(F)$. In this case the map corresponding to $K_4^{(3)}(F)$ exists without assumptions. For $K_6^{(4)}(F)$ we can also work without assumptions, but the situation is a bit more delicate. The complex $\widetilde{\mathcal{M}}_{(n+1)}^{\bullet}(F)$ is a quotient of a complex $\mathcal{M}_{(n+1)}^{\bullet}(F)$. Without assumptions, there is a map $H^2(\mathcal{M}_{(4)}^{\bullet}(F))$ to a quotient of

$K_6^{(4)}(F)$, and the natural map $H^2(\mathcal{M}^\bullet_{(4)}(F)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(F))$ is surjective. More details are given in Section 2.

For the sake of exposition, however, we state our results here usually assuming the Beilinson–Soulé conjecture, while referring to the more technical statements in the body of the paper for the corresponding statements without this assumption.

On the images of the maps to $K_{2n}^{(n+1)}(F)$ we compute the regulator map by pairing the 1-form that is the image of the regulator map in H^1_{dR} with a holomorphic 1-form and integrating over the analytic manifold C_{an}. The image of $K_{2n}^{(n+1)}(C)$ under the regulator map is contained in the subspace $H^1_{\mathrm{dR}}(C_{\mathrm{an}};\mathbb{R}(n))^+$ of $H^1_{\mathrm{dR}}(C_{\mathrm{an}};\mathbb{R}(n))$ consisting of the forms ψ such that $\psi\circ\sigma=\overline{\psi}$, where σ is the involution on C_{an}. Wedging such a form with a holomorphic 1-form on C_{an} and integrating gives a perfect pairing between $H^1_{\mathrm{dR}}(C_{\mathrm{an}};\mathbb{R}(n))^+$and holomorphic 1-forms on C_{an} satisfying $\omega\circ\sigma=\overline{\omega}$, so this completely describes the regulator map. The formula we find corresponding to the symbols $[f]_{n-1}\otimes g$ (see Theorem 3.5) has the form conjectured by Goncharov for such elements, see [4, §7].

The following is part of Theorem 3.5 below, to which we refer the reader for the statement without assumptions, see also Remark 3.7.

THEOREM 1. *Suppose the Beilinson–Soulé conjecture holds for fields of characteristic zero, so there is a map*

$$H^p(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)) \to K_{2n+2-p}^{(n+1)}(F).$$

Let ω be an element in $H^0(C_{\mathrm{an}};\Omega)$ such that $\omega\circ\sigma=\overline{\omega}$, where σ is the involution on C_{an} obtained by letting complex conjugation act on $\mathbb{C}$. Fix an orientation on C_{an} such that σ reverses the orientation. If $\sum_j c_j[f_j]_n\otimes g_j$ is an element of $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$, then the composition of maps

$$H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)) \longrightarrow K_{2n}^{(n+1)}(F) \xrightarrow{\mathrm{reg}} H^1_{\mathrm{dR}}(F;\mathbb{R}(n))^+ \xrightarrow{\int_{C_{\mathrm{an}}}\cdot\wedge\overline{\omega}} \mathbb{R}(1)$$

is given by mapping $[f]_n\otimes g$ to

$$\frac{\pm n2^n}{n+1}\int_{C_{\mathrm{an}}} \log|g|\log^{n-2}|f|(\log|1-f|\,\mathrm{d}\log|f| - \log|f|\,\mathrm{d}\log|1-f|)\wedge\overline{\omega}.$$

Also, on the image of $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$ in $K_{2n}^{(n+1)}(F)$ we compute an approximation to the boundary map ∂ in the above localization sequence, in terms of the complexes $\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)$ and $\widetilde{\mathcal{M}}^\bullet_{(n)}(k(x))$, where x is a closed point of C. This can be carried out completely for $n=2$ and $n=3$, corresponding to $K_4^{(3)}(F)$ and $K_6^{(4)}(F)$, and could probably be done for all $K_{2n}^{(n+1)}(F)$, but the combinatorics get rather complicated already for $K_6^{(4)}(F)$, so we restrict ourself to the cases $K_4^{(3)}(F)$ and $K_6^{(4)}(F)$. For the cases $n=2$ and 3 we prove the following result (which for the sake of exposition is again formulated assuming the Beilinson–Soulé conjecture, see Theorem 4.6 for the result for $n=3$ without this assumption). If x is a closed point of C,

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one can define a map $\delta_x : H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)) \to H^1(\widetilde{\mathcal{M}}^\bullet_{(n)}(k(x)))$ by mapping $[f]_2 \otimes g$ to $\mathrm{ord}_x(g)[f(x)]_2$, with the convention that $[0]_2$ and $[\infty]_2$ are zero.

THEOREM 2. *Let $\delta = \prod_x \delta_x$. Then, assuming the Beilinson–Soulé conjecture for $n = 3$, the diagram*

$$\begin{array}{ccc} H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)) & \longrightarrow & K^{(n+1)}_{2n}(F) \\ \downarrow{\scriptstyle n\delta} & & \downarrow{\scriptstyle \partial} \\ \coprod_{x\in C^{(1)}} H^1(\widetilde{\mathcal{M}}^\bullet_{(n)}(k(x))) & \longrightarrow & \coprod_{x\in C^{(1)}} K^{(n)}_{2n-1}(k(x)) \end{array}$$

commutes for $n = 2$ or 3, up to sign and up to $\partial\Big(K^{(n)}_{2n-1}(k) \cup F^*_{\mathbb{Q}}\Big)$ in the lower right hand corner.

It turns out that the results we obtain for the regulator maps on $K_4^{(3)}(F)$ and $K_6^{(4)}(F)$, as well as the boundary maps in those cases, can be very effectively combined with work of Goncharov. In order to state the results, we introduce the complex $\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C)$ as the total complex of the following double complex.

$$\begin{array}{ccccccc} \widetilde{M}_{(n+1)}(F) & \stackrel{\mathrm{d}}{\to} & \widetilde{M}_{(n)}(F) \otimes F^*_{\mathbb{Q}} & \stackrel{\mathrm{d}}{\to} & \widetilde{M}_{(n-1)}(F) \otimes \bigwedge^2 F^*_{\mathbb{Q}} & \stackrel{\mathrm{d}}{\to} & \cdots \\ \downarrow & & \delta\downarrow & & \delta\downarrow & & \\ 0 & \to & \coprod_x \widetilde{M}_{(n)}(k(x)) & \stackrel{\mathrm{d}}{\to} & \coprod_x \widetilde{M}_{(n-2)}(k(x)) \otimes k(x)^*_{\mathbb{Q}} & \stackrel{\mathrm{d}}{\to} & \cdots \end{array}$$

where both coboundaries have degree 1 and the total complex is a cohomological complex with $\widetilde{M}_{(n+1)}(F)$ in degree 1. The map δ in the diagram is determined by the requirement that $\delta_x([f]_l \otimes g_1 \wedge \cdots \wedge g_{n-l+1})$ equals $\mathrm{ord}_x(g_1)[f(x)]_l \otimes g_2(x) \wedge \cdots \wedge g_{n-l+1}(x)$ if none of the $g_i(x)$ is zero or ∞ for $i = 2, \ldots, n-l+1$, again putting $[0]_l = [\infty]_l = 0$. There is an obvious inclusion $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$, and Theorem 2 above implies that (assuming the Beilinson–Soulé conjecture) under the map $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)) \to K^{(n+1)}_{2n}(F)$ above, $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C))$ is mapped to $K^{(n+1)}_{2n}(C) + K^{(n)}_{2n-1}(k) \cup F^*_{\mathbb{Q}}$ inside $K^{(n+1)}_{2n}(F)$. It also turns out that $K^{(n+1)}_{2n}(C) + K^{(n)}_{2n-1}(k) \cup F^*_{\mathbb{Q}}$ is actually a direct sum $K^{(n+1)}_{2n}(C) \oplus K^{(n)}_{2n-1}(k) \cup F^*_{\mathbb{Q}}$, so that we can use the projection onto the first factor to get a map

$$H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C)) \to K^{(n+1)}_{2n}(C).$$

We call the composition of this map with the map

$$\mathrm{reg} : K^{(n+1)}_{2n}(C) \to H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(n))^+$$

the regulator on $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C))$. If we combine it with the map

$$\int_{C_{\mathrm{an}}} \cdot \wedge \overline{\omega} : H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(n))^+ \to \mathbb{R}(1)$$

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for an element ω of $H^0(C_{\mathrm{an}};\Omega)$ such that $\omega\circ\sigma=\overline{\varpi}$, then this composition is still given by the formulae in Theorem 1 above, see Proposition 3.4.

In [4, §6], Goncharov defined the following complexes $\Gamma(F,n)$ (in degree $1,\dots,n$), given by

$$B_n(F)\to B_{n-1}(F)\otimes F^*_{\mathbb{Q}}\to\dots\to B_2(F)\otimes\bigwedge^{n-2}F^*_{\mathbb{Q}}\to\bigwedge^{n}F^*_{\mathbb{Q}}$$

and, for each $x\in C^{(1)}$, $\Gamma(k(x),n-1)$ (in degrees $1,\dots,n-1$), given by

$$B_{n-1}(k(x))\to\dots\to B_2(k(x))\otimes\bigwedge^{n-3}k(x)^*_{\mathbb{Q}}\to\bigwedge^{n-1}k(x)^*_{\mathbb{Q}}$$

Here for any infinite field F, $B_k(F)$ is a $\mathbb{Q}$-vector space generated by elements $\{f\}_k$ with $f\in F\cup\{\infty\}$, modulo certain (inductively defined) relations, which include $\{0\}_k=\{\infty\}_k=0$. The maps are given by

$$\{f\}_l\otimes g_1\wedge\dots\wedge g_{n-l}\mapsto\{f\}_{l-1}\otimes f\wedge g_1\wedge\dots\wedge g_{n-l}$$

if $l>2$, and by

$$\{f\}_2\otimes g_1\wedge\dots\wedge g_{n-2}\mapsto(1-f)\wedge f\wedge g_1\wedge\dots\wedge g_{n-2}.$$

There is a map

$$\Gamma(F,n)\to\coprod_{x\in C^{(1)}}\Gamma(k(x),n-1)[-1]$$

given by

$$\{f\}_l\otimes g_1\wedge\dots\wedge g_{n-l}\mapsto\{f(x)\}_l\otimes\partial_{n-l,x}(g_1\wedge\dots\wedge g_{n-l})$$

with $\partial_{m,x}$ the unique map $\bigwedge^m F^*_{\mathbb{Q}}\to\bigwedge^{m-1}k(x)^*_{\mathbb{Q}}$ determined by

$$\pi_x\wedge u_1\wedge\dots\wedge u_{l-1}\mapsto u_1(x)\wedge\dots\wedge u_{l-1}(x)$$

$$u_1\wedge\dots\wedge u_l\mapsto 0$$

if all u_i are units at x and π_x is a uniformizer at x. $\Gamma(C,n)$ is defined as the mapping cone of the maps of complexes above. Goncharov also defines complexes $\Gamma'(F,n)$, $\Gamma'(k(x),n-1)$ for $n=3$ and 4, and constructs maps of the corresponding complexes as above. The complexes Γ' have the same shape as the complexes Γ with the same maps between them, but the $B_k(F)$ are replaced by $B'_k(F)$, generated by $F\cup\{\infty\}$, but with *explicit* relations between the generators. $\Gamma'(C,n)$ is defined as the mapping cone, defined by the corresponding Γ' complexes. Goncharov also constructs a map

$$K_{2n}(C)\to H^2(\Gamma'(C,n+1))$$

for $n=2$ or 3, and shows that the Beilinson regulator factors through this map. We summarize part of his results (especially [3, Theorems 5.5 and 5.9]) in a form suitable for our needs.

THEOREM 3 (Goncharov). *For $n = 2$ or 3, the regulator map*

$$K_{2n}^{(n+1)}(C) \to H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(n))^+$$

can be extended over the map $K_{2n}^{(n+1)}(C) \to H^2(\Gamma'(C, n+1))$ to a map $H^2(\Gamma'(C, n+1)) \to H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(n))^+$. For ω a holomorphic 1-form satisfying $\omega \circ \sigma = \overline{\omega}$, the composition

$$H^2(\Gamma'(C, n+1)) \longrightarrow H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(n))^+ \xrightarrow{\int_{C_{\mathrm{an}}} \cdot \wedge \overline{\omega}} \mathbb{R}(1)$$

is given by mapping $\{f\}_n \otimes g$ to

$$c_n \int_C \log|g| \log^{n-2}|f| (\log|1-f| \mathrm{d} \log|f| - \log|f| \mathrm{d} \log|1-f|) \wedge \overline{\omega}$$

for some nonzero rational constant c_n.

We compare the images under the regulator map of $K_{2n}^{(n+1)}(C)$, $H^2(\Gamma'(C, n+1))$ and $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C))$ by showing that there is a map $B_2'(F) \to \widetilde{M}_{(2)}(F)$ given by sending $\{x\}_2$ to $[x]_2$. This gives us maps

$$K_4^{(3)}(C) \to H^2(\Gamma'(C, 3)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)) \to K_4^{(3)}(C),$$

such that if we take the regulator to $H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(2))^+$ from all those groups, the resulting diagram commutes up to nonzero rational factors. For $K_6^{(4)}(C)$ the situation is again somewhat more complicated, but comparing the formulae for the regulators of elements in $H^2(\Gamma'(C, 4))$ and $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$ we can get the following result without assuming the Beilinson–Soulé conjecture, see Corollaries 5.5 and 5.8.

THEOREM 4.

(i) *For $n = 2$, the groups $K_4^{(3)}(C)$, $H^2(\Gamma'(C, 3))$ and $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$ have the same image in $H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(2))^+$ under the regulator map.*
(ii) *For $n = 3$, the groups $K_6^{(4)}(C)$, $H^2(\Gamma'(C, 4))$, and $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$ have the same image in $H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(3))^+$ under the regulator map.*

It should be stressed here that the results of Theorem 4 hold without any assumptions about the Beilinson–Soulé conjecture. Those results in the cases $p = 2$ and $n = 3$ or 4 strongly corroborate a conjecture of Goncharov that there is an isomorphism $H^2(\Gamma(C, n)) \cong K_{2n-p}^{(n)}(C)$ for all p and n, see [4, §6].

Such a description of the image of the regulator map is important for the following reason. The Beilinson conjectures for C as above predict the following.

(1) $K_{2n}^{(n+1)}(C)$ has $\mathbb{Q}$-dimension $r = \text{genus}\,(C)[k:\mathbb{Q}]$ for $n \geqslant 2$ (for $n = 1$, which we shall not study here, the $\mathbb{Q}$-dimension could be larger, depending on the reduction of the curve).

(2) For $n \geqslant 2$, the Beilinson regulator induces an isomorphism

$$K_{2n}^{(n+1)}(C) \otimes_{\mathbb{Q}} \mathbb{R} \to H_{\mathcal{D}}^2(C_{\text{an}}; \mathbb{R}(n+1))^+ \cong H_{\text{dR}}^1(C_{\text{an}}; \mathbb{R}(n))^+,$$

where the $+$ indicates the part of the cohomology where the involution formed by the combination of complex conjugation σ on the space C_{an} and complex conjugation on the coefficients $\mathbb{R}(n+1)$ (resp. $\mathbb{R}(n)$) acts as the identity.

(3) If $\alpha_1, \ldots, \alpha_r$ is a $\mathbb{Q}$-basis of $K_{2n}^{(n+1)}(C)$, let A be the matrix obtained by writing $\text{reg}(\alpha_1), \ldots, \text{reg}(\alpha_r)$ with respect to a basis of $H^1(C_{\text{an}}; \mathbb{Q}(n))^+ \cong \mathbb{Q}^r$. Note that $\det(A)$ is determined up to multiplication by an element in $\mathbb{Q}^*$. Assume that the L-function $L(C, s)$ of C can be analytically continued to the entire complex plane. Then $\det(A)/L^*(C, 1-n)$ is an element in $\mathbb{Q}^*$, where $L^*(C, z)$ is the first nonvanishing coefficient in the power series expansion of $L(C, s)$ around $s = z$.

Clearly, for those conjectures it is important to have a good description of the image of the K-theory under the regulator map, which is one of the aims of this paper. The regulator $\det(A)$ in (3) above is described in this paper in terms of a determinant of integrals of $\text{reg}(\alpha_i) \wedge \overline{\omega}_j$ over C_{an} for holomorphic 1-forms ω_j on C, and using the periods of the ω_j's, see Proposition 3.2 below.

As a concluding remark we mention that according to the Beilinson conjectures for C as above, one should have $K_{2n}(C) \otimes_{\mathbb{Z}} \mathbb{Q} = K_{2n}^{(n+1)}(C)$, and the regulator map should give an injection $K_{2n}(C) \otimes_{\mathbb{Z}} \mathbb{Q} = K_{2n}^{(n+1)}(C) \to H_{\text{dR}}^1(C_{\text{an}}; \mathbb{R}(n))^+$ for $n \geqslant 2$, so that conjecturally we get closer to a description of the even K-groups of C. On the other hand, the regulator vanishes on all $K_{2n}^{(j)}(C)$ with $j \neq n+1$ if $n \geqslant 1$. Hence Theorem 4 above also gives a complete description of the image of the regulator of $K_4(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K_6(C) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The paper is organized as follows. We review the description and the construction of the complexes from [7] in Section 2 below, state some of their properties, and take the opportunity to prove some loose ends needed in the rest of the paper. The cohomology groups of these complexes map to the K-theory of the field F, and in Section 3 we prove a version of Theorem 1 above. We also prove that there is a duality between certain holomorphic 1-forms and a subspace of H_{dR}^1 containing the image of the regulator map, given by pairing the two forms and integrating over the curve. Thus we get a description of the image of the regulator in terms of what we call regulator integrals. Section 4 contains the proof of Theorem 2 above (or rather of its incarnation without assumptions about the Beilinson–Soulé conjecture), and is by far the longest. We give most of the proof for general n, but somewhere along the road the combinatorics simply become too complicated and we restrict ourselves to the cases corresponding to $K_4^{(3)}(F)$ and $K_6^{(4)}(F)$. Even so, the reader may feel the proof is a bit tedious and messy. Yet the order of the acts is planned, and the end of the way inescapable. Finally, in Section 5 we relate our work with that of Goncharov as quoted in Theorem 3 above, proving Theorem 4 above. This way we obtain a complete combinatorical description of the image of $K_4^{(3)}(C)$ and $K_6^{(4)}(C)$ (and as explained above, of $K_4(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K_6(C) \otimes_{\mathbb{Z}} \mathbb{Q}$) under the regulator map, independent of any conjectures. We conclude the section by indicating how such results could be obtained for higher n, but all this would depend rather heavily on conjectures in algebraic K–theory.

Notation. The following notation will be fixed throughout the paper. k is a number field. C is a smooth, geometrically irreducible, proper curve over k. $F = k(C)$ is the field of rational functions on C. $C^{(1)}$ will denote the set of points of C of codimension one, i.e., the set of closed points of C.

In all sections except Section 2, n is a fixed integer at least equal to two. For an Abelian group A, $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. $\mathbb{Q}(m) = (2\pi i)^m \mathbb{Q} \subset \mathbb{C}$ and similarly for $\mathbb{R}(m)$. In the decomposition $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n)$ we let π_{n-1} denote the projection onto the $\mathbb{R}(n-1)$-part. If S is a subset of a vector space V, we shall mean by $\langle S \rangle$ (resp. $\langle S \rangle_{\mathbb{R}}$) the $\mathbb{Q}$ (resp. $\mathbb{R}$) subspace spanned by the elements of S.

Throughout the paper, in integrals and cohomology groups, we write simply C for C_{an}, which is the analytic manifold associated to $C \otimes_{\mathbb{Q}} \mathbb{C}$. Note that by our assumptions, this is a disjoint union of $[k : \mathbb{Q}]$ copies of a Riemann surface of genus the genus of C. Similarly, we shall write $H^1_{\text{dR}}(F)$ for $\varinjlim_{U \subset C} H^1_{\text{dR}}(U_{\text{an}})$, where the limit is over all Zariski open subsets of C.

2. Some Preliminary Results

This section contains a description of the complexes $\mathcal{M}^\bullet_{(n)}(F)$ and $\widetilde{\mathcal{M}}^\bullet_{(n)}(F)$, together with the maps $\varphi^p_{(n)}$ from their H^p to $K^{(n)}_{2n-p}(F)$ under suitable assumptions, as they were constructed in [7]. Apart from that, we also prove or state some results in this context that are useful for the rest of the paper.

We briefly recall the construction of the complexes $\mathcal{M}^\bullet_{(n)}(F)$ and $\widetilde{\mathcal{M}}^\bullet_{(n)}(F)$ in [7, Section 3], where F is a field of characteristic zero. Let $Y = \operatorname{Spec}(\mathrm{F})$, or more generally a Noetherian, quasi-projective separated regular scheme. For convenience, we shall refer to such a scheme as a reasonable regular scheme. Let t be the standard affine coordinate on $\mathbb{P}^1$, and let $X_Y = \mathbb{P}^1_Y \setminus \{t = 1\}$. In [7] a formalism of 'multi-relative' K-theory with weights is developed. To fix ideas, look at the exact sequence in relative K-theory

$$\ldots \to K^{(j)}_{m+1}(\{t = 0, \infty\}) \to K^{(j)}_m(X_Y; \{t = 0, \infty\}) \to K^{(j)}_m(X_Y) \to \ldots .$$

One has $K^{(j)}_n(X_Y) \cong K^{(j)}_n(Y)$ by the homotopy property for K-theory of a reasonable regular scheme, and the map $K^{(j)}_n(X_Y) \to K^{(j)}_n(\{t = 0, \infty\}) \cong K^{(j)}_n(Y)^{\oplus 2}$ is the diagonal map. From this, one gets isomorphisms $K^{(j)}_n(X_Y; \{t = 0, \infty\}) \cong K^{(j)}_{n+1}(Y)$. (We shall apply this isomorphism only in case Y is a Zariski open part of a smooth curve

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over a number field, the Spec of its function field, or the Spec of a number field, in which case all conditions are satisfied.) Iterating this idea one gets 'multi-relative' K-theory, by taking relativity step by step. Let t_i be the coordinate on the i-th copy of X in X^n. Writing $\square^n$ for $\{t_1 = 0, \infty\}, \ldots, \{t_n = 0, \infty\}$, we have a long exact sequence in relative K-theory

$$\cdots \to K^{(j)}_{m+1}(\{t_n = 0, \infty\}; \square^{n-1}) \to K^{(j)}_m(X^n_Y; \square^n) \to$$

$$K^{(j)}_m(X^n_Y; \square^{n-1}) \to K^{(j)}_m(\{t_n = 0, \infty\}; \square^{n-1}) \to \cdots$$

and as before it follows from the homotopy property for K-theory of a reasonable regular scheme Y that $K^{(j)}_m(X^n_Y; \square^n) \cong K^{(j)}_{m+1}(X^{n-1}_Y; \square^{n-1})$ for $m \geqslant 0$. Repeating this, we get $K^{(j)}_m(X^n_Y; \square^n) \cong K^{(j)}_{m+n}(Y)$ for $m \geqslant 0$. Note that there is no obvious choice of this isomorphism, which will result in statements up to sign below.

Let $Y = \operatorname{Spec}(F)$, but drop Y from the notation. Let $U \subset F^* \setminus \{1\}$ be finite. Write $X^k_{\mathrm{loc}} = X^k \setminus \{t_i = u_j,\ u_j \in U,\ i = 1, \ldots, k\}$. One has a fourth quadrant spectral sequence

$$E^{p,q}_1 = \coprod K^{(n-p)}_{-p-q}(X^{n-1-p}_{\mathrm{loc}}; \square^{n-1-p}) \Rightarrow$$

$$K^{(n)}_{-p-q}(X^{n-1}; \square^{n-1}) \cong K^{(n)}_{-p-q+n-1}(F) \tag{2.1}$$

which, if we write $K^{(j)}_{m,l}$ for $K^{(j)}_m(X^l_{\mathrm{loc}}; \square^l)$, looks like

$$\begin{array}{llll}
\vdots & \vdots & \vdots & \\
K^{(n)}_{-q-1,n-1} & \coprod K^{(n-1)}_{-q-2,n-2} & \coprod K^{(n-2)}_{-q-3,n-3} & \cdots \\
K^{(n)}_{-q,n-1} & \coprod K^{(n-1)}_{-q-1,n-2} & \coprod K^{(n-2)}_{-q-2,n-3} & \cdots \\
K^{(n)}_{-q+1,n-1} & \coprod K^{(n-1)}_{-q,n-2} & \coprod K^{(n-2)}_{-q-1,n-3} & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

Here the coproduct for X^{n-1-p}_{loc} corresponds to the codimension p hyperplanes given by p equations of type $t_i = u_i$, $u_i \in U$. If $K^{(j)}_m(Y) = 0$ for $2j \leqslant m$, $m > 0$, all the terms below the row with $q = -n$ vanish, [7, page 221]. Hence if we view this lowest row with the differential of the spectral sequence as a cohomological complex (depending on U)

$$C^\bullet_{(n)}: \quad K^{(n)}_{n,n-1} \to \coprod K^{(n-1)}_{n-1,n-2} \to \coprod K^{(n-2)}_{n-2,n-3} \to \ldots$$

in degrees 1 through n, we get a map

$$H^p(C^\bullet_{(n)}) \to K^{(n)}_{n-p+1}(X^{n-1}; \square^{n-1}) \cong K^{(n)}_{2n-p}(F).$$

This procedure works more generally for Y a reasonable regular scheme, and

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$U \subset \Gamma(Y, \mathcal{O}^*) \setminus \{1\}$ such that for all u_k and u_l in U, $u_k - u_l$ and $u_k - 1$ are invertible on Y if they are not identically zero. Let $G = \mathrm{Spec}(\mathbb{Q}[S, S^{-1}, (1-S)^{-1}])$. From the localization sequence

$$\cdots \to K_n^{(j-1)}(\mathrm{Spec}(\mathbb{Q}))^{\oplus 2} \to K_n^{(j)}(\mathbb{A}^1_{\mathbb{Q}}) \to K_n^{(j)}(G) \to K_{n-1}^{(j-1)}(\mathrm{Spec}\,(\mathbb{Q}))^{\oplus 2} \to \cdots$$

and the facts that $K_n^{(j)}(\mathbb{A}^1_{\mathbb{Q}}) \cong K_n^{(j)}(\mathbb{Q})$ and $K_n^{(j)}(\mathbb{Q}) = 0$ unless $n = 2j-1$ for $j \geqslant 1$, one gets that the conditions about weights above are satisfied for G. One can use the spectral sequence above, with G instead of Y, and $U = \{S\}$, to construct elements $[S]_n \in K_n^{(n)}(X^{n-1}_{G,\mathrm{loc}}; \square^{n-1})$ for $n \geqslant 2$, satisfying $\mathrm{d}[S]_n = \sum_{j=1}^{n-1}(-1)^j[S]_{n-1|t_j=S}$, where we put $[S]_1 = 1 - S$. With some more care, one sees that actually $[S]_n \in K_n^{(n)}(X^{n-1}_{\mathbb{G}_m,\mathrm{loc}}; \square^{n-1})$. Any $u \in F^* \setminus \{1\}$, or more generally any $u \in \Gamma(Y, \mathcal{O}^*)$ such that $1-u$ is also invertible on Y, yields a map $Y \to G$, and hence yields an element $[u]_n \in K_n^{(n)}(X^{n-1}_{Y,\mathrm{loc}}; \square^{n-1})$ by pulling back, with boundary $\mathrm{d}[u]_n = \sum_{j=1}^{n-1}(-1)^j[u]_{n-1|t_j=u}$ in $C^\bullet_{(n)}$.

We now return to the case $Y = \mathrm{Spec}(F)$, $U \subset F^* \setminus \{1\}$ finite, X^n_{loc} as before. Write $K_{(p)}$ for $K_p^{(p)}(X^{p-1}_{Y,\mathrm{loc}}; \square^{p-1})$. For the construction of $\widetilde{\mathcal{M}}^\bullet_{(n)}(F)$ one starts with the complex $C^\bullet_{(n)}$ (starting in degree 1)

$$K_{(n)} \to \left(\coprod_U K_{(n-1)}\right)^{\oplus\binom{n-1}{1}} \to \left(\coprod_{U^2} K_{(n-2)}\right)^{\oplus\binom{n-1}{2}} \to \left(\coprod_{U^{n-1}} K_{(1)}\right)^{\oplus\binom{n-1}{n-1}}.$$

The $\oplus\binom{n-1}{p}$ here corresponds to the number of ways of putting p of the coordinate t_j to a constant in U. For any $u \in U$, we have $[u]_n \in K_n^{(n)}(X^{n-1}_{\mathrm{loc}}; \square^{n-1})$. The element $[u]_2$ has boundary $(1-u)^{-1}_{|t=u}$, and for $n \geqslant 3$ $[u]_n$ has boundary $\sum_{j=1}^{n-1}(-1)^j[u]_{n-1|t_j=u}$. Moreover, $C^\bullet_{(n)}$ carries an action of S_{n-1} by permuting the coordinates, and $[u]_n$ is in fact in the alternating part for this action. Let $(1+I)^* = K_1^{(1)}(X_{\mathrm{loc}}; \square)$, which can be described more explicitly as

$$\left\{F(t) = \prod_i (t - x_i)^{n_i}(t-1)^{-n_i} \text{ such that } x_i \in U \text{ and } \prod_i x_i^{n_i} = 1\right\}_{\mathbb{Q}}.$$

There are $m-1$ cup products

$$(1+I)^* \cup K_{m-1}^{(m-1)}(X^{m-2}_{\mathrm{loc}}; \square^{m-2}) \to K_m^{(m)}(X^{m-1}_{\mathrm{loc}}; \square^{m-1})$$

depending on which of the coordinates on X^{m-1}_{loc} we use for the $(1+I)^*$-factor. We let $(1+I)^* \tilde{\cup} K_{m-1}^{(m-1)}(X^{m-2}_{\mathrm{loc}}; \square^{m-2})$ denote the span of the images of all possibilities. Define

$$\mathrm{symb}_2 = \langle [u]_2 \rangle + (1+I)^* \cup F^*_{\mathbb{Q}} \subset K_2^{(2)}(X_{\mathrm{loc}}; \square)$$

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and for $n \geqslant 3$

$$\mathrm{symb}_n = \langle [u]_n \rangle + (1+I)^* \tilde{\cup}\, \mathrm{symb}_{n-1} \subset K_n^{(n)}(X_{\mathrm{loc}}^{n-1}; \square^{n-1}).$$

We get a subcomplex $C^{\bullet}_{(n),\log}$ of $C^{\bullet}_{(n)}$,

$$\mathrm{symb}_n \to \Big(\coprod_{U'} \mathrm{symb}_{n-1}\Big)^{\oplus\binom{n-1}{1}} \to \Big(\coprod_{U'^2} \mathrm{symb}_{n-2}\Big)^{\oplus\binom{n-1}{2}} \to \Big(\coprod_{U'^{n-1}} F^*_{\mathbb{Q}}\Big)^{\oplus\binom{n-1}{n-1}}.$$

The subcomplex $J^{\bullet}_{(n)}$ of $C^{\bullet}_{(n),\log}$ given by

$$(1+I)^* \tilde{\cup}\, \mathrm{symb}_{n-1} \to \mathrm{d}(\ldots) + \Big(\coprod (1+I)^* \tilde{\cup}\, \mathrm{symb}_{n-2}\Big)^{\oplus\binom{n-1}{1}} \to$$

$$\mathrm{d}(\ldots) + \ldots \to \cdots \to \mathrm{d}(\ldots) + \Big(\coprod (1+I)^* \tilde{\cup} F^*_{\mathbb{Q}}\Big)^{\oplus\binom{n-1}{n-2}} \to \mathrm{d}(\ldots)$$

is acyclic, and we can form the quotient complex $C^{\bullet}_{(n),\log}/J^{\bullet}_{(n)}$. Because S_{n-1} acts on $C^{\bullet}_{(n),\log}$ and $J^{\bullet}_{(n)}$ is stable under the action, we can take the alternating part of this quotient complex, and we get the complex

$$\mathcal{M}^{\bullet}_{(n)}(F): \quad M_{(n)} \to M_{(n-1)} \otimes \langle U \rangle \to \cdots$$
$$\cdots \to M_{(2)} \otimes \bigwedge^{n-2} \langle U \rangle \to F^*_{\mathbb{Q}} \otimes \bigwedge^{n-1} \langle U \rangle,$$

where $\langle U \rangle$ is the (multiplicative) subspace of $F^*_{\mathbb{Q}}$ spanned by U, and

$$M_{(l)} = \frac{\mathrm{symb}_l}{(1+I)^* \tilde{\cup}\, \mathrm{symb}_{l-1}}.$$

(In [7] and [8] we wrote the factors in the tensor product the other way round. We change this notation here to conform with the notation used by Goncharov.) Finally, by taking direct limits over U we get the complex

$$\mathcal{M}^{\bullet}_{(n)}(F): \quad M_{(n)}(F) \to M_{(n-1)}(F) \otimes F^*_{\mathbb{Q}} \to \cdots$$
$$\cdots \to M_{(2)}(F) \otimes \bigwedge^{n-2} F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \bigwedge^{n-1} F^*_{\mathbb{Q}}.$$

So $M_l(F)$ is generated by symbols $[f]_l$ with $f \in F^* \setminus \{1\}$, and the differential is given by

$$\mathrm{d}([f]_l \otimes g_1 \wedge \cdots \wedge g_{n-l}) = [f]_{l-1} \otimes f \wedge g_1 \wedge \cdots \wedge g_{n-l}$$

if $l \geqslant 3$ and

$$(1-f) \otimes f \wedge g_1 \wedge \cdots \wedge g_{n-l}$$

if $l = 2$. The symbol $[1]_l$ also exists, with the relation $[1]_l = 2^{l-1}([1]_l + [-1]_l)$, see [7, Proposition 6.1]. In particular $\mathrm{d}[1]_l = 0$ for all $k \geqslant 2$. By construction, if the Beilinson–Soulé conjecture holds for F, there are maps

$$\varphi^p_{(n)} : H^p(\mathcal{M}^\bullet_{(n)}(F)) \to K^{(p)}_{2n-p}(F)$$

as the composition of

$$H^p(\mathcal{M}^\bullet_{(n)}(F)) \xleftarrow{\simeq} H^p(C^\bullet_{(n),\log}(F)^{\mathrm{alt}})$$

$$\to H^p(C^\bullet_{(n),\log}(F)) \to H^p(C^\bullet_{(n)}(F)) \to K^{(p)}_{2n-p}(F).$$

Finally, the complex $\widetilde{\mathcal{M}}^\bullet_{(n)}(F)$ is obtained by quotienting out the complex $\mathcal{M}^\bullet_{(n)}(F)$ by the subcomplex

$$\begin{aligned} &\langle [u]_n + (-1)^n [1/u]_n \rangle \to \langle [u]_{n-1} + (-1)^{n-1}[1/u]_{n-1} \rangle \otimes F^*_{\mathbb{Q}} \to \cdots \\ &\cdots \to \langle [u]_2 + [1/u]_2 \rangle \otimes \bigwedge^{n-2} F^*_{\mathbb{Q}} \to \mathrm{d}(\ldots). \end{aligned} \tag{2.2}$$

We get the complex $\widetilde{\mathcal{M}}^\bullet_{(n)}(F)$

$$\widetilde{M}_{(n)}(F) \to \widetilde{M}_{(n-1)}(F) \otimes F^*_{\mathbb{Q}} \to \cdots \to \widetilde{M}_{(2)}(F) \otimes \bigwedge^{n-2} F^*_{\mathbb{Q}} \to \bigwedge^{n} F^*_{\mathbb{Q}}.$$

where $\widetilde{M}_{(l)}(F) = M_{(l)}(F)/\langle [u]_l + (-1)^l [1/u]_l \rangle$. The subcomplex (2.2) is acyclic in degrees $n-1$ and n ([7, Remark 3.23]) and is acyclic everywhere if the Beilinson–Soulé conjecture is true (not just for F but for more schemes, see [7, Proposition 3.20]). Note that now the differential at the $(n-1)$-th place is given by

$$\mathrm{d}\big([f]_2 \otimes g_1 \wedge \cdots \wedge g_n\big) = (1-f) \wedge f \wedge g_1 \wedge \cdots \wedge g_n$$

with the other differentials unchanged. If the Beilinson–Soulé conjecture holds more generally, we therefore get a map

$$\tilde{\varphi}^p_{(n)} : H^p(\widetilde{\mathcal{M}}^\bullet_{(n)}(F)) \to K^{(p)}_{2n-p}(F)$$

as the composition of

$$\begin{aligned} &H^p(\widetilde{\mathcal{M}}^\bullet_{(n)}(F)) \xleftarrow{\simeq} H^p(\mathcal{M}^\bullet_{(n)}(F)) \xleftarrow{\simeq} \\ &\quad H^p(C^\bullet_{(n),\log}(F)^{\mathrm{alt}}) \to H^p(C^\bullet_{(n)}(F)) \to K^{(p)}_{2n-p}(F) \end{aligned}$$

Here the leftmost arrow is an isomorphism if the Beilinson–Soulé conjecture is true in general, and the rightmost arrow exists if the Beilinson–Soulé conjecture is true for the K-theory of F. By construction, all arrows from left to right are injective for $p = 1$, if they exist.

The reader may check that, if the Beilinson–Soulé conjecture is true in general, then for an element α in $H^p(\mathcal{M}^\bullet_{(n)}(F))$ (resp. $H^2(\widetilde{\mathcal{M}}^\bullet_{(n)}(F))$), $\varphi^p_{(n)}(\alpha)$ (resp. $\tilde{\varphi}^p_{(n)}(\alpha)$) nat-

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urally lives in $K^{(n)}_{2n-p}(U)$ for U some Zariski open subset of a reasonable regular scheme Y with function field F. This is because the lift of such an element will involve only finitely many elements in F, and the spectral sequence (2.1) will involve only finitely many $t_i = u_j$'s. But then this spectral sequence exists for a suitable open part of Y as well, by leaving out the closed part where $u_i = u_j$ for all i,j such that $u_i \neq u_j$. Moreover, the Beilison–Soulé conjecture implies that the localization map $K^{(n)}_{2n-1}(U) \to K^{(n)}_{2n-1}(F)$ is an injection, which allows one to check that the corresponding map of the complex for the Zariski open part U to the corresponding complex for F is an injection, see [7, Remark 3.17]. We shall only apply this to the case that U is a Zariski open part of the curve C, in which case the injection above is guaranteed by a localization sequence

$$\cdots \coprod_x K^{(n-1)}_{2n-1}(k(x)) \to K^{(n)}_{2n-1}(U) \to K^{(n)}_{2n-1}(F) \to \cdots$$

because $K^{(n-1)}_{2n-1}(k(x)) = 0$ as $k(x)$ is a number field.

In this paper we shall be mainly interested in the case $p = 2$ and $n = 4$, i.e., the target is $K^{(4)}_6(F)$. The leftmost arrow here is a surjection without any assumptions because of the acyclicity of the complex (2.2) in degree 3. The rightmost arrow exists to a quotient $K^{(4)}_6(F)/N$, which is as follows. In the spectral sequence (2.1) all higher differentials leaving $E^{1,-4}_2$ are zero, as they land in $K^{(1)}_2(F)$'s or outside the range of the spectral sequence. So $E^{1,-4}_2 = E^{-1,4}_\infty$ and we get a map $H^2(C^\bullet_{(4)}(F)) = E^{-1,4}_\infty$, a subquotient of $K^{(4)}_6(F)$. In order to determine this more precisely, note that we have a long exact localization sequence

$$\cdots \to \coprod K^{(1)}_3(F) \to K^{(2)}_3(X;\square) \to K^{(2)}_3(X_{\mathrm{loc}};\square) \to \coprod K^{(1)}_2(F) \to \cdots.$$

As $K^{(1)}_3(F)$ and $K^{(1)}_2(F)$ are both zero, we get

$$K^{(2)}_3(X_{\mathrm{loc}};\square) \cong K^{(2)}_3(X;\square) \cong K^{(2)}_4(F).$$

Therefore $E^{-1,4}_\infty \subset K^{(4)}_6(F)/N$ with N generated by $K^{(2)}_4(F) \cup K^{(2)}_2(F)$, and we get a map

$$H^2(\mathcal{M}^\bullet_{(4)}(F)) \to E^{1,-4}_\infty \to K^{(4)}_6(F)/N$$

which does not depend on any assumptions.

In Proposition 4.1 below, we shall introduce maps $\delta = \prod \delta_x$ with

$$\delta_x : \widetilde{\mathcal{M}}^\bullet_{(n)}(F) \to \widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x))[1]$$

given by

$$\delta_x([f]_{n-l} \otimes g_1 \wedge \cdots \wedge g_l) = \mathrm{sp}_{n-l,x}([f]_{n-l}) \otimes \partial_{l,x}(g_1 \wedge \cdots \wedge g_l)$$

in degrees 1 through $n-1$, and by the map $-\partial_{n,x}$ in degree n, where $\mathrm{sp}_{n-l,x}([f]_{n-l}) = [f(x)]_{n-l}$ if $f(x) \neq 0$ or ∞, 0 otherwise, and $\partial_{l,x}$ the unique map from

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$\bigwedge^l F^*_{\mathbb{Q}}$ to $\bigwedge^{l-1} k(x)^*_{\mathbb{Q}}$ determined by

$$\pi_x \wedge u_1 \wedge \cdots \wedge u_{l-1} \longmapsto u_1(x) \wedge \cdots \wedge u_{l-1}(x)$$

$$u_1 \wedge \cdots \wedge u_l \longmapsto 0$$

if all u_i are units at x and π_x is a uniformizer at x. This map obviously gives rise to a map $\widetilde{\mathcal{M}}^\bullet_{(n)}(F) \to \widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x))[1]$ by composition with the natural projection $\mathcal{M}^\bullet_{(n)}(F) \to \widetilde{\mathcal{M}}^\bullet_{(n)}(F)$. Following Goncharov ([4, §6]), we introduce the complexes $\mathcal{M}^\bullet_{(n+1)}(C)$ and $\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C)$, defined to be the total complexes of

$$\begin{array}{ccccccc} M_{(n+1)}(F) & \xrightarrow{\mathrm{d}} & M_{(n)}(F) \otimes F^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} & M_{(n-1)}(F) \otimes \bigwedge^2 F^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} \cdots \\ \downarrow & & \delta \downarrow & & \delta \downarrow & \\ 0 & \to & \coprod \widetilde{M}_{(n)}(k(x)) & \xrightarrow{\mathrm{d}} & \coprod \widetilde{M}_{(n-1)}(k(x)) \otimes k(x)^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} \cdots \end{array}$$

and

$$\begin{array}{ccccccc} \widetilde{M}_{(n+1)}(F) & \xrightarrow{\mathrm{d}} & \widetilde{M}_{(n)}(F) \otimes F^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} & \widetilde{M}_{(n-1)}(F) \otimes \bigwedge^2 F^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} \cdots \\ \downarrow & & \delta \downarrow & & \delta \downarrow & \\ 0 & \to & \coprod \widetilde{M}_{(n)}(k(x)) & \xrightarrow{\mathrm{d}} & \coprod \widetilde{M}_{(n-1)}(k(x)) \otimes k(x)^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} \cdots, \end{array}$$

where both coboundaries have degree 1 and the total complexes are cohomological complexes with $M^\bullet_{(n+1)}(F)$ and $\widetilde{M}^\bullet_{(n+1)}(F)$ in degree 1. There are obvious inclusions of $H^2(\mathcal{M}^\bullet_{(n+1)}(C))$ into $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$, of $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C))$ into $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$, so that the maps $\varphi^2_{(n+1)}$ (resp. $\tilde{\varphi}^2_{(n+1)}$) obviously extend to maps on the cohomology of those complexes.

In [7] regulator maps

$$K_p^{(q)}(X^n_{Y,\mathrm{loc}}; \square^n) \to H^{2q-p}_{\mathcal{D}}(X^n_{Y,\mathrm{loc}}; \square^n; \mathbb{R}(q))$$

to relative Deligne cohomology were defined. We recall that the Deligne cohomology group $H^n_{\mathcal{D}}(X; E; \mathbb{R}(q))$ can be described as the quotient

$$\frac{\left\{ \begin{array}{l} (\omega_n, s_n) \text{ with } \omega_n \text{ in } F^q(D)^n,\ s_n \text{ in } j_* S^{n-1}_X(q-1) \text{ such} \\ \text{that } \omega_{n|E} \equiv 0,\ s_{n|E} \equiv 0 \text{ and } \mathrm{d}s_n = \pi_{q-1}\omega_n \end{array} \right\}}{\left\{ \begin{array}{l} (\mathrm{d}\omega_{n-1}, \pi_{q-1}\omega_{n-1} - \mathrm{d}s_{n-1}) \text{ with } \omega_{n-1} \text{ in } F^q(D)^{n-1}, \\ s_{n-1} \text{ in } j_* S^{n-2}_X(q-1) \text{ such that } \omega_{n-1|E} \equiv 0 \text{ and} \\ s_{n-1|E} \equiv 0 \end{array} \right\}}.$$

(See [7, p. 218].) Here the notation means the following. We write X etc. for the underlying topological complex manifold consisting of the closed points of $X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{C})$. $\overline{X}$ is a compactification of X with complement D such that D and $D \cup E$ are a system of divisors with normal crossings. j is the imbedding of X into $\overline{X}$. $S^\bullet_X(q)$ is the complex of $\mathbb{R}(q)$-valued C^∞-forms on X, $F^q(D)^\bullet$ the complex of $\mathbb{C}$-valued C^∞-forms on $\overline{X}$ of type (p, r) with $p \geqslant q$ and with logarithmic poles along D. (So locally on $\overline{U} \subset \overline{X}$ an element in $F^q(D)^n$ is a sum of elements of the form $\varphi \wedge \psi$ with $\varphi \in \Omega^{\bullet}_{\overline{U}}(D \cap \overline{U})$ of degree $p \geqslant q$, and $\psi \in C^{0,n-p}(\overline{U})$.) Note that if $q > \dim X$, we get a natural isomorphism $H^n_{\mathcal{D}}(\mathbb{R}(q)) \cong H^{n-1}_{\mathrm{dR}}(\mathbb{R}(q-1))$.

The regulator lands in the invariant (or plus) part of Deligne cohomology with respect to the involution given by the combined action of complex conjugation on the underlying topological space (through the action on $\mathbb{C}$ in $X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{C})$) and on the coefficients $\mathbb{R}(q) \subset \mathbb{C}$. This involution acts similarly on $H^{n-1}_{\mathrm{dR}}(\mathbb{R}(q-1))$, and the plus-space in Deligne cohomology is isomorphic to the plus-space in H_{dR} if $q > \dim X$.

The regulator of a cup product in K-theory is given by the cup products of the regulators, (see [7, (22) and (40)], but (40) is flawed by typographical errors). For (ω_p, s_p) in $H^p_{\mathcal{D}}(\mathbb{R}(k))$, (ω_q, s_q) in $H^q_{\mathcal{D}}(\mathbb{R}(l))$, we get that in $H^{p+q}_{\mathcal{D}}(\mathbb{R}(k+l))$

$$(\omega_p, s_p) \cup (\omega_q, s_q) = (\omega_p \wedge \omega_q, s_p \wedge \pi_l \omega_q + (-1)^p (\pi_k \omega_p) \wedge s_q) \tag{2.3}$$

As for the regulator of $[S]_n$, it is given by (ω_n, ϵ_n), with

$$\omega_n = (-1)^{n-1} \mathrm{d}\log \frac{t_1 - S}{t_1 - 1} \wedge \cdots \wedge \mathrm{d}\log \frac{t_{n-1} - S}{t_{n-1} - 1} \wedge \mathrm{d}\log(1 - S). \tag{2.4}$$

Here ϵ_n is an $\mathbb{R}(n-1)$-valued $(n-1)$-form such that $\mathrm{d}\epsilon_n = \pi_{n-1}\omega_n$. (Unfortunately the signs in equation (41) in [7] were wrong, so the formula for ω_{p+1} on page 237 needs a sign $(-1)^p$. This does not change the results of the paper, as it only introduces a similar sign in [7, Proposition 4.1], which was stated up to sign anyway. The correct statement including sign of that Proposition is that the integral evaluated there for ϵ_n using the orientation $\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}y_{n-1}$ equals $(-1)^{n(n+1)/2} (2\pi i)^{(1-n)} (n-1)!\, P_{n,\mathrm{Zag}}(z)$.)

Finally, we have to introduce some polylogarithm functions and state their relations with the present constructions.

Let $\mathrm{Li}_l(z) = \sum_{m=1}^{\infty} z^m/m^l$ for $l \geqslant 1$ and $z \in \mathbb{C}$, $|z| < 1$. Then $\mathrm{Li}_1(z) = -\mathrm{Log}(1-z)$ and $\mathrm{dLi}_{l+1}(z) = \mathrm{Li}_l(z)\mathrm{d}\log z$ for $l \geqslant 1$. The functions Li_l can be continued to multi-valued holomorphic functions on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. Let the Bernoulli numbers B_l be defined by

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{k=0}^{\infty} \frac{B_l}{l!} x^l$$

It is well known that the functions (called P_n resp. $P_{n,\mathrm{Zag}}$ in [7] and [8])

$$P_{n,\mathrm{Zag}}(z) = \pi_{n-1}\left(\sum_{l=0}^{n-1} \frac{(-\log|z|)^l}{l!} \mathrm{Li}_{n-l}(z) \right)$$

and

$$P_n^{\mathrm{mod}}(z) = \pi_{n-1}\left(\sum_{l=0}^{n-1} \frac{2^l B_l}{l!} \log^l |z| \mathrm{Li}_{n-l}(z) \right)$$

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extend to single valued functions on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ with values in $\mathbb{R}(n-1)$, see [11]. The functions P_n^{mod} satisfy the functional equations $P_n^{\mathrm{mod}}(z) + (-1)^n P_n^{\mathrm{mod}}(z^{-1}) = 0$, and extend to continuous functions on $\mathbb{P}^1_{\mathbb{C}}$ with $P_n^{\mathrm{mod}}(0) = P_n^{\mathrm{mod}}(\infty) = 0$.

We have the following relations between the functions $P_{n,\mathrm{Zag}}$:

LEMMA 2.1.

$$\mathrm{d}P_{n,\mathrm{Zag}}(z) = P_{n-1,\mathrm{Zag}}(z)\mathrm{d}\, i \arg z + (-1)^n \frac{\log^{n-1}|z|}{(n-1)!}\pi_{n-1}\mathrm{d}\log(1-z)$$

Proof.

$$\begin{aligned}\mathrm{d}P_{n,\mathrm{Zag}}(z) &= \pi_{n-1}\left(\sum_{l=0}^{n-2}\frac{(-\log|z|)^l}{l!}\mathrm{Li}_{n-1-l}(z)\mathrm{d}\log z\right) + \\ &\quad + \pi_{n-1}\left(-\frac{(-\log|z|)^{n-1}}{(n-1)!}\mathrm{d}\log(1-z)\right) - \\ &\quad - \pi_{n-1}\left(\sum_{l=1}^{n-1}\frac{(-\log|z|)^{l-1}}{(l-1)!}\mathrm{Li}_{n-l}(z)\mathrm{d}\log|z|\right) \\ &= \pi_{n-1}\left(\sum_{l=0}^{n-2}\frac{(-\log|z|)^l}{l!}\mathrm{Li}_{n-1-l}(z)\mathrm{d}\, i\arg z\right) + \\ &\quad + (-1)^n\frac{\log^{n-1}|z|}{(n-1)!}\pi_{n-1}\mathrm{d}\log(1-z)\end{aligned}$$

As in [11, §7] one checks that we have the relations

$$P_{n,\mathrm{Zag}}(z) = \sum_{0 \leqslant 2j < m}\frac{\log^{2j}|z|}{(2j+1)!}P_{m-2j}^{\mathrm{mod}}(z).$$

LEMMA 2.2. *Let C be a complete, smooth, irreducible curve over $\mathbb{C}$ with function field $F = \mathbb{C}(C)$. If $f_1, \ldots, f_l$ are elements of F^*, and c_j are rational numbers such that*

$$\sum_{j=1}^{l} c_j f_j \otimes \cdots \otimes f_j \otimes \big(f_j \wedge (1-f_j)\big) = 0$$

in $\mathrm{Sym}^{n-2}F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}}$, then the function

$$z \longmapsto \sum_{j=1}^{l} c_j P_n^{\mathrm{mod}}(f_j(z))$$

is constant on C.

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Proof. This is done by Zagier in the proof of [11, Proposition 3] for $C = \mathbb{P}^1_{\mathbb{C}}$, which works just as well for any curve as in the statement of the Lemma.

Because $B_0 = 0$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$, we have

$$P_{3,\mathrm{Zag}}(z) = P_3^{\mathrm{mod}}(z) - \frac{1}{6}\log^2|z|\log|1-z|. \tag{2.5}$$

Propositions 4.1, 5.1, Remark 5.2 and Theorem 5.3 of [7] contain the following result.

THEOREM 2.3. *Let k be a number field and let $\sigma_1, \dots, \sigma_r$ be all embeddings of k into $\mathbb{C}$. Then the maps $\varphi^p_{(n)}$ and $\tilde{\varphi}^p_{(n)}$ exist without assumptions. They are injective for $p = 1$, and isomorphisms for (p, n) equal to $(1, 2)$ or $(1, 3)$. Moreover, the composition*

$$H^1(\widetilde{\mathcal{M}}^\bullet_{(n)}(k)) \overset{\tilde{\varphi}^1_{(n)}}{\to} K^{(n)}_{2n-1}(k) \overset{\mathrm{reg}}{\to}$$
$$H^0_{\mathrm{dR}}(\mathrm{Spec}(k \otimes_{\mathbb{Q}} \mathbb{C}); \mathbb{R}(n-1))^+ = \left(\oplus_\sigma \mathbb{R}(n-1)_\sigma\right)^+$$

is given by mapping $[x]_n$ to $\pm(n-1)!\,(P^{\mathrm{mod}}_n(\sigma(x))_\sigma)$.

Finally, we shall need the following result of Borel [1], to which we shall refer as Borel's theorem. Namely, for a number field k the regulator

$$K^{(n)}_{2n-1}(k) \to H^0_{\mathcal{D}}(\mathrm{Spec}\,(k) \otimes_{\mathbb{Q}} \mathbb{C}; \mathbb{R}(n))^+ \cong \left(\oplus_\sigma \mathbb{R}(n-1)_\sigma\right)^+$$

is an injection for $n \geqslant 2$, and induces an isomorphism

$$K^{(n)}_{2n-1}(k) \otimes_{\mathbb{Q}} \mathbb{R} \overset{\sim}{\to} \left(\oplus_\sigma \mathbb{R}(n-1)_\sigma\right)^+. \tag{2.6}$$

Heres σ runs through all embeddings of k into $\mathbb{C}$ as in Theorem 2.3 above.

We shall want the following theorem for the computation of the boundary map under localization.

THEOREM 2.4. *We have a commutative diagram (up to sign)*

$$\begin{array}{ccc} K^{(n+1)}_n(X^n; \square^n) & \longrightarrow & K^{(n+1)}_n(X \times X^{n-1}_{\mathrm{loc}}; \square^n) \\ \downarrow\cong & & \downarrow\cong \\ K^{(n+1)}_{n+1}(X^{n-1}; \square^{n-1}) & \longrightarrow & K^{(n+1)}_{n+1}(X^{n-1}_{\mathrm{loc}}; \square^{n-1}) \end{array}$$

and the image of $\sum_j c_j[f_j]_n \otimes g_j$ in $K^{(n+1)}_n(X^n; \square^n)$ under the map

$$\varphi^2_{(n+1)} : H^2(\mathcal{M}^\bullet_{(n+1)}(F)) \to K^{(n+1)}_n(X^n; \square^n) \cong K^{(n+1)}_{2n}(F)$$

gets mapped to $\pm\sum_j c_j[f_j]_n \cup g_j$ in $K^{(n+1)}_{n+1}(X^{n-1}_{\mathrm{loc}}; \square^{n-1})$, up to terms in $(1+I)^*\tilde{\cup}K^{(n)}_n(X^{n-2}_{\mathrm{loc}}; \square^{n-2})$.

Proof. The proof is rather analogous to the proof of [8, Proposition 3.2].

For computing the image under the regulator map we use integration. Because we shall be integrating forms on non-compact varieties, we need some results about the dependence of the result on the explicit representative chosen for a particular class. This problem was dealt with in Proposition 4.6 of [8], which we now proceed to recall.

Let Y be an algebraic variety of dimension n with compactification $\overline{Y}$ such that the complement of Y is given by D, a divisor with normal crossings. Suppose moreover that there is another divisor D' on $\overline{Y}$ such that the union of D and D' is a divisor with normal crossings. We want to say something about the behaviour close to D of forms on Y that vanish on $Y \cap D'$. Suppose that locally in a compact subset of $\overline{Y}$, D is given by $\prod_{i=1}^{k} x_i = 0$. Let $r_i = |x_i|$, $\theta_i = \arg x_i$. We will consider differential forms β on Y that satisfy the following condition on the compact subset of the chosen neighbourhood of D.

β vanishes on $D' \cap Y$ and can be written as sums of products of $\log r_i$, $\mathrm{d}\log r_i$, $\mathrm{d}\theta_i$, bounded C^∞-functions on Y and the restriction of C^∞-forms on $\overline{Y}$ to Y. (2.7)

PROPOSITION 2.5. *Let Y, $\overline{Y}$, D and D' be as above. Suppose that β_1 and β_2 are two closed n-forms on Y as in (2.7) that represent the same class in relative de Rham cohomology $H^n(Y; D'; \mathbb{R}(j))$. Let ω be a holomorphic or anti-holomorphic n-form on $\overline{Y}$, possibly with logarithmic poles along D'. Then*

$$\int_Y \beta_1 \wedge \omega = \int_Y \beta_2 \wedge \omega.$$

We conclude this section with some remarks on orientations and standard integrals, to be used throughout the paper.

We shall always use the following orientations for the integrals involved: with $t = x + iy$ the standard parameter on $\mathbb{A}^1 \subset \mathbb{P}^1$, $\mathbb{P}^1$ or open parts have orientation given by $\mathrm{d}x \wedge \mathrm{d}y = \frac{-1}{2i}\mathrm{d}t \wedge \mathrm{d}\bar{t}$. On $(\mathbb{P}^1)^n$ or open parts, we use the orientation given by $\frac{-1}{2i}\mathrm{d}t_1 \wedge \mathrm{d}\bar{t}_1 \wedge \cdots \wedge \frac{-1}{2i}\mathrm{d}t_n \wedge \mathrm{d}\bar{t}_n$. On $X^n_{S^1} = X^n \times S^1$ we take the product orientation of the above on X^n with the standard counterclockwise orientation on S^1.

Using Stokes' theorem and the fact that $\mathrm{d}\log((t-c)/(t-1)) \wedge \mathrm{d}\log t = 0$ for $c \in \mathbb{C}$, we find

$$\int_X \mathrm{d}i\arg\frac{t-c}{t-1} \wedge \mathrm{d}\ \log|t| = -\int_X \mathrm{d}\ \log\left|\frac{t-f}{t-1}\right| \wedge \mathrm{d}i\arg t = 2\pi i \log|c|$$

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and

$$\int_X \mathrm{d}i\arg\left(\frac{t-f}{t-1}\right) \wedge \mathrm{d}i\arg t = -\int_X \mathrm{d}\log\left|\frac{t-f}{t-1}\right| \wedge \mathrm{d}\log|t| = 0,$$

hence

$$\int_X \mathrm{d}\log\Big(\frac{t-c}{t-1}\Big) \wedge \mathrm{d}\log\bar{t} = \int_X \mathrm{d}\log\Big|\frac{t-c}{t-1}\Big|^2 \wedge \mathrm{d}\log\bar{t} = 4\pi i \log|c|.$$

We also have the standard integral for ρ_1 a bump function around $t=0$, i.e., $\rho_1 \equiv 1$ around $t=0$ and $\rho_1 \equiv 0$ off $t=0$,

$$\int_X \mathrm{d}(\rho_1(t)\mathrm{d}i\arg t) = -2\pi i.$$

Finally, we shall need the following integral. Let h be a function on $\mathbb{P}^1$ with $h(\infty) - h(0) = 1$. Then

$$\int_X \mathrm{d}h(t) \wedge \mathrm{d}i\arg t = 2\pi i.$$

3. The Regulator Integral

Let C be a smooth, proper, geometrically ireducible curve over the number field k, and let g be its genus. Then C_{an}, the associated complex manifold to $C \otimes_{\mathbb{Q}} \mathbb{C}$, is a disjoint union of $[k:\mathbb{Q}]$ complete algebraic curves C_τ over $\mathbb{C}$ of genus g, indexed by the embeddings of k into $\mathbb{C}$. We fix an orientation on C_{an} such that the involution σ given by complex conjugation on $\mathbb{C}$ in $C \otimes_{\mathbb{Q}} \mathbb{C}$ reverses the orientation. We also introduce the number r defined by $r = [k\colon\mathbb{Q}] \cdot g$.

The goal of this section is to describe the regulator on the image of $\varphi^2_{(n+1)}$ inside $H^1_{\mathrm{dR}}(F;\mathbb{R}(n))^+$. We begin with some remarks on the cohomology groups of C_{an}.

For $\tau : k \to \mathbb{C}$ an embedding, denote C_τ the curve obtained from C by base change from k to $\mathbb{C}$ via τ.

If τ is a real embedding, then σ acts on C_τ, reversing its orientation. $H^1_{\mathrm{dR}}(C_\tau;\mathbb{C})$ is spanned (as a $\mathbb{C}$-vector space) by the holomorphic and the anti-holomorphic forms on C_τ. Then the $\mathbb{R}$-vector space of holomorphic 1-forms ω on C_τ such that $\omega \circ \sigma = \overline{\omega}$ is given by $H^0(C_{\mathbb{R}};\Omega) \cong \mathbb{R}^g$ where $C_{\mathbb{R}}$ is the base change from k to $\mathbb{R}$ via τ. On the other hand, by projecting

$$H^1_{\mathrm{dR}}(C_\tau;\mathbb{C}) \cong H^0(C_{\mathbb{R}};\Omega) \oplus H^0(C_{\mathbb{R}};\overline{\Omega}) \oplus iH^0(C_{\mathbb{R}};\Omega) \oplus iH^0(C_{\mathbb{R}};\overline{\Omega})$$

onto the $\mathbb{R}(n)$ and $\pm$ parts one checks easily that

$$H^1_{\mathrm{dR}}(C_\tau;\mathbb{R}(n))^+ = \pi_n H^0(C_{\mathbb{R}};\Omega) \cong \mathbb{R}^g$$

because the forms remain independent after projection onto the real or imaginary

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parts. We get a pairing $H^1_{dR}(C_\tau; \mathbb{R}(n))^+ \times H^0(C_\mathbb{R}; \Omega) \to \mathbb{R}(1)$ by mapping $(\pi_n\psi, \varphi)$ to

$$\int_{C_\tau} \pi_n\psi \wedge \overline{\varphi} = \frac{1}{2}\int_{C_\tau} \psi \wedge \overline{\varphi}.$$

This pairing takes values in $\mathbb{R}(1)$ because σ reverses the orientation, and $\psi \circ \sigma = \overline{\psi}$ and $\omega \circ \sigma = \overline{\omega}$. It is non-degenerate because of the duality between holomorphic and anti-holomorphic forms on C_τ.

If τ is not a real embedding, σ acts on $C_\tau \coprod C_{\bar\tau}$. Then the holomorphic 1-forms ω such that $\omega \circ \sigma = \overline{\omega}$ are given by the pairs $(\omega, \overline{\omega \circ \sigma})$ with $\omega \in H^0(C_\tau; \Omega) \cong \mathbb{C}^g$. And $H^1_{dR}(C_\tau \coprod C_{\bar\tau}; \mathbb{R}(n))^+$ is given by the pairs $(\psi, \overline{\psi \circ \sigma})$ with $\psi \in H^1_{dR}(C_\tau; \mathbb{R}(n))$ which has $\mathbb{R}$-dimension $2g$. In this case we get a pairing $H^1_{dR}(C_\tau \coprod C_{\bar\tau}; \mathbb{R}(n))^+ \times H^0(C_\tau \coprod C_{\bar\tau}; \Omega)^+ \to \mathbb{R}(1)$ by mapping $((\psi, \overline{\psi \circ \sigma}), (\omega, \overline{\omega \circ \sigma}))$ to

$$\int_{C_\tau} \psi \wedge \overline{\omega} + \int_{C_{\bar\tau}} \overline{\psi \circ \sigma} \wedge \omega \circ \sigma = 2\int_{C_\tau} \psi \wedge \pi_{n+1}\omega.$$

It has values in $\mathbb{R}(1)$ for the same reasons as above. It is non-degenerate because the pairing

$$H^1_{dR}(C_\tau; \mathbb{R}(n)) \times H^1_{dR}(C_\tau; \mathbb{R}(n+1)) \to H^2_{dR}(C_\tau; \mathbb{R}(1)) \cong \mathbb{R}(1)$$

is non-degenerate, and the projection $H^0(C_\tau; \Omega) \to H^1_{dR}(C_\tau; \mathbb{R}(n+1))$ given by mapping ω to $\pi_{n+1}\omega$ is an isomorphism.

We summarize those results in the following Remark.

Remark 3.1. $H^1_{dR}(C; \mathbb{R}(n))^+ \cong \mathbb{R}^r$ with $r = [k:\mathbb{Q}] \cdot g$. Moreover, the holomorphic forms ω on C_{an} such that $\omega \circ \sigma = \overline{\omega}$ form the dual of $H^1_{dR}(C; \mathbb{R}(n))^+$ under the pairing

$$H^1_{dR}(C; \mathbb{R}(n))^+ \times \langle\omega\rangle \to \mathbb{R}(1)$$

defined by sending (ψ, ω) to $\int_C \psi \wedge \overline{\omega}$.

We use this duality in Proposition 3.2 below in order to give the Beilinson regulator in terms of integrals.

In the following formulae we use the notation $\pm$ or $\mp$ where we read either the top or the bottom in all places. The involution σ acts also on $H_1(C; \mathbb{Q})$, so this space splits into a $+$ part and a $-$ part as well. From the pairing with $H^1(C; \mathbb{Q})$ one deduces that both pieces have $\mathbb{Q}$-dimension r, as $H_1(\mathbb{Q})^\pm$ is perpendicular to $H^1(\mathbb{Q})^\mp$. Let $\{s_{1,\pm}, \ldots, s_{r,\pm}\}$ be a basis of $H_1(C; \mathbb{Q})^\pm$, and let $\{s^*_{1,\pm}, \ldots, s^*_{r,\pm}\}$ in $H^1(C; \mathbb{Q})$ be its dual base, so that $\int_{s_{m,\pm}} s^*_{k,\pm} = \delta_{mk}$. Let $T^\pm_{k,l} = (\int_{s_{k,\pm}} \omega_l)$, so $\omega_l = \sum_k (T^+_{k,l} s^*_{k,+} + T^-_{k,l} s^*_{k,-})$. Write $\mathbb{R}(+)$ for $\mathbb{R}(0)$ and $\mathbb{R}(-)$ for $\mathbb{R}(1)$, and similarly for $\pi_\pm$. As $\overline{\omega_l} = \omega_l \circ \sigma$, we get $\int_{s_{k,\pm}} \overline{\omega_l} = \int_{s_{k,\pm}} \omega_l \circ \sigma = \int_{\sigma(s_{k,\pm})} \omega_l = \pm \int_{s_{k,\pm}} \omega_l$, and hence $T^\pm$ has entries in $\mathbb{R}(\pm)$. Therefore $T^\pm_{kl} = \int_{s_{k,\pm}} \pi_\pm \omega_l$. In particular, $\pi_+\omega_l = \sum_k T^+_{k,l} s^*_{k,+}$ and $\pi_-\omega_l = \sum_k T^-_{k,l} s^*_{k,-}$.

PROPOSITION 3.2. *Suppose the Beilinson regulator maps the elements* $\alpha_1, \dots, \alpha_r$ *in* $K_{2n}^{(n+1)}(C)$ *to* $\psi_1, \dots, \psi_r$ *in* $H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+$. *Let* $\omega_1, \dots, \omega_r$ *and* $T^{\pm}$ *be as above, and let* $R_{k,l} = \int_C \psi_k \wedge \overline{\omega_l}$. *Then the Beilinson regulator of* $\alpha_1, \dots, \alpha_r$ *is given by*

$$c_{n+1} = \frac{\det(R)}{(2\pi i)^{nr}\det(T^{\pm})}.$$

where we take $-$ *for n even, and* $+$ *for n odd.*

Proof. In this proof, let us write $\pm$ and $\mp$ where we mean that we take the top for n even, the bottom for n odd. Note that $\psi_k \circ \sigma = \pm\psi_k$ if $\psi_k \in H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+$. Therefore we can define the $\mathbb{R}(n)$-valued matrix M by $\psi_k = \sum_m M_{k,m} s^*_{m\pm}$. Then by definition, $c_{n+1} = (2\pi i)^{-nr}\det(M)$. As σ reverses the orientation, $\omega_l \circ \sigma = \overline{\omega_l}$ and $\psi_k \circ \sigma = \overline{\psi_k} = (-1)^n \psi_k$ imply that

$$R_{k,l} = \int_C \psi_k \wedge \overline{\omega_l} = -\int_C \psi_k \circ \sigma \wedge \overline{\omega_l \circ \sigma} = -\int_C \overline{\psi_k} \wedge \omega_l = -\overline{R_{k,l}}.$$

Therefore $R_{k,l}$ is purely imaginary, and

$$\begin{aligned} R_{k,l} &= -\int_C \psi_k \wedge \pi_{\mp}\omega_l \\ &= -\sum_n T^{\mp}_{n,l} \int_C \psi_k \wedge s^*_{n,\mp} \\ &= -\sum_{m,n} M_{k,m} T^{\mp}_{n,l} \int_C s^*_{m,\pm} \wedge s^*_{n,\mp} \\ &= -\sum_{m,n} M_{k,m} A_{m,n} T^{\mp}_{n,l} \end{aligned}$$

with $A_{m,n} = \int_C s^*_{m,\pm} \wedge s^*_{n,\mp}$. As $\det(A)$ expresses the non–degeneracy of the pairing $H^1(C; \mathbb{Q}) \times H^1(C; \mathbb{Q}) \to H^2(C; \mathbb{Q})$, it is an element of $\mathbb{Q}^*$. Hence taking determinants we find that $\det(R) = \det(M)\det(T^{\mp})$ up to a factor in $\mathbb{Q}^*$. So we get that the regulator c_{n+1} of $\alpha_1, \dots, \alpha_r$ is given by

$$c_{n+1} = \frac{\det(R)}{(2\pi i)^{nr}\det(T^{\mp})}.$$

DEFINITION 3.3. If ω is a holomorphic 1-form on C_{an} such that $\omega \circ \sigma = \overline{\omega}$, we call the map $H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+ \to \mathbb{R}(1)$ given by $\psi \mapsto \int_C \psi \wedge \overline{\omega}$ the regulator integral associated to ω. We shall use the same terminology if we precede this with the regulator map from $K_{2n}^{(n+1)}(C)$ to $H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+$.

The regulator integral has the advantage that it can be factored over larger groups than just $K_{2n}^{(n+1)}(C)$:

PROPOSITION 3.4. *Let* ω *be a holomorphic 1-form on C. Then the regulator integral* $K_{2n}^{(n+1)}(C) \to H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+ \to \mathbb{R}(1)$ *associated to* ω *extends naturally over* *the maps*

$$K_{2n}^{(n+1)}(C) \to K_{2n}^{(n+1)}(F) \to K_{2n}^{(n+1)}(F)/K_{2n-1}^{(n)}(k) \cup F_{\mathbb{Q}}^*.$$

Proof. Let β be a class in $H^1_{\mathrm{dR}}(F;\mathbb{R}(n))^+ = \lim_{\overrightarrow{U\subset C}} H^1_{\mathrm{dR}}(U;\mathbb{R}(n))^+$. Then using the fact that $H^1_{\mathrm{dR}}(U;\mathbb{C})$ can be computed using forms with logarithmic singularities, one sees that β has a representative ψ as in (2.7), and we extend the map by mapping β to $\int_C \psi \wedge \overline{\varpi}$. Proposition 2.5 shows that this does not depend on our choice of ψ, hence is well defined.

As for the last map, note that the regulator of $\alpha \cup f$ for $\alpha \in K_{2n-1}^{(n)}(k)$ and $f \in F$ is given by $(0, c) \cup (\mathrm{d}\log f, \log|f|) = (0, c\mathrm{d}i\arg f)$ in Deligne cohomology, hence by $\mathrm{d}i\arg f$ in H^1_{dR}. Then the regulator integral becomes $\int_C c\mathrm{d}i\arg f \wedge \overline{\varpi} = \int_C c\mathrm{d}\log|f| \wedge \overline{\varpi} = 0$ as one easily checks using Stokes' theorem.

The rest of the section is devoted to rewriting the integrals occurring in Proposition 3.2 on the image of $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$. We shall in fact prove the following Theorem.

THEOREM 3.5. *Suppose the Beilinson–Soulé conjecture holds for F, so there is a map $H^p(\mathcal{M}^\bullet_{(n+1)}(F)) \to K_{2n+2-p}^{(n+1)}(F)$ as explained in Section 2. If $\sum_j c_j[f_j]_n \otimes g_j$ is an element of $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$, then the regulator integral*

$$H^2(\mathcal{M}^\bullet_{(n+1)}(F)) \longrightarrow K_{2n}^{(n+1)}(F) \xrightarrow{\mathrm{reg}} H^1_{\mathrm{dR}}(F;\mathbb{R}(n)) \xrightarrow{\int_C \cdots \wedge \overline{\varpi}} \mathbb{R}(1)$$

is given by mapping $[f]_n \otimes g$ to

$$\pm 2^n \int_C \log|g| \log^{n-1}|f| \mathrm{d}\log|1-f| \wedge \overline{\varpi}$$

For an element $\sum_j c_j[f_j]_n \otimes g_j$ in $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$, the total sum is the same if we map $[f]_n \otimes g$ to

$$\pm 2^n \frac{n}{n+1} \int_C \log|g| \log^{n-2}|f|\, \theta(1-f, f) \wedge \overline{\varpi}, \tag{3.1}$$

where $\theta(f,g) = \log|f|\,\mathrm{d}\log|g| - \log|g|\,\mathrm{d}\log|f|$. This last integral is zero on symbols $\langle [f]_n + (-1)^n[1/f]_n\rangle \otimes g$, hence factors through the map $H^2(\mathcal{M}^\bullet_{(n+1)}(F)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$.

Remark 3.6. If the curve is an elliptic curve E, the integrals occurring in Theorem 3.5 can be rewritten using Fourier transformation. This gives expressions for the regulator integral in terms of non-classical Eisenstein series (see e.g., [5, Theorem 3.4] for the case $n = 2$ and [3, Theorem 5.8] for arbitrary n). It seems that for $n = 2$ those Eisenstein series were first considered by Deninger in [2].

The rest of this section is devoted to the proof of Theorem 3.5. We first make the isomorphism $H^{n+1}_{\mathrm{dR}}(X^n_C; \square^n; \mathbb{R}(n))^+ \cong H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+$ explicit. Namely, let h be a real-valued C^∞-function on $\mathbb{P}^1_{\mathbb{C}}$ such that $h(\infty) - h(0) = 1$. Then the isomorphism

$$H^{n-1}_{\mathrm{dR}}(Y; \mathbb{R}(m)) \xrightarrow{\sim} H^n_{\mathrm{dR}}(X_Y; \square; \mathbb{R}(m))$$

is given by sending ψ to $\psi \wedge \mathrm{d}h$, and similarly for the $+$ parts if h is symmetric with respect to complex conjugation on $\mathbb{P}^1_{\mathbb{C}}$. As

$$\int_X \mathrm{d}h \wedge \mathrm{d}\log \bar{t} = -\int_X \mathrm{d}h \wedge \mathrm{d}i \arg t = -2\pi i,$$

this means that for ψ in $H^1_{\mathrm{dR}}(C; \mathbb{C})$,

$$\begin{aligned} &\pm \int_C \psi \wedge \overline{\omega} \\ &= (2\pi i)^{-n} \int_{X^n_C} \psi \wedge \mathrm{d}h(t_1) \wedge \cdots \wedge \mathrm{d}h(t_n) \wedge \mathrm{d}\log \overline{t_1} \wedge \cdots \wedge \mathrm{d}\log \overline{t_n} \wedge \overline{\omega} \\ &= (-2\pi i)^{-n} \int_{X^n_C} \psi \wedge \mathrm{d}h(t_1) \wedge \cdots \wedge \mathrm{d}h(t_n) \wedge \mathrm{d}i \arg t_1 \wedge \cdots \wedge \mathrm{d}i \arg t_n \wedge \overline{\omega}. \end{aligned}$$

Now let $\alpha = \sum_j c_j[f_j]_n \otimes g_j$ be an element of $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$, with image ψ in $H^1_{\mathrm{dR}}(F; \mathbb{R}(n))^+$ under the composition

$$H^2(\mathcal{M}^\bullet_{(n+1)}(F)) \to K^{(n+1)}_{2n}(F) \to H^1_{\mathrm{dR}}(F; \mathbb{R}(n))^+.$$

From Theorem 2.4 we have a commutative diagram

$$\begin{array}{ccc} K_n^{(n+1)}(X^n; \square^n) & \longrightarrow & K_n^{(n+1)}(X \times X^{n-1}_{\mathrm{loc}}; \square^n) \\ \downarrow \cong & & \downarrow \cong \\ K_{n+1}^{(n+1)}(X^{n-1}; \square^{n-1}) & \longrightarrow & K_{n+1}^{(n+1)}(X^{n-1}_{\mathrm{loc}}; \square^{n-1}), \end{array}$$

and the image of $\alpha = \sum_j c_j[f_j]_n \otimes g_j$ under $\varphi^2_{(n+1)}$ in $K_n^{(n+1)}(X^n; \square^n)$ under the maps in this diagram will be mapped to $\alpha' = \pm \sum_j c_j[f_j]_n \cup g_j$ in $K^{(n+1)}_{n+1}(X^{n-1}_{\mathrm{loc}}; \square^{n-1})$, modulo $(1+I)^* \tilde{\cup} K^{(n)}_n(X^{n-2}_{\mathrm{loc}}; \square^{n-2})$. One has the corresponding diagram in Deligne cohomology, which is equal to the de Rham cohomology in all cases. Hence we have $\mathrm{reg}(\alpha)$ in $H^{n+1}_{\mathrm{dR}}(X^n; \square^n; \mathbb{R}(n))^+$, corresponding to ψ' in $H^n_{\mathrm{dR}}(X^{n-1}; \square^{n-1}; \mathbb{R}(n))^+$as well as ψ in $H^1_{\mathrm{dR}}(F; \mathbb{R}(n))^+$ under the relativity isomorphisms. ψ' in turn maps to $\mathrm{reg}(\alpha')$ in $H^n_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}}; \square^{n-1}; \mathbb{R}(n))^+$. Therefore, if $\overline{\omega}$ is any holomorphic 1-form on C, and $\Theta = \mathrm{d}\log \overline{t_1} \wedge \cdots \wedge \mathrm{d}\log \overline{t_{n-1}}$, we want to compute

$$\pm \int_C \psi \wedge \overline{\omega} = \frac{1}{(2\pi i)^{n-1}} \int_{X^{n-1} \times C} \psi' \wedge \Theta \wedge \overline{\omega} = \frac{1}{(2\pi i)^{n-1}} \int_Z \mathrm{reg}(\alpha') \wedge \Theta \wedge \overline{\omega}$$

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where Z is a localization of X_C^{n-1} and $\mathrm{reg}(\alpha')$ is a form satisfying (2.7) on a suitable compactification of Z, $\overline{Z}$. We can obtain $\overline{Z}$ from $(\mathbb{P}^1)^{n-1} \times C$ by repeatingly blowing up, obtaining a Z which has a Zariski open part isomorphic to $X_{U,\mathrm{loc}}^{n-1}$ for some Zariski open U of C. To see that the last equality sign is true, we have to check that the conditions of Proposition 2.5 apply, i.e., that on $\overline{Z}\,\mathrm{reg}(\alpha')$ can be written as in (2.7) and that (the pullback to $\overline{Z}$ of) $\mathrm{d}\log\overline{t_1} \wedge \cdots \wedge \mathrm{d}\log\overline{t_{n-1}} \wedge \overline{\omega}$ has poles of order one along the strict transform of $\square^{n-1}$ and no poles elsewhere. This one easily checks explicitly, cf. the computations on page 608 of [6].

Because we can take $\overline{Z}$ to be a blowup of $(\mathbb{P}^1)_C^{n-1}$, which is isomorphic to $(\mathbb{P}^1)_U^{n-1}$ for some Zariski open part U of C, we can compute simply on $X_{U,\mathrm{loc}}^{n-1}$ without changing the value of the integral.

As the regulator of $[f]_n \cup g$ is given by the product

$$\begin{aligned}&(\omega_n, \epsilon_n) \cup (\mathrm{d}\log g, \log|g|)\\ &\quad= (\omega_n \wedge \mathrm{d}\log g, \epsilon_n \wedge \mathrm{d}\, i \arg g + (-1)^n \log|g| \pi_n \omega_n),\end{aligned}$$

we find that the regulator integral is given by

$$(2\pi i)^{1-n} \sum_j c_j \int_{\overline{Z}} \big(\epsilon_n(f_j) \wedge \mathrm{d}\, i \arg g_j + (-1)^n \log|g_j| \pi_n \omega_n(f_j)\big) \wedge \Theta \wedge \overline{\omega},$$

which equals

$$(2\pi i)^{1-n} \sum_j c_j \int_{\overline{Z}} \big(\epsilon_n(f_j) \wedge \mathrm{d}\log|g_j| + (-1)^n \log|g_j| \pi_n \omega_n(f_j)\big) \wedge \Theta \wedge \overline{\omega}$$

as $\mathrm{d}(i \arg g - \log|g|) \wedge \overline{\omega}(f) = 0$ on $X^{n-1} \times C$. Adding

$$0 = (-1)^n (2\pi i)^{1-n} \sum_j c_j \int_{\overline{Z}} \mathrm{d}(\log|g| \epsilon_n(f_j)) \wedge \Theta \wedge \overline{\omega}$$

we obtain

$$(-1)^n (2\pi i)^{1-n} \sum_j c_j \int_{\overline{Z}} \log|g_j|\, \omega_n(f_j) \wedge \Theta \wedge \overline{\omega}$$

Remembering that

$$\omega = \omega_n(f) = (-1)^{n-1} \mathrm{d}\log\frac{t_1 - f}{t_1 - 1} \wedge \cdots \wedge \mathrm{d}\log\frac{t_{n-1} - f}{t_{n-1} - 1} \wedge \mathrm{d}\log(1 - f)$$

we can compute each of the terms in this sum as

$$\begin{aligned}
&\pm (2\pi i)^{1-n} \int_C \left(\int_X \mathrm{d}\log\frac{t-f_j}{t-1} \wedge \mathrm{d}\log\bar{t} \right)^{n-1} \log|g_j| \mathrm{d}\log(1-f_j) \wedge \overline{\varpi} \\
&\quad = \pm 2^{n-1} \int_C \log|g_j| \log^{n-1}|f_j| \mathrm{d}\log(1-f_j) \wedge \overline{\varpi} \\
&\quad = \pm 2^{n} \int_C \log|g_j| \log^{n-1}|f_j| \mathrm{d}\log|1-f_j| \wedge \overline{\varpi}
\end{aligned}$$

because $\int_X \log\frac{t-f}{t-1} \wedge \mathrm{d}\log\bar{t} = 2\pi i \log|f|^2$ and $\mathrm{d}\log\overline{(1-f)} \wedge \overline{\varpi} = 0$ on C. We can rewrite the resulting total sum

$$\pm \sum_j c_j 2^n \int_C \log|g_j| \log^{n-1}|f_j| \mathrm{d}\log|1-f_j| \wedge \overline{\varpi}$$

in terms of

$$\int_C \log|g| \log^{n-2}|f| (\log|1-f| \mathrm{d}\log|f| - \log|f| \mathrm{d}\log|1-f|) \wedge \overline{\varpi}$$

by adding

$$-\sum_j c_j \frac{1}{n+1} \int_C \mathrm{d}\big(\log|g_j| \log^{n-1}|f_j| \log|1-f_j|\big) \wedge \overline{\varpi}$$

and

$$\sum_j c_j \frac{1}{n+1} \int_C \log^{n-2}|f_j| \log|1-f_j|\, \theta(f_j, g_j) \wedge \overline{\varpi}.$$

Note that this does not change the value of the integral, as the first term vanishes by Stokes' theorem, and the second because we take sums of terms corresponding to an element in $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$, and the form vanishes identically after summing up the terms. This yields the integral given in (3.1). One checks immediately by writing it out that each of the terms vanishes on $([f]_n + (-1)^n[1/f]_n) \otimes g$.

Remark 3.7. For $n+1 = 3$ or 4, i.e., the cases $K_4^{(3)}(F)$ and $K_6^{(4)}(F)$, we get the results of Theorem 3.5 without any assumptions. If $n+1=3$, we have the map $\tilde{\varphi}^2_{(3)} : H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(F)) \underset{\sim}{\leftarrow} H^2(\mathcal{M}^\bullet_{(3)}(F)) \to K_4^{(3)}(F)$ without any assumptions, see Section 2, so the Theorem gives the formulae for the regulator on the image of $\tilde{\varphi}^2_{(3)}$. For $n+1=4$, the map $\varphi^2_{(4)}$ from $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(F))$ to $K_6^{(4)}(F)/N$ (with N generated by $K_4^{(2)}(F) \cup K_2^{(2)}(F)$, see Section 2) exists without assumptions, and the regulator factors through this as the regulator map to Deligne cohomology respects the product structure, and vanishes on $K_4^{(2)}(F)$. Hence it factors through this quotient to give us

$$H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(F)) \to H^2_{\mathcal{D}}(F, \mathbb{R}(3))^+ \cong H^1_{\mathrm{dR}}(F, \mathbb{R}(2))^+$$

with the formulae for the regulator integrals as given in Theorem 3.5. Note also that in this case the map $H^2(\mathcal{M}^\bullet_{(4)}(F)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(F))$ is a surjection, which follows from the acyclicity of the complex (2.2) in degrees 3 and 4 as mentioned just after (2.2).

4. The Boundary Under Localization

If x is a closed point of the curve C, $K_{2n}(k(x))$ is torsion as $k(x)$ is a number field. Hence the localization sequence for the K-theory of the curve takes the form

$$0 \to K_{2n}^{(n+1)}(C) \to K_{2n}^{(n+1)}(F) \to \coprod_{x\in C^{(1)}} K_{2n-1}^{(n)}(k(x)) \to \cdots .$$

This section is devoted to the computation of the boundary map on the image in $K_{2n}^{(n+1)}(F)$ of $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$ or $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$ in this localization sequence for $n = 3$. The method chosen probably works for all $n \geqslant 2$ (with the case $n = 2$ already done in [8]), but at some stage the combinatorics become too complicated in general and we restrict ourselves to the case $n = 3$.

Recall that in [8, Corollary 5.4] it was proved that we have a commutative diagram (up to sign and up to $\partial(K_3^{(2)}(k) \cup F^*_{\mathbb{Q}})$ in the lower right hand corner)

$$\begin{array}{ccc} H^2(\mathcal{M}^\bullet_{(3)}(F)) & \xrightarrow{\varphi^2_{(3)}} & K_4^{(3)}(F) \\ \downarrow 2\delta & & \downarrow \partial \\ \coprod_{x\in C^{(1)}} H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(k(x))) & \xrightarrow{\sim} & \coprod_{x\in C^{(1)}} K_3^{(2)}(k(x)) \end{array}$$

Note that the lower horizontal arrow is an isomorphism by Theorem 2.3. For generalizing this to $n = 3$, we need some preliminary results. The following result was proved in [8] for $n = 2$.

PROPOSITION 4.1. *There is a map*

$$\delta: \quad \widetilde{\mathcal{M}}^\bullet_{(n)}(F) \to \coprod_{x\in C^{(1)}} \widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x))[1]$$

given by

$$\delta_x([f]_{n-l} \otimes g_1 \wedge \cdots \wedge g_l) = \mathrm{sp}_{n-l,x}([f]_{n-l}) \otimes \partial_{l,x}(g_1 \wedge \cdots \wedge g_l)$$

for $l = 1, \ldots, n-1$, *and*

$$\delta_x(g_1 \wedge \cdots \wedge g_n) = -\partial_{n,x}(g_1 \wedge \ldots \wedge g_n)$$

for $l = n$, *where* $\mathrm{sp}_{n-l,x}([f]_{n-l}) = [f(x)]_{n-l}$ *if* $f(x) \neq 0$ *or* ∞, *0 otherwise, and* $\partial_{l,x}$ *the*

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unique map from $\bigwedge^l F^*_{\mathbb{Q}}$ *to* $\bigwedge^{l-1} k(x)^*_{\mathbb{Q}}$ *determined by*

$$\pi_x \wedge u_1 \wedge \cdots \wedge u_{l-1} \mapsto u_1(x) \wedge \cdots \wedge u_{l-1}(x)$$

$$u_1 \wedge \cdots \wedge u_l \mapsto 0$$

if all u_i *are units at* x *and* π_x *is a uniformizer at* x. *This gives rise to maps*

$$\delta: \quad H^m(\widetilde{\mathcal{M}}^\bullet_{(n)}(F)) \to \coprod_{x\in C^{(1)}} H^{m-1}(\widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x))).$$

Remark 4.2. We get induced maps

$$\delta: \quad \mathcal{M}^\bullet_{(n)}(F) \to \coprod_{x\in C^{(1)}} \widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x))[1]$$

and

$$\delta: \quad H^m(\mathcal{M}^\bullet_{(n)}(F)) \to \coprod_{x\in C^{(1)}} H^{m-1}(\widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x)))$$

by composing the natural projection $\mathcal{M}^\bullet_{(n)}(F) \to \widetilde{\mathcal{M}}^\bullet_{(n)}(F)$ with δ.

Proof. Let $x \in C$ be a closed point in our curve over the number field k. Fix a uniformizer π_x around x. We shall in fact construct the map $\mathrm{sp}_{n,x} : M_{(n)}(F) \to \widetilde{M}_{(n)}(k(x))$, and then observe that it factors through the projection $M_{(n)}(F) \to \widetilde{M}_{(n)}(F)$.

Assume we have a map $\mathrm{sp}_{n-1,x} : M_{(n-1)}(F) \to \widetilde{M}_{(n-1)}(k(x))$ given by mapping $[f]_{n-1}$ to $[f(x)]_{n-1}$ if $f(x) \neq 0$ or ∞, and 0 otherwise. This was done for $n-1=2$ in the proof of [8, Proposition 5.1], and is the case where one has to work with $\widetilde{M}_{(n-1)}(k(x))$ rather than $M_{(n-1)}(k(x))$. We then have a diagram

$$\begin{array}{ccc} \mathbb{Q}[F^*] & \longrightarrow & M_{(n-1)}(F) \otimes F^*_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \widetilde{M}_{(n)}(k(x)) & \longrightarrow & \widetilde{M}_{(n-1)}(k(x)) \otimes k^*_{\mathbb{Q}} \end{array}$$

where $\mathbb{Q}[F^*]$ is the free $\mathbb{Q}$-vector space on elements of F^*, the vertical maps are $f \mapsto \mathrm{sp}_{n,x}([f(x)]_n)$ and $[f]_n \otimes g \mapsto \mathrm{sp}_{n-1,x}([f]_{n-1}) \otimes \overline{g(x)}$, with $\overline{g(x)} = g\pi_x^{-\mathrm{ord}_x(g)}{}_{|x}$, π_x a uniformizer at x. It is obvious that the diagram commutes. To show that it factors through $M_{(n)}(F)$ observe that if α goes to zero in $M_{(n)}(F)$ then $\mathrm{sp}_{n,x}(\alpha)$ defines an element in $K^{(n)}_{2n-1}(k(x))$. As $k(x)$ is a number field, we can verify that the element is zero by computing the regulator map, given by Theorem 2.3. To this end, consider all embeddings of $k(x)$ into $\mathbb{C}$, i.e., tensor the curve C over $\mathbb{Q}$ with $\mathbb{C}$. Then we have that $P_n^{\mathrm{mod}}(\alpha)$ is constant, see Lemma 2.2. Specializing to a point y where it can be done directly (which means that y should lie in some Zariski open part, see Section

2), we find 0, so the regulator vanishes. Then we use that the regulator does not change if we replace y with x, and continuity, to see that $P_n^{\mathrm{mod}}(\mathrm{sp}_{n,x}(\alpha)) = P_n^{\mathrm{mod}}(\mathrm{sp}_{n,y}(\alpha)) = 0$ because P_n^{mod} is continuous at 0 and ∞ and has value 0. Hence $\mathrm{sp}_{n,x}(\alpha) = 0$ in $\widetilde{M}_{(n)}(k(x))$.

This map gives us the map $\mathrm{sp}_{n,x} : M_{(n)}(F) \to \widetilde{M}_{(n)}(k(x))$, obviously factoring through $\widetilde{M}_{(n)}(F)$. It is then easy to check that we get maps of complexes

$$\mathcal{M}^\bullet_{(n)}(F) \longrightarrow \widetilde{\mathcal{M}}^\bullet_{(n)}(F) \xrightarrow{\delta_x} \widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x))[1]$$

with δ_x given by mapping $[f]_{n-l} \otimes g_1 \wedge \cdots \wedge g_l$ to $\mathrm{sp}_{n-l,x}([f(x)]_{n-l}) \otimes \partial_{l,x}$ for $l = 1, \ldots, n-1$ and $-\partial_{n,x}(g_1 \wedge \ldots \wedge g_n)$ for $l = n$, with $\partial_{l,x}$ the unique map from $\bigwedge^l F^*_{\mathbb{Q}}$ to $\bigwedge^{l-1} k(x)^*_{\mathbb{Q}}$ given by

$$\pi_x \wedge u_1 \wedge \cdots \wedge u_{l-1} \mapsto u_1(x) \wedge \cdots \wedge u_{l-1}(x)$$

$$u_1 \wedge \cdots \wedge u_l \mapsto 0$$

if all u_i are units at x. From this we get maps

$$\delta_x : H^m(\mathcal{M}^\bullet_{(n)}(F)) \to H^{m-1}(\widetilde{\mathcal{M}}^\bullet_{(n-1)}(k(x)))$$

as claimed in the proposition.

We can now introduce the complexes $\mathcal{M}^\bullet_{(n+1)}(C)$ and $\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C)$, by defining them to be the total complexes of the double complexes

$$\begin{array}{ccccccc}
M_{(n+1)}(F) & \xrightarrow{d} & M_{(n)}(F) \otimes F^*_{\mathbb{Q}} & \xrightarrow{d} & M_{(n-1)}(F) \otimes \bigwedge^2 F^*_{\mathbb{Q}} & \xrightarrow{d} & \cdots \\
\downarrow & & \delta \downarrow & & \delta \downarrow & & \\
0 & \to & \coprod \widetilde{M}_{(n)}(k(x)) & \xrightarrow{d} & \coprod \widetilde{M}_{(n-1)}(k(x)) \otimes k(x)^*_{\mathbb{Q}} & \xrightarrow{d} & \cdots
\end{array}$$

and

$$\begin{array}{ccccccc}
\widetilde{M}_{(n+1)}(F) & \xrightarrow{d} & \widetilde{M}_{(n)}(F) \otimes F^*_{\mathbb{Q}} & \xrightarrow{d} & \widetilde{M}_{(n-1)}(F) \otimes \bigwedge^2 F^*_{\mathbb{Q}} & \xrightarrow{d} & \cdots \\
\downarrow & & \delta \downarrow & & \delta \downarrow & & \\
0 & \to & \coprod \widetilde{M}_{(n)}(k(x)) & \xrightarrow{d} & \coprod \widetilde{M}_{(n-1)}(k(x)) \otimes k(x)^*_{\mathbb{Q}} & \xrightarrow{d} & \cdots
\end{array}$$

where both coboundaries have degree 1 and the total complexes are cohomological complexes with $M^\bullet_{(n+1)}(F)$ and $\widetilde{M}^\bullet_{(n+1)}(F)$ in degree 1. There are obvious inclusions of $H^2(\mathcal{M}^\bullet_{(n+1)}(C))$ into $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$ and of $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(C))$ into $H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F))$, so that the maps $\varphi^2_{(n+1)}$ resp. $\tilde{\varphi}^2_{(n+1)}$ obviously extend to maps on the cohomology of those complexes.

COROLLARY 4.3. *Under the map* $\varphi^2_{(3)}$, $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$ *is mapped to* $K_4^{(3)}(C) + K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}$ *inside* $K_4^{(3)}(F)$.

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Remark 4.4. The exact sequence

$$0 \to K_4^{(3)}(C) \to K_4^{(3)}(C) + K_3^{(2)}(k) \cup F_{\mathbb{Q}}^* \to \partial\Big(K_3^{(2)}(k) \cup F_{\mathbb{Q}}^*\Big)$$

is split. Namely, because $K_3^{(2)}(k) \cup k^* = 0$ as $K_4^{(3)}(k) = 0$ (remember that k is a number field), the cup product factors through $K_3^{(2)}(k) \otimes F_{\mathbb{Q}}^*/k_{\mathbb{Q}}^*$. The boundary map factors through this as well, and is hence given by $\partial\alpha \cup f = -\partial f \cup \alpha = -\mathrm{div}(f) \cup \alpha$. Because the divisor map is injective on $F_{\mathbb{Q}}^*/k_{\mathbb{Q}}^*$, the boundary map is injective as well. The corresponding result holds for$K_{2n}^{(n+1)}(C)$ and $K_{2n-1}^{(n)}(k) \cup F_{\mathbb{Q}}^*$.

COROLLARY 4.5. *$\varphi_{(3)}^2 : H^2(\mathcal{M}_{(3)}^{\bullet}(C)) \to K_4^{(3)}(C) + K_3^{(2)}(k) \cup F_{\mathbb{Q}}^*$ can be lifted to a map $\varphi_{(3)}^2 : H^2(\mathcal{M}_{(3)}^{\bullet}(C)) \to K_4^{(3)}(C)$ by changing $\varphi_{(3)}^2(\alpha)$ with elements in $K_3^{(2)}(k) \cup F_{\mathbb{Q}}^*$.*

We now serve the main course in this section:

THEOREM 4.6. *For $n = 3$, the diagram*

$$\begin{array}{ccc} H^2(\mathcal{M}_{(4)}^{\bullet}(F)) & \xrightarrow{\varphi_{(4)}^2} & K_6^{(4)}(F)/K_4^{(2)}(F) \cup K_2^{(2)}(F) \\ \big\downarrow{\scriptstyle 3\delta} & & \big\downarrow{\scriptstyle \partial} \\ \coprod_{x \in C^{(1)}} H^1(\widetilde{\mathcal{M}}_{(3)}^{\bullet}(k(x))) & \xrightarrow{\sim} & \coprod_{x \in C^{(1)}} K_5^{(3)}(k(x)) \end{array}$$

commutes up to sign and up to $\partial\Big(K_5^{(3)}(k) \cup F_{\mathbb{Q}}^*\Big)$ in the lower right hand corner. (Note that the map $\varphi_{(4)}^2$ exists without assumptions as explained in Section 2, and that the lower isomorphism is part of Theorem 2.3.)

Remark 4.7. Note that from the localization sequence

$$0 \to K_4^{(2)}(C) \to K_4^{(2)}(F) \to \coprod K_3^{(1)}(k(x))$$

we get an isomorphism $K_4^{(2)}(F) \cong K_4^{(2)}(C)$ as $K_3^{(1)}(k(x)) = 0$. Hence on $K_4^{(2)}(F) \cup K_2^{(2)}(F)$, ∂_x is given by mapping $\alpha \cup \beta$ to the cup product of $\alpha(x)$ with the boundary at x of β, i.e., zero as $\alpha(x) \in K_4^{(2)}(k(x)) = 0$ as $k(x)$ is a number field. (Of course that would be zero for any field for which the Beilinson–Soulé conjecture holds.) So in fact

$$K_4^{(2)}(F) \cup K_2^{(2)}(F) \subset K_6^{(4)}(C) \subset K_6^{(4)}(F).$$

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COROLLARY 4.8. *Under* $\varphi^2_{(4)}$, $H^2(U_{(4)}(C))$ *is mapped to*

$$K_6^{(4)}(C)/K_2^{(4)}(F)\cup K_2^{(2)}(F)+K_5^{(3)}(k)\cup F^*_{\mathbb{Q}}$$

inside $K_6^{(4)}(F)/K_4^{(2)}(F)\cup K_2^{(2)}(F)$.

COROLLARY 4.9. *Using Remark 4.4, the map*

$$\varphi^2_{(4)}: H^2(\mathcal{M}^\bullet_{(4)}(C))\to K_6^{(4)}(C)/K_2^{(2)}(F)\cup K_2^{(2)}(F)+K_5^{(3)}(k)\cup F^*_{\mathbb{Q}}$$

in Corollary 4.8 can be lifted to a map

$$\varphi^2_{(4)}: H^2(\mathcal{M}^\bullet_{(4)}(C))\to K_6^{(4)}(C)/K_4^{(2)}(F)\cup K_2^{(2)}(F)$$

by changing $\varphi^2_{(4)}(\alpha)$ *with elements in* $K_5^{(3)}(k)\cup F^*_{\mathbb{Q}}$.

Large parts of the proof of Theorem 4.6 work for general n, and we give most of it in this context. However, although the method employed probably works for all n, the combinatorics at a certain stage get rather out of hand, so we restrict our attention to $n=3$ at some point.

Starting with $\alpha=\sum_j c_j[f_j]_n\otimes g_j$ in $H^2(\mathcal{M}^\bullet_{(n+1)}(F))$ we begin with creating an element α_1 in $K_{n+1}^{(n+1)}(X^{n-1}_{F,\mathrm{loc}};\square^{n-1})$. Let $\{A_1,\ldots,A_l\}$ in F^* be a basis of $\langle f_j,g_j\rangle\subset F^*_{\mathbb{Q}}$ obtained by first choosing a basis of $\langle f_j\rangle$ among the f_j's and then extending to a basis of $\langle f_j,g_j\rangle$. Write

$$f_j=\prod_k A_k^{s_{kj}}\qquad\text{and}\qquad g_j=\prod_k A_k^{t_{kj}}\tag{4.1}$$

in $F^*_{\mathbb{Q}}$. Let

$$F_j(t)=\frac{t-f_j}{t-1}\prod\left(\frac{t-A_k}{t-1}\right)^{-s_{kj}}\in(1+I)^*.$$

Let $J=(i_1i_2\cdots i_k)$ with all $i_j\in\{1,\ldots,n-1\}$ be a sequence of distinct elements, and let $J^{\mathrm{ord}}=(j_1j_2\cdots j_k)$ be the ordered version of J, i.e., J and J^{ord} have the same elements, and $j_1<j_2<\cdots<j_k$. We shall write $(-1)^J$ for the sign of the permutation $\binom{i_1\,\cdots\,i_k}{j_1\,\cdots\,j_k}$, and $(-1)^{j\in J}$ to mean $(-1)^l$ if $j=i_l$. If J_1 and J_2 are disjoint tuples, we write J_1J_2 for their juxtaposition. If $j\in J$, write $J\setminus\{j\}$ for the $(|J|-1)$-tuple obtained by deleting j from J. Note that if $1,\ldots,n-1$ are all in J then, by adding j into the j-th position of $J\setminus\{j\}$, and moving it up front and then to its position in J, we find

$$(-1)^{J\setminus\{j\}}=(-1)^{j-1}(-1)^{(j)J\setminus\{j\}}=(-1)^j(-1)^{j\in J}(-1)^J.\tag{4.2}$$

For a set $I\subset\{1,\ldots,n-1\}$ we identify I with the ordered tuple it defines by ordering its elements, and similarly for its complement $I_c=\{1,\ldots,n-1\}\setminus I$.

If $I = \{i_1 \dots, i_k\}$ and $I_c = \{j_1, \dots, j_{n-1-k}\}$ are as above, let $F_j^I = F_j(t_{i_1}) \cup \dots \cup F_j(t_{i_k})$ and $[f_j]^{I_c}_{|I_c|+1} = [f_j]_{n-k}$ seen as element of $K^{(n-k)}_{n-k}(X^{n-1-k}_{\mathrm{loc}}; \square^{n-1-k})$ with coordinates $t_{j_1}, \dots, t_{j_{n-1-k}}$. Then we let

$$\begin{aligned} &\mathrm{d}_i^A F_j^I \cup [f_j]^{I_c}_{n-1-k} \cup g_j \\ &\quad = \text{ contribution to the boundary at } t_i = A_k\text{'s coming from the } F_j \\ &\quad = (-1)^{i \in II_c} \sum_k s_{kj} F_j^{I\setminus\{i\}} \cup [f_j]^{I_c}_{|I_c|+1} \cup g_{j|t_i = A_k} \end{aligned}$$

(noting that $(-1^{i\in II_c})$ if $i \in I)$ if $i \in I$ and zero otherwise. Similarly, we define

$$\begin{aligned} &\mathrm{d}_i^f F_j^I \cup [f_j]^{I_c}_{|I_c|+1} \cup g_j \\ &\quad = \text{ contribution to the boundary at } t_i = f_j\text{'s coming from the } F_j \\ &\quad = -(-1)^{i \in II_c} F_j^{I\setminus\{i\}} \cup [f_j]^{I_c}_{|I_c|+1} \cup g_{j|t_i = f_j} \end{aligned}$$

if $i \in I$ and zero otherwise, and

$$\begin{aligned} &\mathrm{d}_i^{[]} F_j^I \cup [f_j]^{I_c}_{|I_c|+1} \cup g_j \\ &\quad = \text{ contribution to the boundary at } t_i = f_j\text{'s coming from the } [f]_{|I_c|} \\ &\quad = (-1)^{i \in II_c} F_j^I \cup [f_j]^{I_c \setminus \{i\}}_{|I_c\setminus\{i\}|+1} \cup g_{j|t_i = f_j}, \end{aligned}$$

(because $(-1)^{|I|}(-1)^{i\in I_c} = (-1)^{i\in II_c}$ if $i \in I_c$) if $i \notin I$ and zero otherwise. Note that $\mathrm{d}_i = \mathrm{d}_i^A + \mathrm{d}_i^f + \mathrm{d}_i^{[]}$. In the commutative diagram

$$\begin{array}{ccc} K_n^{(n+1)}(X^n; \square^n) & \xrightarrow{\cong} & K_{n+1}^{(n+1)}(X^{n-1}; \square^{n-1}) \\ \downarrow & & \downarrow \\ K_n^{(n+1)}(X \times X^{n-1}_{\mathrm{loc}}; \square^n) & \xrightarrow{\cong} & K_{n+1}^{(n+1)}(X^{n-1}_{\mathrm{loc}}; \square^{n-1}) \end{array}$$

we know that $\varphi^2_{(n+1)}$ maps $\sum_j c_j [f_j]_n \otimes g_j$ to $\pm \sum_j c_j [f_j]_n \cup g_j$ (modulo $(1+I)^* \tilde{\cup} K_n^{(n)}(X^{n-2}_{\mathrm{loc}}; \square^{n-2})$) in $K^{(n+1)}_{n+1}(X^{n-1}_{\mathrm{loc}}; \square^{n-1})$, see Theorem 2.4. The complex (from a spectral sequence analogous to (2.1))

$$K^{(n+1)}_{n+1}(X^{n-1}_{\mathrm{loc}}; \square^{n-1}) \to \coprod K_n^{(n)}(X^{n-2}_{\mathrm{loc}}; \square^{n-2}) \to \dots \to K_2^{(2)}(F)$$

has the acyclic subcomplex

$$\begin{aligned} &(1+I)^* \tilde{\cup} K_n^{(n)}(X^{n-2}_{\mathrm{loc}}; \square^{n-2}) \to \\ &\quad \mathrm{d}(\cdots) + \coprod (1+I)^* \tilde{\cup} K_{n-1}^{(n-1)}(X^{n-3}_{\mathrm{loc}}; \square^{n-3}) \to \cdots \\ &\quad \cdots \to \mathrm{d}(\cdots) + \coprod (1+I)^* \tilde{\cup} K_2^{(2)}(F) \to \mathrm{d}(\cdots) \end{aligned}$$

with quotient complex

$$\frac{K^{(n+1)}_{n+1}(X^{n-1}_{\mathrm{loc}}; \square^{n-1})}{(1+I)^* \tilde{\cup} K_n^{(n)}(X^{n-2}_{\mathrm{loc}}; \square^{n-2})} \to \frac{K_n^{(n)}(X^{n-2}_{\mathrm{loc}}; \square^{n-2})}{(1+I)^* \tilde{\cup} K_{n-1}^{(n-1)}(X^{n-3}_{\mathrm{loc}}; \square^{n-3})} \otimes F^*_{\mathbb{Q}} \to \dots \tag{4.3}$$

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(The proof that the subcomplex is acyclic is completely analogous to the proof of Lemma 3.7 in [7] which is based on the fact that the subcomplex is closed under multiplication by $(1+I)^*$ and every element in the subcomplex contains at least one factor in $(1+I)^*$, see also [7, Remark 3.10].) We know that $\alpha = \sum_j c_j[f_j]_n \otimes g_j$ is mapped to zero under the map in (4.3), and want to lift it back (uniquely because of the acyclicity of the subcomplex) to α_1 in the kernel of $K_{n+1}^{(n+1)}(X_{\mathrm{loc}}^{n-1}; \square^{n-1}) \to \coprod K_n^{(n)}(X_{\mathrm{loc}}^{n-2}; \square^{n-2})$. Note that this lift is the restriction to $K_{n+1}^{(n+1)}(X_{\mathrm{loc}}^{n-1}; \square^{n-1})$ of the image under $\varphi_{(n+1)}^2$ in $K_{n+1}^{(n+1)}(X_{\mathrm{loc}}^{n-1}; \square^{n-1})$. The same proof works over some suitable Zariski open part of C.

PROPOSITION 4.10. *If*

$$\sum_j c_j[f_j]_n \cup g_j \in \frac{K_{n+1}^{(n-1)}(X_{\mathrm{loc}}^{n-1}; \square^{n-1})}{(1+I)^*\tilde{\cup} K_n^{(n)}(X_{\mathrm{loc}}^{n-2}; \square^{n-2})}$$

has trivial boundary $\sum_j c_j[f_j]_{n-1} \cup g_j \otimes f_j$ *in*

$$\frac{K_n^{(n)}(X_{\mathrm{loc}}^{n-2}; \square^{n-2})}{(1+I)^*\tilde{\cup} K_{n-1}^{(n-1)}(X_{\mathrm{loc}}^{n-3}; \square^{n-3})} \otimes F_{\mathbb{Q}}^*$$

(resp. $K_2^{(2)}(F) \otimes F_{\mathbb{Q}}^*$ *for* $n=2$*) then*

$$\alpha_1 = \sum_{I \subset \{1,\ldots,n-1\}} (-1)^{II_c} \sum_j c_j F_j^I \cup [f_j]_{n-|I|}^{I_c} \cup g_j$$

in $K_{n+1}^{(n+1)}(X_{\mathrm{loc}}^{n-1}; \square^{n-1})$ *has trivial boundary in* $\coprod K_n^{(n)}(X_{\mathrm{loc}}^{n-2}; \square^{n-2})$*. Here the sum is over all subsets* I *of* $\{1, \ldots, n-1\}$ *seen as tuples in ascending order.*

Proof. The proof will be by induction on n. We need the following lemma.

LEMMA 4.11. *With* $f_j = \prod_k A_k^{s_{kj}}$ *as before,*

$$\sum_j c_j s_{kj}[f_j]_{n-1} \cup g_j = 0 \quad \text{in} \quad \frac{K_n^{(n)}(X_{\mathrm{loc}}^{n-2}; \square^{n-2})}{(1+I)^*\tilde{\cup} K_{n-1}^{(n-1)}(X_{\mathrm{loc}}^{n-3}; \square^{n-3})}$$

(resp. $K_2^{(2)}(F) \otimes F_{\mathbb{Q}}^*$ *for* $n=2$*).*

Proof. Write the boundary in

$$\frac{K_n^{(n)}(X_{\mathrm{loc}}^{n-2}; \square^{n-2})}{(1+I)^*\tilde{\cup} K_{n-1}^{(n-1)}(X_{\mathrm{loc}}^{n-3}; \square^{n-3})} \otimes F_{\mathbb{Q}}^*$$

in terms of $\cdots \otimes A_k$'s, and collect terms, remembering that the A_k's form a basis of $\langle f_j, g_j \rangle \subset F_{\mathbb{Q}}^*$, so they are independent in $F_{\mathbb{Q}}^*$.

We now compute the boundary of α_1, doing it for all t_i. For $n=2$ one checks easily that α_1 has boundary $\sum_{j,k} c_j s_{kj}(1-f_j) \cup g_{j|t=A_k}$, which is zero as one sees by writing out the boundary of $\sum_j c_j[f_j]_2 \cup g_j$ in terms of our chosen basis for $\langle f_j, g_j\rangle \subset F_{\mathbb{Q}}^*$. For the higher n's, as $\mathrm{d}_i = \mathrm{d}_i^A + \mathrm{d}_i^f + \mathrm{d}_i^{[\,]}$, we get three contributions. We start with the d_i^A-component. We find

$$\sum_{\substack{I\subset\{1,\ldots,n-1\}\\ i\in I}} (-1)^{i\in II_c}(-1)^{II_c}\sum_{j,k} c_j s_{kj} F^{I\setminus\{i\}} \cup [f_j]^{I_c}_{|I_c|+1} \cup g_{j\,|t_i=A_k}$$
$$= (-1)^i \sum_{J\subset\{1,\ldots,\hat{\imath},\ldots,n-1\}} (-1)^{JJ_c}\sum_{j,k} c_j s_{kj} F_j^J \cup [f_j]^{J_c}_{|J_c|+1} \cup g_{j\,|t_i=A_k}$$

by letting $J = I\setminus\{i\}$, and taking J_c in $\{1,\ldots,\hat{\imath},\ldots,n-1\}$, and using (4.2). By induction on n and Lemma 4.11, this equals zero.

For the contribution from d_i^f we get

$$-\sum_{\substack{I\subset\{1,\ldots,n-1\}\\ i\in I}} (-1)^{i\in II_c}(-1)^{II_c}\sum_{j} c_j F^{I\setminus\{i\}} \cup [f_j]^{I_c\setminus\{i\}}_{|I_c\setminus\{i\}|+1} \cup g_{j\,|t_i=f_j}$$
$$= -(-1)^i \sum_{J\subset\{1,\ldots,\hat{\imath},\ldots,n-1\}} (-1)^{JJ_c}\sum_{j} c_j F_j^J \cup [f_j]^{J_c}_{|J_c|+1} \cup g_{j\,|t_i=A_k}$$

again by letting $J = I\setminus\{i\}$, and taking J_c in $\{1,\ldots,\hat{\imath},\ldots,n-1\}$, and using (4.2).

For $\mathrm{d}_i^{[\,]}$ we get a contribution

$$\sum_{\substack{I\subset\{1,\ldots,n-1\}\\ i\notin I}} (-1)^{i\in II_c}(-1)^{II_c}\sum_{j} c_j F^{I} \cup [f_j]^{I_c\setminus\{i\}}_{|I_c\setminus\{i\}|+1} \cup g_{j\,|t_i=f_j}$$
$$= (-1)^i \sum_{J\subset\{1,\ldots,\hat{\imath},\ldots,n-1\}} (-1)^{JJ_c}\sum_{j} c_j F^J \cup [f_j]^{J_c}_{|J_c|+1} \cup g_{j\,|t_i=f_j}$$

with $J = I \subset \{1,\ldots,\hat{\imath},\ldots,n-1\}$, and taking J_c in $\{1,\ldots,\hat{\imath},\ldots,n-1\}$ as before, and using (4.2) again. Obviously, the contributions of d_i^f and $\mathrm{d}_i^{[\,]}$ cancel.

Because $k(x)$ is a number field and the regulator is injective (up to torsion) on its K-theory, we can compute the boundary at the level of Deligne cohomology, so we now turn towards the regulator level. Consider the following commutative diagram with horizontal maps being the regulators (into de Rham cohomology as it is equal to the Deligne cohomology in all cases considered).

$$\begin{array}{ccc}
H^2(\widetilde{\mathcal{M}}^\bullet_{(n+1)}(F)) & & \\
\downarrow & & \\
K_n^{(n+1)}(X^n;\square^n) & \longrightarrow & H^{n+1}_{\mathrm{dR}}(X^n;\square^n;\mathbb{R}(n+1))^+ \\
\downarrow\wr & & \downarrow\wr \\
K_{n+1}^{(n+1)}(X^{n-1};\square^{n-1}) & \longrightarrow & H^{n}_{\mathrm{dR}}(X^{n-1};\square^{n-1};\mathbb{R}(n))^+ \\
\downarrow & & \downarrow \\
K_{n+1}^{(n+1)}(X^{n-1}_{F,\mathrm{loc}};\square^{n-1}) & \longrightarrow & H^{n}_{\mathrm{dR}}(X^{n-1}_{U,\mathrm{loc}};\square^{n-1};\mathbb{R}(n))^+
\end{array}$$

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We shall determine the regulator of α_1 in $H^n_{\mathrm{dR}}(X^{n-1}_{U,\mathrm{loc}};\square^{n-1};\mathbb{R}(n))^+$and then lift it back to $H^n_{\mathrm{dR}}(X^{n-1}_U;\square^{n-1};\mathbb{R}(n))^+$. We begin with finding the indeterminacy in the lift. In order to simplify notation, write X_{loc} for$X_{U,\mathrm{loc}} = X_U\setminus\{t=f_j\}$ with U a suitable Zariski open part as before, $H^p_{\mathrm{dR}}(X^q_{\mathrm{loc}};s)$ for $H^p_{\mathrm{dR}}(X^q_{\mathrm{loc}};\square^q;\mathbb{R}(s))$, and consider the spectral sequence

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H^n_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}};n) & \coprod H^{n-1}_{\mathrm{dR}}(X^{n-2}_{\mathrm{loc}};n-1) & \coprod H^{n-2}_{\mathrm{dR}}(X^{n-3}_{\mathrm{loc}};n-2) & \cdots \\
H^{n-1}_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}};n) & \coprod H^{n-2}_{\mathrm{dR}}(X^{n-2}_{\mathrm{loc}};n-1) & \coprod H^{n-3}_{\mathrm{dR}}(X^{n-3}_{\mathrm{loc}};n-2) & \cdots \\
H^{n-2}_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}};n) & \coprod H^{n-3}_{\mathrm{dR}}(X^{n-2}_{\mathrm{loc}};n-1) & \coprod H^{n-4}_{\mathrm{dR}}(X^{n-3}_{\mathrm{loc}};n-2) & \cdots \\
\vdots & \vdots & \vdots &
\end{array} \tag{4.4}$$

converging to $H^*_{\mathrm{dR}}(X^{n-1};\mathbb{R}(n))$.

LEMMA 4.12. $H^n_{\mathrm{dR}}(X^k_{U,\mathrm{loc}};\square^k)=0$ if $n<k$.

Proof. For $k=0$ this is obvious. For $k\geqslant 1$ we have a spectral sequence (with notation as in (4.4))

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H^n_{\mathrm{dR}}(X^k_{\mathrm{loc}};j) & \coprod H^{n-1}_{\mathrm{dR}}(X^{k-1}_{\mathrm{loc}};j-1) & \coprod H^{n-2}_{\mathrm{dR}}(X^{k-2}_{\mathrm{loc}};j-2) & \cdots \\
H^{n-1}_{\mathrm{dR}}(X^k_{\mathrm{loc}};j) & \coprod H^{n-2}_{\mathrm{dR}}(X^{k-1}_{\mathrm{loc}};j-1) & \coprod H^{n-3}_{\mathrm{dR}}(X^{k-2}_{\mathrm{loc}};j-2) & \cdots \\
H^{n-2}_{\mathrm{dR}}(X^k_{\mathrm{loc}};j) & \coprod H^{n-3}_{\mathrm{dR}}(X^{k-1}_{\mathrm{loc}};j-1) & \coprod H^{n-2}_{\mathrm{dR}}(X^{k-2}_{\mathrm{loc}};j-2) & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

converging to $H^*_{\mathrm{dR}}(X^k_U;\square^k;\mathbb{R}(j))$. We see that the only contributions to $H^n_{\mathrm{dR}}(X^k_U;\square^k;\mathbb{R}(j))$ will come from $H^{n-2p}_{\mathrm{dR}}(X^{k-p}_{U,\mathrm{loc}};\square^{k-p})$'s, which are zero by induction for $p\geqslant 1$. The boundaries leaving $H^n_{\mathrm{dR}}(X^k_{U,\mathrm{loc}};\square^j)$ land in $H^{n-2p+1}_{\mathrm{dR}}(X^{k-p}_{U,\mathrm{loc}};\square^{k-p};\mathbb{R}(j-p))$'s for $p\geqslant 1$, which are also zero by induction. Therefore we get isomorphisms

$$H^n_{\mathrm{dR}}(X^k_{U,\mathrm{loc}};\square^k;\mathbb{R}(j)) \cong H^n_{\mathrm{dR}}(X^k_U;\square^k;\mathbb{R}(j)) \cong H^{n-k}_{\mathrm{dR}}(U;\mathbb{R}(j))=0.$$

Lemma 4.12 shows that in (4.4) there are only two terms contributing to $H^n_{\mathrm{dR}}(X^{n-1};\square^{n-1};\mathbb{R}(n))$, so we have a short exact sequence

$$0\to E_2^+\to H^n_{\mathrm{dR}}(X^{n-1};\square^{n-1};\mathbb{R}(n))^+\to E_3^+\to 0$$

with E_3^+ the $+$ part of the $E_\infty=E_3$ term at the position in the spectral sequence of $H^n_{\mathrm{dR}}(X^{n-1}_{U,\mathrm{loc}};\square^{n-1};\mathbb{R}(n))$, and E_2^+ the $+$ part of the $E_\infty=E_2$ term at the $\coprod H^{n-2}_{\mathrm{dR}}(X^{n-2}_{U,\mathrm{loc}};\square^{n-2};\mathbb{R}(n-1))$ position. Because $H^n_{\mathrm{dR}}(X^{n-1};\square^{n-1};\mathbb{R}(n))^+$ is alternating for the action of S_{n-1} (acting on everything by permuting the coordinates), E_2^+ and E_3^+ are alternating as well. As we are looking at the regulator of an element coming from X_U^{n-1}, it of course survives in the spectral sequence to E_∞ and we can consider its projection in E_3^+. We move on to determining the E_2-term. For this we introduce

$$\mathbb{R}_c = \left\langle \mathrm{d}\, i \arg \prod_j \left(\frac{t-f_j}{t-1} \right)^{n_j} \text{such that} \prod_j f_j \in k^* \right\rangle_{\mathbb{R}}$$

inside $H^1_{\mathrm{dR}}(X_{U,\mathrm{loc}}; \square; \mathbb{R}(1))^+$. Note that an element in $\mathbb{R}c$ is determined completely by its residue at the $t - f_j$'s.

LEMMA 4.13. *The map*

$$H^0_{\mathrm{dR}}(U; \mathbb{R}(0))_{|t=f} \to H^2_{\mathrm{dR}}(X_U; \square\,; \mathbb{R}(1)) \cong H^1_{\mathrm{dR}}(U; \mathbb{R}(1))$$

maps 1 *to* $\pm \mathrm{d}\, i \arg f$.

Proof. Consider the situation $U = \mathbb{G}_m$ and $f = S$ (S the coordinate on $\mathbb{G}_m$). We then have the exact sequence in relative de Rham cohomology

$$\cdots \to H^0(X_{\mathrm{loc}}) \to H^0(U)^{\oplus 2} \to H^1(X_{\mathrm{loc}}; \square) \to H^1(X_{\mathrm{loc}}) \to H^1(U)^{\oplus 2} \to \cdots$$

As $H^1(X_{\mathrm{loc}}) = \langle \mathrm{d} i \arg(t-S)/(t-1) \rangle_{\mathbb{R}} \oplus \langle \mathrm{d} i \arg S \rangle_{\mathbb{R}}$, the last map in the above sequence is injective. From the corresponding sequence with X instead of X_{loc} one then gets that

$$H^1(X; \square\,) \cong H^1(X_{\mathrm{loc}}; \square)$$

as $H^0(X_{\mathrm{loc}}) \cong H^0(X)$. Consider the localization sequence

$$H^1(X; \square) \xrightarrow{\sim} H^1(X_{\mathrm{loc}}; \square) \to H^0(U) \to H^2(X_U; \square) \to H^2(X_{\mathrm{loc}}; \square).$$

Note that $H^2(X_{U,\mathrm{loc}}; \,) \xrightarrow{\sim} H^2(X_{U,\mathrm{loc}})$ because $H^2(\square) = 0$. From the commutative diagram

$$\begin{array}{ccc} H^2(X_U; \square\,) & \longrightarrow & H^2(X_U) \\ \downarrow & & \downarrow \\ H^2(X_{U,\mathrm{loc}}; \square\,) & \xrightarrow{\cong} & H^2(X_{U,\mathrm{loc}}) \end{array}$$

we see that the map $H^2(X_U; \square) \to H^2(X_{U,\mathrm{loc}}; \square)$ is the zero map because $H^2(X_U) \cong H^2(U) = 0$. Hence $H^0(U) \xrightarrow{\sim} H^2(X_U; \square) \xrightarrow{\sim} H^1(U)$. All this works with cohomology with $\mathbb{Z}$-coefficients, which gives the statement for $\mathbb{G}_m$. By pulling back to our original U via f we get the corresponding statement for f and U.

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LEMMA 4.14.

$$H^n(X^n_{U,\mathrm{loc}};\square^n;\mathbb{R}(n))^{\mathrm{alt}}\cong\left(\bigoplus_{k=0}^{n}R_c^k\cup H^{n-k}(X_k^{n-k};\square^{n-k};\mathbb{R}(n-k))\right)^{\mathrm{alt}}$$

Proof. For $n=0$ this is obvious, or for $n=1$ consider the localization sequence

$$0\to H^1_{\mathrm{dR}}(X_U;\square;\mathbb{R}(1))\to H^1_{\mathrm{dR}}(X_{U,\mathrm{loc}};\square;\mathbb{R}(1))\to$$

$$\coprod_j H^0_{\mathrm{dR}}(U;\mathbb{R}(0))_{|t=f_j}\xrightarrow{\varphi}H^2(X_U;\square;\mathbb{R}(1))\to\cdots$$

The map $H^0_{\mathrm{dR}}(U;\mathbb{R}(0))_{|t=f_j}\to H^2_{\mathrm{dR}}(X_U;\square;\mathbb{R}(1))\cong H^1_{\mathrm{dR}}(U;\mathbb{R}(1))$ maps 1 to $\pm\mathrm{d}\,i\arg f_j$ by Lemma 4.13. Hence $\coprod_j a_j$ is in the kernel of φ if and only if $\sum_j a_j\mathrm{d}\,i\arg f_j=0$, which means that $\coprod_j a_j$ is the image of $\sum_j a_j\mathrm{d}i\arg\frac{t-f_j}{t-1}$ in $H^1_{\mathrm{dR}}(X_{U,\mathrm{loc}};\square;\mathbb{R}(1))$. So if we show this is in R_c, we are done. We have an exact sequence

$$0\to k^*\to F^*\to\{\mathrm{d}i\arg f_j\}$$

as one sees by considering the residue versus the divisor map. Tensoring with $\mathbb{R}$ we get

$$0\to k^*\otimes_{\mathbb{Z}}\mathbb{R}\to F^*\otimes_{\mathbb{Z}}\mathbb{R}\to\langle\mathrm{d}i\arg f_j\rangle_{\mathbb{R}}$$

from which it follows that $\sum_j a_j\mathrm{d}i\arg\frac{t-f_j}{t-1}$ is in R_c, e.g., by considering a $\mathbb{Q}$-basis of $\mathbb{R}$. Because R_c injects into $\coprod_j H^0_{\mathrm{dR}}(U;\mathbb{R}(0))_{|t=f_j}$ under the residue, we get the statement for $n=1$.

For $n\geqslant 2$, we use induction. We have a spectral sequence (with notation as in (4.4))

$$\begin{array}{ccc}
\vdots & \vdots & \vdots\\
H^n_{\mathrm{dR}}(X^n_{\mathrm{loc}};n)^{\mathrm{alt}} & \left(\coprod H^{n-1}_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}};n-1)\right)^{\mathrm{alt}} & \left(\coprod H^{n-2}_{\mathrm{dR}}(X^{n-2}_{\mathrm{loc}};n-2)\right)^{\mathrm{alt}}\\
H^{n-1}_{\mathrm{dR}}(X^n_{\mathrm{loc}};n)^{\mathrm{alt}} & \left(\coprod H^{n-2}_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}};n-1)\right)^{\mathrm{alt}} & \left(\coprod H^{n-3}_{\mathrm{dR}}(X^{n-2}_{\mathrm{loc}};n-2)\right)^{\mathrm{alt}}\\
H^{n-2}_{\mathrm{dR}}(X^n_{\mathrm{loc}};n)^{\mathrm{alt}} & \left(\coprod H^{n-3}_{\mathrm{dR}}(X^{n-1}_{\mathrm{loc}};n-1)\right)^{\mathrm{alt}} & \left(\coprod H^{n-4}_{\mathrm{dR}}(X^{n-2}_{\mathrm{loc}};n-2)\right)^{\mathrm{alt}}\\
\vdots & \vdots & \vdots
\end{array}$$

converging to $H^*_{\mathrm{dR}}(X^n_{\mathrm{loc}};\square^n;\mathbb{R}(n))^{\mathrm{alt}}\cong H^*_{\mathrm{dR}}(X^n_{\mathrm{loc}};\square^n;\mathbb{R}(n))$. Introducing the notation H^k_{loc} for $H^k_{\mathrm{dR}}(X^k_{U,\mathrm{loc}};\square^k;\mathbb{R}(k))$, by Lemma 4.12 everything below the line

$$H^n_{\mathrm{loc}}\to\left(\coprod H^{n-1}_{\mathrm{loc}}\right)^{\mathrm{alt}}\to\left(\coprod H^{n-2}_{\mathrm{loc}}\right)^{\mathrm{alt}}\to\cdots\qquad(4.5)$$

vanishes. Write H^k for $H^k_{\mathrm{dR}}(X^k_{U,\mathrm{loc}};\square^k;\mathbb{R}(k))^{\mathrm{alt}}$. The subcomplex of the correspond-

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ing non-alternating row given by

$$\oplus_{k=1}^{n} R_c^k \tilde{\cup} H^{n-k} \to \mathrm{d}(\ldots) + \coprod \oplus_{k=1}^{n-1} R_c^k \tilde{\cup} H^{n-1-k} \to \\ \mathrm{d}(\ldots) + \coprod \oplus_{k=1}^{n-2} R_c^k \tilde{\cup} H^{n-2-k} \to \cdots \tag{4.6}$$

is acyclic as in the proof of Lemma 3.7 of [7] (because it is closed under multiplication by elements in R_c and the boundary is injective on R_c), so hence is its alternating part. Taking the quotient of the complex in (4.5) and the alternating part of its subcomplex (4.6) yields by induction on n the row

$$\left(\frac{H^n_{\mathrm{dR}}(X^n_{U,\mathrm{loc}}; \square^n; \mathbb{R}(n))}{\oplus_{k=1}^{n} R_c^k \tilde{\cup} H^{n-k}}\right)^{\mathrm{alt}} \to \\ H^{n-1}_{\mathrm{dR}}(X^{n-1}_k; \square^{n-1}; \mathbb{R}(n-1)) \otimes_{\mathbb{Q}} F^*_{\mathbb{Q}}/k^*_{\mathbb{Q}} \to \cdots \tag{4.7}$$

Obviously the last map is zero. Because the composition

$$H^{n-1}_{\mathrm{dR}}(X^{n-1}_k; \square^{n-1}; \mathbb{R}(n-1)) \otimes_{\mathbb{Q}} F^*_{\mathbb{Q}}/k^*_{\mathbb{Q}} \to H^{n+1}_{\mathrm{dR}}(X^n_U; \square^n; \mathbb{R}(n))^{\mathrm{alt}} \\ \cong H^1(U; \mathbb{R}(n))$$

maps $c \wedge \mathrm{d}h(t_1) \wedge \cdots \wedge \mathrm{d}h(t_{n-1}) \otimes f_j$ to $c\mathrm{d}i \arg f_j$ this is an injection. So the first map in (4.7) must also be zero, giving an identification

$$H^n_{\mathrm{dR}}(X^n_k; \square^n; \mathbb{R}(k))^{\mathrm{alt}} \cong H^n_{\mathrm{dR}}(X^n_U; \square^n; \mathbb{R}(k))^{\mathrm{alt}} \\ \cong \left(\frac{H^n_{\mathrm{dR}}(X^n_{U,\mathrm{loc}}; \square^n; \mathbb{R}(n))}{\oplus_{k=1}^{n} R_c^k \cup H^{n-k}}\right)^{\mathrm{alt}}$$

from which the result is immediate because $\oplus_{k=1}^{n} R_c^k \cup H^{n-k}$ injects under the residue into $\coprod H^{n-1}$.

Remark 4.15. The proof shows that $E_2 = E_2^{\mathrm{alt}}$ is generated by the $H^{n-2}_{\mathrm{dR}}(X^{n-1}_{\mathbb{C}}; \square^{n-1}; \mathbb{R}(n-1))_{|t_i=f_j}$. Hence by Lemma 4.13, a lift from $E_3 = E_3^{\mathrm{alt}}$ to $H^n_{\mathrm{dR}}(X^{n-1}_U; \square^{n-1}; \mathbb{R}(n))^{\mathrm{alt}} \cong H^1_{\mathrm{dR}}(U; \mathbb{R}(n))$ is determined up to $\mathbb{R}(n-1)\langle \mathrm{d}\, i \arg f_j \rangle_{\mathbb{R}} \wedge \mathrm{d}h(t_1) \wedge \cdots \wedge \mathrm{d}h(t_{n-1})$, corresponding to $\mathbb{R}(n-1)\langle \mathrm{d}\, i \arg f_j \rangle_{\mathbb{R}}$ under this isomorphism.

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Note that we can compute the residue as follows. There is a commutative diagram

$$\begin{array}{ccc} K_{n+1}^{(n+1)}(X_U^{n-1};\square^{n-1}) & \longrightarrow & K_n^{(n)}(X_{k(x)}^{n-1};\square^{n-1}) \\ \downarrow \text{reg} & & \downarrow \text{reg} \\ H^n_{\text{dR}}(X_U^{n-1};\square^{n-1};\mathbb{R}(n)) & \longrightarrow & H^{n-1}_{\text{dR}}(X_{k(x)}^{n-1};\square^{n-1};\mathbb{R}(n)) \\ \downarrow \wr & & \downarrow \wr \\ H^1_{\text{dR}}(U;\mathbb{R}(n)) & \longrightarrow & H^0_{\text{dR}}(k(x);\mathbb{R}(n-1)) \end{array}$$

If ψ is in $H^1_{\text{dR}}(U; \mathbb{R}(n))$ and x is a point not in U, then $\text{res}_x(\psi)$ in $H^0_{\text{dR}}(k(x); \mathbb{R}(n-1))$ is given by

$$\pm\frac{1}{(2\pi i)^n}\int_{X^{n-1}\times S^1_x} \psi\wedge \mathrm{d}h(t_1)\wedge\cdots\wedge \mathrm{d}h(t_{n-1})\wedge \mathrm{d}\, i\arg t_1\wedge\cdots\wedge \mathrm{d}\, i\arg t_{n-1}$$

for S^1_x a circle around x.

We can also replace U with the closed set of C by leaving out small (open) discs around the point $x\notin U$, without changing either the cohomology groups involved or the values of the integrals. We shall assume that from now on, so in particular U is compact.

LEMMA 4.16. *Suppose ψ_1 and ψ_2 in $H^{n+1}_{\text{dR}}(X_U^{n-1};\square^{n-1};\mathbb{R}(n-1))$ satisfy condition (2.7). Then with $\overline{\varpi}=\mathrm{d}\log\overline{t_1}\wedge\cdots\wedge \mathrm{d}\log\overline{t_{n-1}}$, we have an equality*

$$\int_{X^{n-1}\times S^1_x}\psi_1\wedge\overline{\varpi}=\int_{X^{n-1}\times S^1_x}\psi_2\wedge\overline{\varpi}$$

The same holds if we replace $\overline{\varpi}$ with $\mathrm{d}\, i\arg t_1\wedge\cdots\wedge \mathrm{d}\, i\arg t_{n-1}$.

Proof. The proof of Proposition 4.6 of [8] shows that $\psi_1-\psi_2=\mathrm{d}\gamma$, where γ satisfies the conditions in (2.7) on a suitable blowup of $(\mathbb{P}^1_C)^{n-1}$, isomorphic to this over a suitable Zariski open part of C. With that, one checks easily using integration in each fibre, that $\int_{X^{n-1}\times S^1_x}\mathrm{d}\gamma\wedge\overline{\varpi}=0$ as the holomorphic form has a zero along $t_i=1$ for every i. Hence the result follows from Stokes' theorem.

Remark 4.17. Note that if we represent the image of E_2^+ inside $H^n_{\text{dR}}(X_U^{n-1};\square^{n-1};\mathbb{R}(n))^+$ by forms given by

$$H^0_{\text{dR}}(U;\mathbb{R}(n-1))^+\cdot\langle \mathrm{d}\, i\arg f_j\rangle_{\mathbb{R}}\wedge \mathrm{d}h(t_1)\wedge\cdots\wedge \mathrm{d}h(t_{n-1}),$$

then

$$\coprod_{x\notin U}\int_{X^{n-1}\times S^1_x}\psi\wedge \mathrm{d}\, i\arg t_1\wedge\cdots\wedge \mathrm{d}\, i\arg t_{n-1}$$

converges and maps E_2^+ to $H^0_{\text{dR}}(U;\mathbb{R}(n-1))^+\cdot\text{res}\langle \mathrm{d} i\arg f_j\rangle$ inside $\coprod_{x\notin U}\mathbb{R}(n-1)$.

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Proof. The computation of E_2^+ was carried out in the proof of Lemma 4.14. It is generated by the pushforward of the alternating version of $H^{n-2}_{\mathrm{dR}}(X_k^{n-1};\square^{n-1};\mathbb{R}(n-1))_{|t_i=f_j}$. The rest is just a matter of integration and Lemma 4.13.

Remark 4.18. Note that by Borel's theorem (see (2.6)) the image of

$$H^0_{\mathrm{dR}}(U;\mathbb{R}(n-1))^+\cdot\mathrm{res}\langle \mathrm{d}i\arg f_j\rangle$$

is exactly $\coprod_{x\notin U}\int_{S^1}\mathrm{reg}(K^{(n)}_{2n-1}(k))\langle \mathrm{d}i\arg f_j\rangle_{\mathbb{R}}$.

We now have the regulator $\mathrm{reg}(\alpha_1)$ of α_1 in $H^n_{\mathrm{dR}}(X^{n-1}_{U,\mathrm{loc}};\square^{n-1};\mathbb{R}(n))$. If we lift it back to $\psi\in H^n_{\mathrm{dR}}(X_U^{n-1};\square^{n-1};\mathbb{R}(n))$ satisfying (2.7) then by Remark 4.17 and Lemma 4.16

$$-\frac{1}{(2\pi i)^n}\coprod_{x\notin U}\int_{X^{n-1}\times S^1_x}\psi\wedge \mathrm{d}\,i\arg t_1\wedge\cdots\wedge \mathrm{d}\,i\arg t_{n-1} \tag{4.8}$$

differs from the boundary

$$-\frac{1}{(2\pi i)^n}\coprod_{x\notin U}\int_{X^{n-1}\times S^1_x}\mathrm{reg}(\alpha)\wedge \mathrm{d}\,i\arg t_1\wedge\cdots\wedge \mathrm{d}\,i\arg t_{n-1} \tag{4.9}$$

by an element in

$$\frac{1}{2\pi i}\coprod_{x\notin U}\int_{S^1_x}\mathrm{reg}(K^{(n)}_{2n-1}(k))\langle \mathrm{d}i\arg f_j\rangle_{\mathbb{R}}.$$

Because we shall see that all values in (4.8) and necessarily (4.9) are in the image of $K^{(n)}_{2n-1}(k(x))$, it follows that $\psi=\mathrm{reg}(\alpha)+\mathrm{reg}(\beta)$ for some β in $\langle f_j\rangle\cup K^{(n)}_{2n-1}(k)$. From this we get Theorem 4.6.

We now turn our attention to the explicit lift of $\mathrm{reg}(\alpha_1)$, starting out in general, but specializing to the case $n=3$ at some stage.

LEMMA 4.19. *Suppose for* $k=0,\ldots,n-1$ *we have* $(n-1-k)$*-forms* $\varphi_k(t_{k+1},\ldots,t_{n-1})$, *with* $\varphi_{k-1}(0,t_{k+1},\ldots,t_{n-1})=\mathrm{d}\varphi_k(t_{k+1},\ldots,t_{n-1})$ *for* $k=1,\ldots,n-1$. *Assume moreover that each* φ_k *is alternating for the action of* S_{n-k-1}, *and* φ_k *vanishes for* $t_j=\infty$, $j=k+1,\ldots,n-1$. *Let* $\rho(t)$ *be a bump form around* $t=0$. *Then the form*

$$\sum_{I\subset\{1,\ldots,n-1\}}(-1)^{|I|}(-1)^{II_c}D^I\varphi_{|I|}(t^I_c)$$

is an alternating form vanishing at $t_j=0$ *for* $k=1,\ldots,n-1$. *Here the sum is over all ordered tuples* I *of* $\{1,\ldots,n-1\}$, *the complement* I_c *is given the ascending*

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ordering, $t^{I_c} = t_{j_1}, \cdots, t_{j_k}$ *if* $I_c = \{j_1, \ldots, j_k\}$ *with* $j_1 < \cdots < j_k$, *and* $D^J\varphi = \mathrm{d}\big(\rho(t_{j_1})\mathrm{d}\big(\rho(t_{j_2})\cdots\mathrm{d}\big(\rho(t_{j_k})\varphi\big)\cdots\big)\big)$ *if* $J = (j_1 \cdots j_k)$.

Proof. For σ in S_{n-1}, we have that

$$\begin{aligned}
&\sigma^*\big((-1)^{II_c} D^I \varphi_{|I|}(t^{I_c})\big)\\
&\quad = (-1)^{II_c} D^{\sigma(I)} \varphi_{|\sigma(I)|}(t^{\sigma(I_c)})\\
&\quad = (-1)^{\sigma}(-1)^{\sigma(I)\sigma(I_c)} D^{\sigma(I)} \varphi_{|\sigma(I)|}(t^{\sigma(I_c)})\\
&\quad = (-1)^{\sigma}(-1)^{\sigma(I)(\sigma(I))_c} D^{\sigma(I)} \varphi_{|\sigma(I)|}(t^{\sigma(I)_c})
\end{aligned}$$

by replacing $\sigma(I_c)$ with $(\sigma(I))_c$, i.e., ordering it. This shows the form is alternating. Note that $\big(D^I\varphi\big)_{|t_j=0} = 0$ unless $j \notin I$ or is its last element. Therefore, when restricting to $t_j = 0$ we get

$$\sum_{\substack{J\subset\{1,\ldots,\hat{j},\ldots,n-1\}\\ I=J(j)}} (-1)^{|J|+1}(-1)^{J(j)J_c{}^{\mathrm{ord}}} D^J \mathrm{d}\varphi_{|J|+1}(t^{J_c{}^{\mathrm{ord}}})+$$

$$+\sum_{\substack{J\subset\{1,\ldots,\hat{j},\ldots,n-1\}\\ I=J}} (-1)^{|J|}(-1)^{J((j)J_c)^{\mathrm{ord}}} D^J \varphi_{|J|}(t^{((j)J_c)^{\mathrm{ord}}})$$

with $t_j = 0$, and $J_c{}^{\mathrm{ord}}$ taken in $\{1, \ldots \hat{j}, \ldots, n-1\}$. Using (4.2) and the conditions on the φ_k's we have

$$\begin{aligned}
&(-1)^{J(j)J_c{}^{\mathrm{ord}}} = (-1)^j(-1)^{|J|+1}(-1)^{JJ_c{}^{\mathrm{ord}}}\\
&\mathrm{d}\varphi_{|J|+1}(t^{J_c{}^{\mathrm{ord}}}) = \varphi_{|J|}(0, t^{J_c{}^{\mathrm{ord}}})\\
&\varphi_{|J|}(t^{((j)J_c)^{\mathrm{ord}}})_{|t_j=0} = -(-1)^{j\in((j)J_c)^{\mathrm{ord}}}\varphi_{|J|}(0, t^{J_c{}^{\mathrm{ord}}})\\
&(-1)^{J((j)J_c)^{\mathrm{ord}}} = (-1)^j(-1)^{j\in((j)J_c)^{\mathrm{ord}}}(-1)^{|J|}(-1)^{JJ_c{}^{\mathrm{ord}}}
\end{aligned}$$

so everything cancels.

It turns out that for applying Lemma 4.19 with $\varphi_0 = \epsilon_n$, writing down the forms involved is quite messy. We therefore assume from now on that $n = 3$. (The case $n = 2$ was done before in [8].) In this case we shall carry out the lift explicitly.

We need some identities between forms (all f's, g's are functions on C).

$$\begin{aligned}
&\mathrm{d}i\arg f_1 \wedge \mathrm{d}i\arg f_2 = -\mathrm{d}\log|f_1| \wedge \mathrm{d}\log|f_2|\\
&\mathrm{d}i\arg f_1 \wedge \mathrm{d}\log|f_2| = -\mathrm{d}\log|f_1| \wedge \mathrm{d}i\arg f_2
\end{aligned}$$

Both identities follow by considering the real or imaginary parts of $\mathrm{d}\log f_1 \wedge \mathrm{d}\log f_2 = 0$. We let

$$\sigma(f_1, f_2) = \log|f_1|\mathrm{d}i\arg f_2 - \log|f_2|\mathrm{d}i\arg f_1,$$

so $\mathrm{d}\sigma(f_1, f_2) = 0$.

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Let $U = \mathbb{G}_m \setminus \{1\}$, and let $X_{U,\mathrm{loc}} = X_U \setminus \{t = S\}$. We want to write down the explicit elements $(\omega_n, \epsilon_n) \in H^n_{\mathcal{D}}(X^{n-1}_{\mathrm{loc},U}; \square^{n-1}; \mathbb{R}(n))^+$ that are the images of $[S]_n$ under the regulator for $n = 1$, 2 and 3, see Section 2.

For $n = 1$, we have $\tilde{\epsilon}_1 = \epsilon_1 = \log|1 - f|$ and $\omega_1 = \mathrm{d}\log(1 - f)$. For $n = 2$, we have

$$\omega_2 = -\mathrm{d}\log\frac{t-f}{t-1} \wedge \mathrm{d}\log(1-f)$$

To find ϵ_2, let

$$\tilde{\epsilon}_2 = \log|1-f|\mathrm{d}i\arg\left(\frac{t-f}{t-1}\right) - \log\left|\frac{t-f}{t-1}\right|\mathrm{d}i\arg(1-f)$$

and if we specialize this to $t = 0$, we find this equals $\mathrm{d}\eta_1^{(2)}$ with $\eta_1^{(2)} = -P_{2,\mathrm{Zag}}(f)$. Then $\epsilon_2 = \tilde{\epsilon}_2 - \mathrm{d}\left(\rho(t)\eta_1^{(2)}\right)$ with $\rho(t)$ a bump form around $t = 0$. (This is the correct ϵ_2, see [8] or the explanation after (2.4).)

Finally, for $n = 3$, we have that

$$\omega_3 = \mathrm{d}\log\frac{t_1-f}{t_1-1} \wedge \mathrm{d}\log\frac{t_2-f}{t_2-1} \wedge \mathrm{d}\log(1-f).$$

In order to find ϵ_3, let

$$\begin{aligned}\tilde{\epsilon}_3 = \log|1-f|\mathrm{d}i\arg\left(\frac{t_1-f}{t_1-1}\right) \wedge \mathrm{d}i\arg\left(\frac{t_2-f}{t_2-1}\right) + \\ + \frac{2}{3!}\log|1-f|\mathrm{d}\log\left|\frac{t_1-f}{t_1-1}\right| \wedge \mathrm{d}\log\left|\frac{t_2-f}{t_2-1}\right| - \\ - \log\left|\frac{t_1-f}{t_1-1}\right|\mathrm{d}i\arg(1-f) \wedge \mathrm{d}i\arg\left(\frac{t_2-f}{t_2-1}\right) - \\ - \frac{2}{3!}\log\left|\frac{t_1-f}{t_1-1}\right|\mathrm{d}\log|1-f| \wedge \mathrm{d}\log\left|\frac{t_2-f}{t_2-1}\right| + \\ + \log\left|\frac{t_2-f}{t_2-1}\right|\mathrm{d}i\arg(1-f) \wedge \mathrm{d}i\arg\left(\frac{t_1-f}{t_1-1}\right) + \\ + \frac{2}{3!}\log\left|\frac{t_2-f}{t_2-1}\right|\mathrm{d}\log|1-f| \wedge \mathrm{d}\log\left|\frac{t_1-f}{t_1-1}\right|.\end{aligned}$$

Specializing to $t_1 = 0$ we find after some computation that we get $\mathrm{d}\eta_1^{(3)}$ where $\eta_1^{(3)}(t_2)$ is given by

$$\begin{aligned}-P_{2,\mathrm{Zag}}(f)\mathrm{d}i\arg\left(\frac{t_2-f}{t_2-1}\right) - \frac{2}{3}\log|1-f|\log\left|\frac{t_2-f}{t_2-1}\right|\mathrm{d}\log|f| - \\ - \frac{1}{3}\log|1-f|\log|f|\mathrm{d}\log\left|\frac{t_2-f}{t_2-1}\right|.\end{aligned}$$

Finally, putting $t_2 = 0$ in $\eta_1^{(3)}$ we find that we get $\mathrm{d}\eta_2^{(3)}$ with

$$\eta_2^{(3)}(f) = -P_{3,\mathrm{Zag}}(f) - \frac{1}{2}\log^2|f|\log|1-f|.$$

Putting everything together as in Lemma 4.19, we put

$$\begin{aligned}\epsilon_3 = \tilde{\epsilon}_3 &- \mathrm{d}[\rho(t_1)\eta_1^{(3)}(t_2)] + \mathrm{d}[\rho(t_2)\eta_1^{(3)}(t_1)] + \\ &+ \mathrm{d}[\rho(t_1)\mathrm{d}[\rho(t_2)\eta_2^{(3)}(f)]] - \mathrm{d}[\rho(t_2)\mathrm{d}[\rho(t_1)\eta_2^{(3)}(f)]].\end{aligned}$$

We check that (ω_3, ϵ_3) is the class of the regulator of $[f]_3$. Let (ω_3, ϵ_3') be the regulator. Then $\epsilon_3 - \epsilon_3' \in H^2_{\mathrm{dR}}(X^2_{U,\mathrm{loc}}; \square^2; \mathbb{R}(2))^+$. By (4.4) and Lemma 4.12, $H^2_{\mathrm{dR}}(X^2; \square^2; \mathbb{R}(2))^+$ can be computed as the kernel of the map

$$H^2_{\mathrm{dR}}(X^2_{U,\mathrm{loc}}; \square^2; \mathbb{R}(2))^{+,\mathrm{alt}} \overset{\mathrm{res}}{\to} \left(\coprod H^1_{\mathrm{dR}}(X^1_{U,\mathrm{loc}}; \mathbb{R}(1))^+\right)^{\mathrm{alt}},$$

so if we can show that both ϵ_3 and ϵ_3' have residue $\epsilon_{2|t_1=S} - \epsilon_{2|t_2=S}$ then they differ by an element of $H^2_{\mathrm{dR}}(X^2; \square^2; \mathbb{R}(2))^+ \cong \mathbb{R}(2)$, and we can check that they are the same by specializing to a fixed value and integrating. From the exact sequence

$$0 \to H^1_{\mathrm{dR}}(X_U; \square; \mathbb{R}(1))^+ \to H^1_{\mathrm{dR}}(X^1_{U,\mathrm{loc}}; \square^1; \mathbb{R}(1))^+ \to H^0_{\mathrm{dR}}(U; \mathbb{R}(0))^+$$

we see, as $H^1_{\mathrm{dR}}(X_U; \square; \mathbb{R}(1))^+ \cong H^0_{\mathrm{dR}}(U; \mathbb{R}(1))^+ = 0$ that we do not lose any information about the residue in $H^1_{\mathrm{dR}}(X^1_{U,\mathrm{loc}}; \square^1; \mathbb{R}(1))^+$ by specializing S to a constant. Therefore, assuming $S = c$ is constant in $\mathbb{Q}$ such that $\rho(c) = 0$ from now on, we find at $t_1 = c$ that the residue is

$$\log|1-c|\mathrm{d}i\arg\left(\frac{t_2 - c}{t_2 - 1}\right) + \mathrm{d}(P_{2,\mathrm{Zag}}(c)\rho_1(t_2)) = \epsilon_2(t_2, c)$$

as desired. So $\epsilon_3 - \epsilon_3'$ is an element of $H^2_{\mathrm{dR}}(X^2; \square^2; \mathbb{R}(2))^+ \cong \mathbb{R}(2)$. Note that again in order to check that they are identical, we can specialize to $S = c$, so that we can check that they are the same class by computing

$$\int_{X^2} \epsilon_3(t_1, t_2, c) \wedge \mathrm{d}i\arg t_1 \wedge \mathrm{d}i\arg t_2 = (2\pi i)^2 2P_{3,\mathrm{Zag}}(c).$$

Because this is the answer for ϵ_3 (see [7, Proposition 4.1] with the correct sign as mentioned just after (2.4)), we conclude that $\epsilon_3' = \epsilon_3$.

Recall that we had the element $\alpha = \sum_j c_j[f_j]_3 \otimes g_j$ in $H^2(M^\bullet_{(4)}(F))$. Before writing down the corresponding regulator, we deduce some identities. We have the identity

$$\sum_j c_j[f_j]_2 \otimes (f_j \wedge g_j) = 0. \tag{4.10}$$

By applying $\mathrm{d} \otimes id$ to it and writing it with respect to our basis of $\langle f_j, g_j \rangle$ we find that for all k

$$\sum_j c_j s_{kj}(1 - f_j) \otimes (f_j \otimes g_j - g_j \otimes f_j) = 0. \tag{4.11}$$

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By using our basis once more we find that for all k and l

$$\sum_j c_j s_{kj} s_{lj} (1-f_j) \otimes g_j = \sum_j c_j s_{kj} t_{lj} (1-f_j) \otimes f_j \tag{4.12}$$

We shall also need

$$\begin{aligned}\sum_{j,k} c_j s_{kj} [f_j]_2 \otimes g_j &= \sum_{j,k,l} c_j s_{kj} t_{lj} [f_j]_2 \otimes A_l \\ &= \sum_{j,k,l} c_j s_{lj} t_{kj} [f_j]_2 \otimes A_l = \sum_{j,k} c_j [f_j]_2 \otimes f_j = 0\end{aligned}$$

in $H^2(\mathcal{M}^\bullet_{(3)}(F))$, where we used (4.10) and used our basis again. Similarly we see from (4.10) that for all k and l

$$\sum_j c_j s_{kj} t_{lj} [f_j]_2 = \sum_j c_j s_{lj} t_{kj} [f_j]_2. \tag{4.13}$$

Note also that the map $f \mapsto P_{2,\mathrm{Zag}}(f)$ factors though $\widetilde{M}_{(2)}(F)$, so that an element of $\widetilde{M}_{(2)}(F)$ gives rise to a continuous function on C, differentiable where f has no zeros or poles or assumes the value 1.

From Proposition 4.10 we obtain the element

$$\begin{aligned}\sum_j c_j [f_j]_3 \cup g_j + \sum_j c_j F_j(t_1) \cup [f_j]_2(t_2) \cup g_j - \\ - \sum_j c_j F_j(t_2) \cup [f_j]_2(t_1) \cup g_j + \\ + \sum_j c_j F_j(t_1) \cup F_j(t_2) \cup [f_j]_1 \cup g_j\end{aligned}$$

in $K_4^{(4)}(X^2_{\mathrm{loc}}; \square^2)$, with

$$F_j(t) = \frac{t-f_j}{t-1} \prod \left(\frac{t-A_k}{t-1}\right)^{-s_{kj}}$$

in $(1+I)^*$ if $f_j = \prod_k A_k^{s_{kj}}$ as in (4.1). According to (2.3), it has regulator

$$\begin{aligned}&\sum_j c_j \epsilon_3(t_1, t_2, f_j) \wedge \mathrm{d}i \arg g_j - \log|g_j| \pi_3 \omega_3(t_1, t_2, f_j) - \\ &- \sum_j c_j \mathrm{d}i \arg F_j(t_1) \wedge \big(\epsilon_2(t_2, f_j) \wedge \mathrm{d}i \arg g_j + \log|g_j| \pi_2 \omega_2(t_2, f_j)\big) + \\ &+ \sum_j c_j \mathrm{d}i \arg F_j(t_2) \wedge \big(\epsilon_2(t_1, f_j) \wedge \mathrm{d}i \arg g_j + \log|g_j| \pi_2 \omega_2(t_1, f_j)\big) + \\ &+ \sum_j c_j \mathrm{d}i \arg F_j(t_1) \wedge \mathrm{d}i \arg F_j(t_2) \wedge \sigma(1-f_j, g_j)\end{aligned} \tag{4.14}$$

in $H^3_{\mathrm{dR}}(X^3_{U,\mathrm{loc}}; \square^3; \mathbb{R}(3))^+$, which we want to lift back to a form in

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$H^3_{\mathrm{dR}}(X^3_U; \square^3; \mathbb{R}(3))^+$. $\tilde{\epsilon}_3 \wedge \mathrm{d}i \arg g - \log|g|\pi_3\omega_3$ can be rewritten as

$$\begin{aligned}
&\mathrm{d}i\arg\left(\frac{t_1-f}{t_1-1}\right)\wedge \mathrm{d}i\arg\left(\frac{t_2-f}{t_2-1}\right)\wedge\sigma(1-f,g)+\\
&\quad+\mathrm{d}\Bigg[\left(\frac{1}{3}\log|g|\mathrm{d}i\arg(1-f)+\frac{1}{6}\sigma(g,1-f)\right)\wedge\\
&\quad\wedge\left(\log\left|\frac{t_1-f_j}{t_1-1}\right|\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log\left|\frac{t_1-f_j}{t_1-1}\right|\right)+\\
&\quad+\log|g|\log\left|\frac{t_1-f}{t_1-1}\right|\mathrm{d}\log|1-f|\wedge\mathrm{d}i\arg\left(\frac{t_2-f}{t_2-1}\right)-\\
&\quad-\log|g|\log\left|\frac{t_2-f}{t_2-1}\right|\mathrm{d}\log|1-f|\wedge\mathrm{d}i\arg\left(\frac{t_1-f}{t_1-1}\right)\Bigg].
\end{aligned}$$

For

$$\mathrm{d}i\arg F(t_1)\wedge(\tilde{\epsilon}_2(t_2,f)\wedge\mathrm{d}i\arg g+\log|g|\pi_2\omega_2(t_2,f))$$

we obtain

$$\begin{aligned}
&\mathrm{d}i\arg F(t_1)\wedge\mathrm{d}i\arg\left(\frac{t_2-f}{t_2-1}\right)\wedge\sigma(1-f,g)+\\
&\quad+\mathrm{d}\left[-\log|g|\log\left|\frac{t_2-f}{t_2-1}\right|\mathrm{d}\log|1-f|\wedge\mathrm{d}i\arg F(t_1)\right].
\end{aligned}$$

Putting all this together, we get that the form in (4.14), after replacing all ϵ's with $\tilde{\epsilon}$'s, is given by

$$\begin{aligned}
\mathrm{reg}^{\sim} = \sum_j c_j\mathrm{d}\Bigg(&\frac{1}{6}\log\left|\frac{t_1-f_j}{t_1-1}\right|\sigma(g_j,1-f_j)\wedge\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\\
&-\frac{1}{6}\log\left|\frac{t_2-f_j}{t_2-1}\right|\sigma(g_j,1-f_j)\wedge\mathrm{d}\log\left|\frac{t_1-f_j}{t_1-1}\right|+\\
&+\log|g_j|\log\left|\frac{t_1-f_j}{t_1-1}\right|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg\left(\frac{t_2-f_j}{t_2-1}\right)+\\
&+\frac{1}{3}\log|g_j|\log\left|\frac{t_1-f_j}{t_1-1}\right|\mathrm{d}i\arg(1-f_j)\wedge\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\\
&-\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg\left(\frac{t_1-f_j}{t_1-1}\right)-\\
&-\frac{1}{3}\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}i\arg(1-f_j)\wedge\mathrm{d}\log\left|\frac{t_1-f_j}{t_1-1}\right|+\\
&+\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg F_j(t_1)-\\
&-\log|g_j|\log\left|\frac{t_1-f_j}{t_1-1}\right|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg F_j(t_2)\Bigg)+
\end{aligned}$$

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$$+\sum_{j,k,l} c_j s_{kj} s_{lj} \mathrm{d}i \arg\left(\frac{t_1 - A_k}{t_1 - 1}\right) \wedge \mathrm{d}i \arg\left(\frac{t_2 - A_l}{t_2 - 1}\right) \wedge \sigma(1 - f_j, g_j)$$

Using (4.12), the last part can be written as

$$\begin{aligned}
&\sum_{j,k,l} c_j s_{kj} t_{lj} \mathrm{d}i \arg\left(\frac{t_1 - A_k}{t_1 - 1}\right) \wedge \mathrm{d}i \arg\left(\frac{t_2 - A_l}{t_2 - 1}\right) \wedge \sigma(1 - f_j, f_j) \\
&\quad = \sum_{j,k,l} c_j s_{kj} t_{lj} \mathrm{d}\left(-P_{2,\mathrm{Zag}}(f_j) \mathrm{d}i \arg\left(\frac{t_1 - A_k}{t_1 - 1}\right) \wedge \mathrm{d}i \arg\left(\frac{t_2 - A_l}{t_2 - 1}\right)\right)
\end{aligned}$$

because $\sigma(1 - f, f) = -\mathrm{d}P_{2,\mathrm{Zag}}(f)$. Therefore $\widetilde{\mathrm{reg}}$ above is equal to $\mathrm{d}\psi_1(t_1, t_2)$ with $\psi_1(t_1, t_2)$ given by

$$\begin{aligned}
&\sum_j c_j \left(\frac{1}{6} \log\left|\frac{t_1 - f_j}{t_1 - 1}\right| \sigma(g_j, 1 - f_j) \wedge \mathrm{d}\log\left|\frac{t_2 - f_2}{t_2 - 1}\right| - \right. \\
&\quad - \frac{1}{6} \log\left|\frac{t_2 - f_j}{t_2 - 1}\right| \sigma(g_j, 1 - f_j) \wedge \mathrm{d}\log\left|\frac{t_1 - f_j}{t_1 - 1}\right| + \\
&\quad + \log|g_j| \log\left|\frac{t_1 - f_j}{t_1 - 1}\right| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg\left(\frac{t_2 - f_j}{t_2 - 1}\right) + \\
&\quad + \frac{1}{3} \log|g_j| \log\left|\frac{t_1 - f_j}{t_1 - 1}\right| \mathrm{d}i \arg(1 - f_j) \wedge \mathrm{d}\log\left|\frac{t_2 - f_j}{t_2 - 1}\right| - \\
&\quad - \log|g_j| \log\left|\frac{t_2 - f_j}{t_2 - 1}\right| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg\left(\frac{t_1 - f_j}{t_1 - 1}\right) - \\
&\quad - \frac{1}{3} \log|g_j| \log\left|\frac{t_2 - f_j}{t_2 - 1}\right| \mathrm{d}i \arg(1 - f_j) \wedge \mathrm{d}\log\left|\frac{t_1 - f_j}{t_1 - 1}\right| + \\
&\quad + \log|g_j| \log\left|\frac{t_2 - f_j}{t_2 - 1}\right| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg F_j(t_1) - \\
&\quad \left. - \log|g_j| \log\left|\frac{t_1 - f_j}{t_1 - 1}\right| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg F_j(t_2)\right) - \\
&\quad - \sum_{j,k,l} c_j s_{kj} t_{lj} P_{2,\mathrm{Zag}}(f_j) \mathrm{d}i \arg\left(\frac{t_1 - A_k}{t_1 - 1}\right) \wedge \mathrm{d}i \arg\left(\frac{t_2 - A_l}{t_2 - 1}\right).
\end{aligned}$$

Observing that

$$\begin{aligned}
&\sum_j c_j \left(\log|g_j| \log|f_j| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg\left(\frac{t_2 - f_j}{t_2 - 1}\right) - \right. \\
&\quad \left. - \log|g_j| \log|f_j| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg F_j(t_2)\right) \\
&\quad = \sum_{j,k} c_j s_{kj} \log|g_j| \log|f_j| \mathrm{d}\log|1 - f_j| \wedge \mathrm{d}i \arg\left(\frac{t_2 - A_k}{t_2 - 1}\right),
\end{aligned}$$

we find for $\psi_1(0, t_2)$

$$\sum_j c_j\Bigg(\frac{1}{6}\log|f_j|\sigma(g_j, 1-f_j)\wedge \mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-$$
$$-\frac{1}{6}\log\left|\frac{t_2-f_j}{t_2-1}\right|\sigma(g_j, 1-f_j)\wedge \mathrm{d}\log|f_j|+$$
$$+\frac{1}{3}\log|g_j|\log|f_j|\mathrm{d}i\arg(1-f_j)\wedge \mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-$$
$$-\frac{1}{3}\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}i\arg(1-f_j)\wedge \mathrm{d}\log|f_j|-$$
$$-\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|1-f_j|\wedge \mathrm{d}i\arg f_j+$$
$$+\sum_k s_{kj}\log|g_j|\log|f_j|\mathrm{d}\log|1-f_j|\wedge \mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)-$$
$$-\sum_k s_{kj}P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg g_j\wedge \mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)\Bigg).$$

On the other hand, from writing down the explicit ϵ_n for $n = 1, 2$ and 3, we know that if we put $t_1 = 0$ in $\mathrm{reg}^{\sim}$, we get dcor_1 with $\mathrm{cor}_1(t_2)$ given by

$$\sum_j c_j\left(\eta_1^{(3)} - \eta_1^{(2)}\mathrm{d}i\arg F_j(t_2)\right)\wedge \mathrm{d}i\arg g_j$$
$$= \sum_j c_j\Bigg(-P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg\left(\frac{t_2-f_j}{t_2-1}\right) - \frac{2}{3}\log|1-f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|f_j|-$$
$$-\frac{1}{3}\log|1-f_j|\log|f_j|\mathrm{d}\log\left|\frac{t_2-f_2}{t_2-1}\right| + P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg F_j(t_2)\Bigg)\wedge \mathrm{d}i\arg g_j$$
$$= \sum_j c_j\Bigg(\sum_k s_{kj} - P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)-$$
$$-\frac{2}{3}\log|1-f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|f_j| - \frac{1}{3}\log|1-f_j|\log|f_j|\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|\Bigg)\wedge \mathrm{d}i\arg g_j.$$

So $\psi_1(0, t_2) - \mathrm{cor}_1(t_2)$ equals

$$\sum_j c_j\Bigg(\sum_k s_{kj}P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)+$$
$$+\frac{2}{3}\log|1-f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|f_j|+$$

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$$
\begin{aligned}
&+\frac{1}{3}\log|1-f_j|\log|f_j|\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|\Bigg)\wedge \mathrm{d}i\arg g_j+\\
&+\sum_j c_j\Bigg(\frac{1}{6}\log|f_j|\sigma(g_j,1-f_j)\wedge\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\\
&-\frac{1}{6}\log\left|\frac{t_2-f_j}{t_2-1}\right|\sigma(g_j,1-f_j)\wedge\mathrm{d}\log|f_j|+\\
&+\frac{1}{3}\log|g_j|\log|f_j|\mathrm{d}i\arg(1-f_j)\wedge\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\\
&-\frac{1}{3}\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}i\arg(1-f_j)\wedge\mathrm{d}\log|f_j|-\\
&-\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg f_j+\\
&+\sum_k s_{kj}\log|g_j|\log|f_j|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)-\\
&-\sum_k s_{kj}P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg g_j\wedge\mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)\Bigg)\\
&=\sum_{j,k}c_jt_{kj}\big(-2P_{2,\mathrm{Zag}}(f_j)\mathrm{d}i\arg f_j+\log^2|f_j|\mathrm{d}\log|1-f_j|\big)\wedge\mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)+\\
&+\sum_j c_j\Bigg(\frac{1}{6}\log|f_j|\sigma(g_j,1-f_j)\wedge\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\\
&-\frac{1}{6}\log\left|\frac{t_2-f_j}{t_2-1}\right|\sigma(g_j,1-f_j)\wedge\mathrm{d}\log|f_j|+\\
&+\frac{1}{3}\log|g_j|\log|f_j|\mathrm{d}i\arg(1-f_j)\wedge\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|-\\
&-\frac{1}{3}\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}i\arg(1-f_j)\wedge\mathrm{d}\log|f_j|-\\
&-\log|g_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|1-f_j|\wedge\mathrm{d}i\arg f_j+\\
&+\frac{2}{3}\log|1-f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}\log|f_j|\wedge\mathrm{d}i\arg g_j+\\
&+\frac{1}{3}\log|1-f_j|\log|f_j|\mathrm{d}\log\left|\frac{t_2-f_j}{t_2-1}\right|\wedge\mathrm{d}i\arg g_j\Bigg)
\end{aligned}
$$

where we used (4.10) and (4.11), as well as the fact that mapping $[f]_2$ to $P_{2,\mathrm{Zag}}(f)$ factors through $M_{(2)}(F)$ (following (4.13)) and (4.13). Using integration by parts to get $\log\left|\frac{t_2-f_j}{t_2-1}\right|$ out, we find that this equals $\mathrm{d}\psi_2(t_2)$ with

$$
\psi_2(t_2)=\sum_j c_j\Bigg(\sum_k -2t_{kj}P_{3,\mathrm{Zag}}(f_j)\mathrm{d}i\arg\left(\frac{t_2-A_k}{t_2-1}\right)-
$$

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$$-\frac{1}{6}\log|f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\sigma(g_j,1-f_j)-$$
$$-\frac{1}{3}\log|g_j|\log|f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}i\arg(1-f_j)+$$
$$+\frac{1}{3}\log|1-f_j|\log|f_j|\log\left|\frac{t_2-f_j}{t_2-1}\right|\mathrm{d}i\arg g_j\Big)$$

as all the other elements then cancel using only that π_1 of a holomorphic 2-form on U vanishes identically.

We now have the (alternating) forms $\mathrm{reg}^{\sim}(t_1,t_2)$, $\psi_1(t_1,t_2)$, $\psi_2(t_2)$ and $\mathrm{cor}_1(t_2)$, all of which become identically zero if we put $t_2=\infty$. If we let $\mathrm{cor}_2=\sum_j c_j\eta_2^{(3)}(f_j)\wedge\mathrm{d}i\arg g_j$ (which does not depend on t_1 or t_2), we have that they satisfy the relations

$$\mathrm{reg}^{\sim}(t_1,t_2)=\mathrm{d}\psi_1(t_1,t_2)$$
$$\mathrm{reg}^{\sim}(0,t_2)=\mathrm{dcor}_1(t_2)$$
$$\mathrm{cor}_1(0)=\mathrm{dcor}_2$$
$$\psi_1(0,t_2)-\mathrm{cor}_1(t_2)=\mathrm{d}\psi_2(t_2).$$

We now consider the following form, where $\rho_1(t)$ is a bump form around $t=0$, $\rho_2(t_1,t_2)$ a bump form around the collection of $t_i=f_j$ or $t_i=1$ (symmetric with respect to interchanging t_1 and t_2). It is easy to choose them in such a way that of $\rho_1(t_1)$, $\rho_1(t_2)$ and $\rho_2(t_1,t_2)$ at most two are nonzero at the same time. Let reg be the form (satisfying (2.7)) given by

$$\begin{aligned}&\mathrm{reg}^{\sim}-\mathrm{d}[\rho_1(t_1)\mathrm{cor}_1(t_2)]+\mathrm{d}[\rho_1(t_2)\mathrm{cor}_1(t_1)]+\\&\quad+\mathrm{d}\big[\rho_1(t_1)\mathrm{d}[\rho_1(t_2)\mathrm{cor}_2]\big]-\mathrm{d}\big[\rho_1(t_2)\mathrm{d}[\rho_1(t_1)\mathrm{cor}_2]\big]-\\&\quad-\mathrm{d}\big[\rho_2(t_1,t_2)\psi_1(t_1,t_2)\big]+\\&\quad+\mathrm{d}\big[\rho_2(t_1,t_2)\big(\rho_1(t_1)\mathrm{cor}_1(t_2)-\rho_1(t_2)\mathrm{cor}_1(t_1)\big)\big]+\\&\quad+\mathrm{d}\big[\rho_2(t_1,t_2)\mathrm{d}[\rho_1(t_1)\psi_2(t_2)]\big]-\mathrm{d}\big[\rho_2(t_1,t_2)\mathrm{d}[\rho_1(t_2)\psi_2(t_1)]\big].\end{aligned}\tag{4.15}$$

Here the first five terms are the original regulator as in (4.14), the remaining form the d of some 2-form vanishing at $t_i=0,\infty$, lifting the regulator back to X_V^2, in fact to $(\mathbb{P}^1)_V^2$. (For checking that this is the case, note that the product $\rho_2(t_1,t_2)\rho_1(t_1)\rho_1(t_2)$ is identically zero by our choice of ρ_2, so that at most two of them are nonzero at any point of X_U^2.)

We now proceed to computing the integral in (4.8). Write Ξ for $\mathrm{d}i\arg t_1\wedge\mathrm{d}i\arg t_2$. Some calculations using the formulae at the end of Section 2 give the following integrals.

$$\int_{X^2\times S^1_x} \epsilon_3(t_1,t_2,f)\wedge \mathrm{d}i\,\arg g\wedge \Xi = (2\pi i)^2\int_{S^1_x} 2P_{3,\mathrm{Zag}}(f)\mathrm{d}i\,\arg g,$$
$$\int_{X^2\times S^1_x} -\log|g|\pi_3\omega_3(t_1,t_2,f)\wedge\Xi = (2\pi i)^2\int_{S^1_x}\log^2|f|\log|g|\mathrm{d}i\,\arg(1-f),$$
$$\int_{X^2\times S^1_x}\mathrm{d}i\,\arg F(t_1)\wedge\epsilon_2(t_2,f)\wedge\mathrm{d}i\,\arg g\wedge\Xi = 0,$$
$$\int_{X^2\times S^1_x}\mathrm{d}i\,\arg F(t_1)\wedge\log|g|\pi_2\omega_2(t_2,f)\wedge\Xi = 0,$$
$$\int_{X^2\times S^1_x}\mathrm{d}i\,\arg F(t_1)\wedge\mathrm{d}i\,\arg F(t_2)\wedge\sigma(1-f,g)\wedge\Xi = 0,$$
$$\int_{X^2\times S^1_x} -\mathrm{d}[\rho_2(t_1,t_2)\psi_1(t_1,t_2)]\wedge\Xi$$
$$= -4\pi i\int_{X^1\times S^1_x}\rho_2(0,t)\psi_1(0,t)\wedge\mathrm{d}\,i\,\arg t,$$
$$\int_{X^2\times S^1_x}\mathrm{d}[\rho_2(t_1,t_2)(\rho_1(t_1)\mathrm{cor}_1(t_2)-\rho_1(t_2)\mathrm{cor}_1(t_1))]\wedge\Xi$$
$$= 4\pi i\int_{X^1\times S^1_x}\rho_2(0,t)\mathrm{cor}_1(t)\wedge\mathrm{d}i\,\arg t,$$
$$\int_{X^2\times S^1_x}\mathrm{d}[\rho_2(t_1,t_2)\mathrm{d}[\rho_1(t_1)\psi_2(t_2)]]-\mathrm{d}[\rho_2(t_1,t_2)\mathrm{d}[\rho_1(t_2)\psi_2(t_1)]]\wedge\Xi$$
$$= 4\pi i\int_{X^1\times S^1_x}\rho_2(0,t)\big(\psi_1(0,t)-\mathrm{cor}_1(t)\big)\wedge\mathrm{d}i\,\arg t +$$
$$+(2\pi i)^2\int_{S^1_x}4P_{2,\mathrm{Zag}}(f)\mathrm{d}i\,\arg g.$$

(For those computations, note that S^1_x has dimension one, so that a lot of the contributions actually vanish identically on $X^2_{S^1_x}$.) The first five lines here suffice to compute the contribution of the first two lines in (4.15), as those equal (4.14).

Putting everything together we find that (4.8) equals

$$\frac{-1}{2\pi i}\int_{S^1_x}\sum_j c_j\big(6P_{3,\mathrm{Zag}}(f_j)\mathrm{d}i\,\arg g_j+\log^2|f_j|\log|g_j|\mathrm{d}i\,\arg(1-f_j)\big). \tag{4.16}$$

We now rewrite the form in (4.16) in order to compute its residues. Using (2.5) and (4.12), we find

$$\sum_j c_j\big(6P_3^{\mathrm{mod}}(f_j)\mathrm{d}i\,\arg g_j+\log^2|f_j|\sigma(g_j,1-f_j)\big)$$
$$=\sum_j c_j\big(6P_3^{\mathrm{mod}}(f_j)\mathrm{d}i\,\arg g_j+\log|f_j|\log|g_j|\sigma(f_j,1-f_j)\big).$$

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Subtracting

$$\sum_j \mathrm{d}c_j\big(\log|f_j|\log|g_j|P_{2,\mathrm{Zag}}(f_j)\big)$$
$$= \sum_j c_j\big(2\log|g_j|P_{2,\mathrm{Zag}}(f_j)\mathrm{d}\log|f_j| + \log|f_j|\log|g_j|\sigma(f_j, 1-f_j)\big)$$

this transforms into

$$\sum_j c_j\big(6P_3^{\mathrm{mod}}(f_j)\mathrm{d}i\,\arg g_j - 2\log|g_j|P_2^{\mathrm{mod}}(f_j)\mathrm{d}\log|f_j|\big). \tag{4.17}$$

Hence the integral in (4.16) yields $-6\sum_j c_j\,\mathrm{ord}_x(g_j)P_3^{\mathrm{mod}}(f_j(x))$, which is the regulator of the element $\pm 3\sum_j c_j\delta_x([f_j]_3\otimes g_j)$, see Theorem 2.3. Because the regulator is injective on $K_5^{(3)}(k(x))$, this proves that the diagram

$$\begin{array}{ccc} H^2(\mathcal{M}^\bullet_{(4)}(F)) & \longrightarrow & K_6^{(4)}(F)/K_4^{(2)}(F)\cup K_2^{(2)}(F) \\ \big\downarrow 3\delta & & \big\downarrow \partial \\ \coprod_{x\in C^{(1)}} H^1(\widetilde{\mathcal{M}}^\bullet_{(3)}(k(x))) & \xrightarrow{\sim} & \coprod_{x\in C^{(1)}} K_5^{(3)}(k(x)) \end{array}$$

commutes (up to sign and up to $\partial\Big(K_5^{(3)}(k)\cup F^*_{\mathbb{Q}}\Big)$ in the lower right hand corner), thus proving Theorem 4.6.

Remark 4.20. Because the form ψ appearing in (4.17) (or (4.16)) has the same residue (modulo the residue of the regulator of $K_5^{(3)}(k)\cup F^*_{\mathbb{Q}}$) as the regulator of α in the localization sequence

$$0 \to H^1_{\mathrm{dR}}(C;\mathbb{R}(3))^+ \to H^1_{\mathrm{dR}}(F;\mathbb{R}(3))^+ \to \coprod H^0_{\mathrm{dR}}(k(x);\mathbb{R}(2))^+,$$

they differ by an element in $H^1_{\mathrm{dR}}(C;\mathbb{R}(3))^+ + \mathrm{reg}(K_5^{(3)}(k)\cup F^*_{\mathbb{Q}})$. Using integration by parts and Stokes' theorem it is not hard to check that

$$\int_C \psi\wedge\overline{\omega} = \int_C \mathrm{reg}(\alpha)\wedge\overline{\omega}$$

with the last given by Theorem 3.5. By Remark 3.1 ψ is an explicit representative of the regulator of $\sum_j c_j[f_j]_3\otimes g_j$ in $H^1_{\mathrm{dR}}(C;\mathbb{R}(3))$, modulo the regulator of $K_5^{(3)}(k)\cup F^*_{\mathbb{Q}}$.

5. Connections with Goncharov's Work

In this section, we start with showing how the work in the previous two sections, together with the work of Goncharov ([3] and [5], see [4, Section 8] for an overview of the results without proofs) leads to a complete description of the image of the regulator map on $K_4^{(3)}(C)$ and $K_6^{(4)}(C)$. In particular, this proves a conjecture of Goncharov for those cases ([5, Conjecture 1.5] or [3, Conjecture 1.6]). We also sketch how, assuming some conjectures, the relation with results as in Goncharov's work for higher K-groups would work out.

In [4, §6], Goncharov defined the following complexes $\Gamma(F, n)$ (in degree $1, \ldots, n$), given by

$$B_n(F) \to B_{n-1}(F) \otimes F^*_{\mathbb{Q}} \to \ldots \to B_2(F) \otimes \bigwedge^{n-2} F^*_{\mathbb{Q}} \to \bigwedge^{n} F^*_{\mathbb{Q}}$$

and, for each $x \in C^{(1)}$, $\Gamma(k(x), n-1)$ (in degrees $1, \ldots, n-1$), given by

$$B_{n-1}(k(x)) \to \ldots \to B_2(k(x)) \otimes \bigwedge^{n-3} k(x)^*_{\mathbb{Q}} \to \bigwedge^{n-1} k(x)^*_{\mathbb{Q}}$$

Here for any infinite field F, $B_k(F)$ is a $\mathbb{Q}$-vector space generated by elements $\{f\}_k$ with $f \in F \cup \{\infty\}$, modulo certain (inductively defined) relations. All maps are given by

$$\{f\}_k \otimes g_1 \wedge \cdots \wedge g_{n-k} \mapsto \{f\}_{k-1} \otimes f \wedge g_1 \wedge \cdots \wedge g_{n-k}.$$

There is a map

$$\Gamma(F, n) \to \coprod_{x \in C^{(1)}} \Gamma(k(x), n-1)[-1] \tag{5.1}$$

given by

$$\{f\}_k \otimes g_1 \wedge \cdots \wedge g_{n-k} \mapsto \{f(x)\}_k \otimes \partial_{n-k,x}(g_1 \wedge \cdots \wedge g_{n-k})$$

with $\partial_{m,x}$ the unique map $\bigwedge^m F^*_{\mathbb{Q}} \to \bigwedge^{m-1} k(x)^*_{\mathbb{Q}}$ determined as in Proposition 4.1

$$\pi_x \wedge u_1 \wedge \cdots \wedge u_{k-1} \mapsto u_1(x) \wedge \cdots \wedge u_{k-1}(x)$$
$$u_1 \wedge \cdots \wedge u_k$$

if all u_i are units at x and π_x is a uniformizer at x. $\Gamma(C, n)$ is defined as the mapping cone of (5.1). Goncharov also defines complexes $\Gamma'(F, n)$, $\Gamma'(k(x), n-1)$ for $n = 3$ and 4, and constructs maps as in (5.1). The complexes Γ' have the same shape as the complexes Γ with the same maps between them, but the $B_k(F)$ are replaced by $B'_k(F)$, generated by $F \cup \{\infty\}$, but with *explicit* relations between the generators. $\Gamma'(C, n)$ is defined as the mapping cone, defined by the corresponding Γ' complexes in (5.1). Goncharov also constructs a map

$$K_{2n}(C) \to H^2(\Gamma'(C, n+1)) \tag{5.2}$$

for $n = 2$ or 3, and shows that the Beilinson regulator factors through this map. We summarize part of his results in a form suitable for our needs

THEOREM 5.1 (Goncharov). *Let ω be a global holomorphic 1-form on C. Then for $n = 2$ or 3, the regulator map*

$$K^{(n+1)}_{2n}(C) \to H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+$$

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can be extended over the map $K_{2n}^{(n+1)}(C) \to H^2(\Gamma'(C, n+1))$ *to a map* $H^2(\Gamma'(C, n+1)) \to H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+$. *For* ω *a holomorphic 1-form such that* $\omega \circ \sigma = \overline{\omega}$, *the composition*

$$H^2(\Gamma'(C, n+1)) \longrightarrow H^1_{\mathrm{dR}}(C; \mathbb{R}(n))^+ \xrightarrow{\int_{C_{\mathrm{an}}} \cdots \wedge \overline{\omega}} \mathbb{R}(1)$$

is given by mapping $\{f\}_n \otimes g$ *to*

$$c_n \int_C \log|g| \log^{n-2}|f| (\log|1-f| \mathrm{d}\log|f| - \log|f| \mathrm{d}\log|1-f|) \wedge \overline{\omega}$$

for some nonzero rational constant c_n.

For $n = 2$, this is proved in [5]. There the map (5.2) is constructed at the end of Section 2.7. The extension of the regulator is given just before Theorem 3.1, which states that the extension of the regulator coincides with Beilinson's regulator on $K_4^{(3)}(C)$. Finally, Theorem 3.3 gives the formula for the regulator integral. For $n = 3$, the corresponding results can be found in [3], namely Theorems 4.2, 5.3 and 5.5.

LEMMA 5.2. *There is a map*

$$B_2'(F) \to \widetilde{M}_{(2)}(F)$$

given by sending $\{x\}_2$ *to* $[x_2]$.

Proof. $B_2'(F)$ is a free $\mathbb{Q}$-vector space on elements $\{x\}_2$ with x in $F^* \setminus \{1\}$, modulo the relations

$$\{x\}_2 + \{y\}_2 + \left\{\frac{1-x}{1-xy}\right\}_2 + \{1-xy\}_2 + \left\{\frac{1-y}{1-xy}\right\}_2 = 0$$

It is known (see [10, Lemmas 1.2 and 1.4]) that one then also has the following relations

$$\{x\}_2 + \{1-x\}_2 = 0 \qquad \text{and} \qquad \{x\}_2 + \{1/x\}_2 = 0$$

We have to show that the corresponding relations hold in $\widetilde{M}_{(2)}(F)$. We start with the last two. The relation $[x_2] + [1/x]_2 = 0$ holds in $\widetilde{M}_{(2)}(F)$ by definition. The element $[x_2] + [1-x]_2$ lies in $H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(F))$ and is a pullback from an element in $H^1(\widetilde{\mathcal{M}}^\bullet_{(2)}(\mathbb{Q}(t))$, which injects into $K_3^{(2)}(\mathbb{Q}(t))$ by Theorem 2.3. But $K_3^{(2)}(\mathbb{Q}(t)) = K_3^{(2)}(\mathbb{Q}) = 0$ so this element is zero. For the actual five-term relation, observe that modulo the last two relations (for $[\cdots]_2$ instead of $\{\cdots\}_2$), the first corresponds

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to the relation in $M_{(2)}(F)$ given by

$$-[x^{-1}]_2-[1-y]_2+\left[\frac{1-xy}{x}\right]_2+\left[\frac{x}{1-xy}\right]_2-\left[1-\frac{1-x}{1-xy}\right]_2-$$
$$-\left[\frac{1}{1-xy}\right]_2+\left[\frac{1-y}{1-xy}\right]_2.$$

The construction of the complexes as sketched in Section 2 gives that the lift of those elements are given by

$$\sum_j [f_j]_2+\sum_j (1-f_j)\cup F_j(t)$$

where F_j is the function expressing f_j in elements of a chosen basis in $\{f_j\}$. In order to show that those equal zero, we work universally, i.e., we work over the base scheme

$$Z=\operatorname{Spec}(\mathbb{Q}[X,Y,(1-X)^{-1},(1-Y)^{-1},(1-XY)^{-1}])$$

and we want to show that we are pulling back a universal element in $K_3^{(2)}(Z)\cong K_3^{(2)}(\mathbb{Q})=0$ via the map $x\mapsto X$, $y\mapsto Y$. We can pull back directly where all $f_j\neq 1$, i.e., pull back from the open part Z' of Z where $1-X-XY\neq 0$. But $K_3^{(2)}(Z')=0$ as well. If $1-x-xy=0$, using the relations $\{x\}_2+\{1-x\}_2$ and $\{x\}_2+\{1/x\}_2$ which we know already, the relation reduces to to $\{x^2\}_2=2\{x\}_2+2\{-x\}_2$. One proves this one in a similar way over $\operatorname{Spec}(\mathbb{Q}[X,(1-X^2)^{-1}])$.

Remark 5.3. In fact $B_2(F)=B_2'(F)$. Namely, let F be any infinite field, and suppose $\alpha\in\operatorname{Ker}\left(\mathrm{d}:\mathbb{Q}[F(T)\cup\{\infty\}]\to\bigwedge^2 F(t)^*_{\mathbb{Q}}\right)$. By Suslin's work [10], this yields an element in $K_3^{(2)}(F(t))\cong K_3^{(2)}(F)$. But modulo the five term relations, we can rewrite this to $\alpha\equiv\beta$ with $\beta\in K_3^{(2)}(F)\cong\operatorname{Ker}\left(\mathrm{d}:B_2(F)\to\bigwedge^2 F^*_{\mathbb{Q}}\right)$. Then $\alpha(0)\equiv\beta$ modulo the relations, as one checks by a case by case check depending on the zeroes and poles of the functions involved. Of course this works for $\alpha(1)$ as well, so in total $\alpha(0)-\alpha(1)$ is in the (degenerate) relations in B_2, hence is zero in $B_2(F)$.

We use Lemma 5.2 to link our results with Theorem 5.1, beginning with the case $n=2$.

THEOREM 5.4. *The maps in (5.2), Lemma 5.2 and* $\varphi^2_{(3)}$ *give maps*

$$K_4^{(3)}(C)\to H^2(\Gamma'(C,3))\to H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))\to K_4^{(3)}(C)+K_3^{(2)}(k)\cup F^*_{\mathbb{Q}}.$$

Viewing this last group as inside $K_4^{(3)}(F)$*, the composition of those maps with the regulator integral associated to* ω *is given by Theorem 3.5 on* $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$ *and by Theorem 5.1 on* $H^2(\Gamma'(C,3))$*. In particular, all those groups, as well as the group* $H^2(\mathcal{M}^\bullet_{(3)}(C))\cong H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$*, have the same image in* $\mathbb{R}(1)$ *under the regulator integral associated with* ω.

Proof. That we get the map from $H^2(\Gamma'(C,3))$ to $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$ is clear from Lemma 5.2. The regulator integral on $H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$ was stated in Theorem 3.5, and that the composition of this with the map $H^2(\Gamma'(C,3)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$ coincides with the formulae in Theorem 5.1 up to a nonzero rational number is clear. Then, fixing ω, we get that the images in $\mathbb{R}(1)$ have the relations:

$$\begin{aligned}&\mathrm{Image}(K_4^{(3)}(C)) \subseteq \mathrm{Image}(H^2(\Gamma'(C,3))) \subseteq \mathrm{Image}(H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C)))\\ &\quad\subseteq \mathrm{Image}(K_4^{(3)}(C) + K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}) = \mathrm{Image}(K_4^{(3)}(C))\end{aligned}$$

because the regulator integral vanishes on $K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}$ by Proposition 3.4.

COROLLARY 5.5. *The groups* $K_4^{(3)}(C)$, $H^2(\Gamma'(C,3))$ *and, in case* $K_3^{(2)}(k) = 0$, *the group* $H^2(\mathcal{M}^\bullet_{(3)}(C)) \cong H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(C))$, *have the same image in* $H^1_{\mathrm{dR}}(C;\mathbb{R}(2))^+$ *under the regulator map. The same holds true without assuming that* $K_3^{(2)}(k) = 0$ *if we use the modified version of* $\varphi^3_{(2)}$ *as described in Corollary 4.5.*

Proof. This is clear from Theorem 5.4, as the regulator integrals form the dual space of $H^1_{\mathrm{dR}}(C;\mathbb{R}(2))^+$, see Remark 3.1, and the difference between using $\varphi^2_{(3)}$ and its modification lies in $K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}$, on which the regulator integral vanishes by Proposition 3.4.

We now turn towards $n = 3$. As described in Section 2, the natural map $H^2(\mathcal{M}^\bullet_{(4)}(F)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(F))$ is a surjection, so we get a surjection $H^2(\mathcal{M}^\bullet_{(4)}(C)) \to H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$. In particular, those two groups have the same image under the regulator integral as in Theorem 3.5 and Remark 3.7.

We recall the definition of the group $B_3'(k)$: it is the free $\mathbb{Q}$-vector space with generators $\{x\}_3$ for $x \in F^* \setminus \{1\}$, and relations

$$\sum_{i=1}^{3}\left(\{\alpha_i\}_3 + \{\beta_i\}_3 - \left\{\frac{\beta_i}{\alpha_{i-1}}\right\}_3 + \left\{\frac{\beta_i}{\alpha_{i-1}\alpha_i}\right\}_3 + \left\{\frac{\alpha_i\beta_{i-1}}{\beta_{i+1}}\right\}_3 + \right.$$
$$\left. + \left\{-\frac{\beta_i}{\alpha_i\beta_{i-1}}\right\}_3 - \left\{\frac{\alpha_i\alpha_{i-1}\beta_{i+1}}{\beta_i}\right\}_3\right) - 3\{1\}_3 + \{-\alpha_1\alpha_2\alpha_3\}_3. \tag{5.3}$$

Here $\beta_i = 1 - \alpha_i(1-\alpha_{i-1})$ with indices taken modulo 3.

THEOREM 5.6. *Let C be a smooth, proper, geometrically irreducible curve over the number field k, with function field F. Then the groups*

$$K_6^{(4)}(C),\ H^2(\Gamma'(C,4)),\ H^2(\mathcal{M}^\bullet_{(4)}(C)) \text{ and } H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$$

all have the same image under the regulator integral, given by Theorem 3.5.

Proof. By Corollary 4.8, $H^2(\mathcal{M}^\bullet_{(4)}(C))$ maps to

$$K_6^{(4)}(C) + K_5^{(3)}(k) \cup F^*_{\mathbb{Q}}/K_4^{(2)}(F) \cup K_2^{(2)}(F)$$

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inside $K_6^{(4)}(F)/K_4^{(2)}(F) \cup K_2^{(2)}(F)$. By Proposition 3.4, the regulator integral is zero on $K_5^{(3)}(k) \cup F_{\mathbb{Q}}^*$, and it also kills $K_4^{(2)}(F) \cup K_2^{(2)}(F)$ according to Remark 3.7, whence

$$\mathrm{Image}(H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))) = \mathrm{Image}(H^2(\mathcal{M}^\bullet_{(4)}(C))) \subset \mathrm{Image}(K_6^{(4)}(C))$$

in $\mathbb{R}(1)$.

From Goncharov's work as quoted in Theorem 5.1 we get a map

$$K_6^{(4)}(C) \to H^2(\Gamma'(C,4)).$$

By Lemma 5.2 we have a map $B_2'(F) \to \widetilde{M}_{(2)}(F)$. In [7, p. 241] a map $B_3'(k(x)) \to \widetilde{M}_{(3)}(k(x))$ was created by mapping $\{y\}_3$ to $[y]_3$. This is well defined because d is zero on the relations in (5.3) so they give rise to an element in $H^1(\widetilde{\mathcal{M}}^\bullet_{(3)}(k(x))) \subset K_5^{(3)}(k(x))$. Because the regulator for the embedding σ of $k(x)$ into $\mathbb{C}$ is given by mapping $[y]_3$ to a nonzero multiple of $P_3^{\mathrm{mod}}(\sigma(y))'$ the function P_3^{mod} vanishes on the elements in (5.3) and the regulator is injective, the elements in (5.3) go to zero in $\widetilde{M}_{(3)}(k(x))$ and our map is well-defined. Using those two maps we see that if $\sum_j c_j\{f_j\}_3 \otimes g_j$ is an element of $H^2(\Gamma'(C,4))$, then $\sum_j c_j[f_j]_3 \otimes g_j$ is an element of $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$. So we get the inclusion of images under the regulator integrals

$$\begin{aligned}&\mathrm{Image}(K_6^{(4)}(C)) \subseteq \mathrm{Image}(H^2(\Gamma'(C,4))) \subseteq \\ &\quad \subseteq \mathrm{Image}(H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))) \subseteq \mathrm{Image}(K_6^{(4)}(C)).\end{aligned}$$

Remark 5.7. If we use the lifted version

$$\varphi^2_{(4)} : H^2(\mathcal{M}^\bullet_{(4)}(C)) \to K_6^{(4)}(C)/K_4^{(2)}(F) \cup K_2^{(2)}(F)$$

as in Corollary 4.9, followed by the regulator map to $H^1_{\mathrm{dR}}(C; R(3))^+$, the resulting total map factors through the projection from $H^2(\mathcal{M}^\bullet_{(4)}(C))$ to $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$. To see this, note that the kernel of this projection consists of elements α of the form $\sum_j n_j([f_j]_3 - [1/f_j]_3 \otimes g_j$, with $\sum_j n_j([f_j]_2 + [1/f_j]_2 \otimes f_j \wedge g_j = 0$ in $M_{(2)}(F) \otimes \bigwedge^2 F_{\mathbb{Q}}^*$. By Remark 3.1, it is enough to check that the regulator integrals all vanish on the regulator of such α, but this is part of Theorem 3.5 and Remark 3.7.

Using the lifted versions of $\varphi^2_{(4)}$ and Remark 5.7, we get the following Corollary.

COROLLARY 5.8. *The groups* $K_6^{(4)}(C)$, $H^2(\Gamma'(C,4))$, $H^2(\mathcal{M}^\bullet_{(4)}(C))$ *and* $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$ *have the same image in* $H^1_{\mathrm{dR}}(C; \mathbb{R}(3))^+$ *under the regulator map.*

Proof. This is immediate from Theorem 5.6 because the regulator integrals are dual to $H^1_{\mathrm{dR}}(C; \mathbb{R}(3))^+$, see Remark 3.1, and the regulator integrals vanish on $K_5^{(3)}(k) \cup F_{\mathbb{Q}}^*$ by Proposition 3.4, so they do not notice the modification due to the lift.

Remark 5.9. Theorem 5.4 and Theorem 5.6 (or Corollary 5.5 and Corollary 5.8) give, in principle, a (difficult) combinatorical description of the image of $K_4^{(3)}(C)$ resp. $K_6^{(4)}(C)$ under the regulator map, in terms of the groups B'_n involved in $H^2(\Gamma'(C,3))$ resp. $H^2(\Gamma'(C,4))$. According to the Beilinson conjectures, for $n \geqslant 2$, the regulator map from $K_{2n}^{(n+1)}(C)$ to $H^1_{\mathrm{dR}}(C;\mathbb{R}(n))^+$ should be an injection, so that this conjecturally gives a combinatorical description of those K-groups as well.

Remark 5.10. One could try to check the explicit relations of $B'_3(F)$ in $\widetilde{\mathcal{M}}_{(3)}(F)$ along the lines of Lemma 5.2, in order to get a map from $H^2(\Gamma'(C,4))$ to $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$. Due to the size of the relations involved, the author has not tried to do this. Note also that that would still not give us a map from $K_6^{(4)}(C) \to K_6^{(4)}(C) + K_5^{(3)}(k) \cup F^*_{\mathbb{Q}}$ similar to the maps in Theorem 5.4, as the map $\varphi^2_{(4)}$ from $H^2(\widetilde{\mathcal{M}}^\bullet_{(4)}(C))$ to $K_6^{(4)}(C) + K_5^{(3)}(k) \cup F^*_{\mathbb{Q}}$ depends on the Beilinson–Soulé conjecture as explained in Section 2. Thus the results for $n = 3$ are necessarily weaker than those for $n = 2$.

Remark 5.11. One can give a more general proof of the existence of a map $B_n(F) \to \widetilde{M}_{(n)}(F)$ for all $n \geqslant 2$, but it becomes dependent on conjectures. Namely, assume that

(1) F is the function field of a smooth, projective, geometrically irreducible variety Z over the number field k;
(2) $B_n(F)$ is a quotient of the free $\mathbb{Q}$-vector space on elements $\{x\}_n$ with $x \in F^* \setminus \{1\}$, with relations $\sum_j c_j\{f_j(x_1, \cdots, x_m)\}_n = 0$ for rational numbers c_j, and rational functions f_j on Z with coefficients in a number field k. Assume moreover that there exists a Zariski open part U of Z such that for all y closed in U, the function $\sum_j c_j P_n^{\mathrm{mod}}(\sigma(f_j(y)))$ vanishes identically for all embeddings of $k(y)$ into $\mathbb{C}$;
(3) the Beilinson–Soulé conjecture is true for general fields of characteristic zero: $K_n^{(p)}(F) = 0$ if $2p - n \leqslant 0$ and $n > 0$;
(4) For a smooth, geometrically irreducible variety Z over a number field k, $K_{2n-1}^{(n)}(Z) \cong K_{2n-1}^{(n)}(k)$ by pullback from the base (which is part of the Beilinson conjectures).

Then proceeding by induction, assume that we have defined a map $B_{n-1}(F) \to \widetilde{\mathcal{M}}_{(n-1)}(F)$ by $\{x\}_{n-1} \mapsto [x]_{n-1}$, so that the diagram

$$\begin{array}{ccc}
<\{x\}_n, x \in F^* \setminus \{1\}> & \longrightarrow & B_{n-1}(F) \otimes F^*_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\widetilde{M}_{(n)}(F) & \longrightarrow & \widetilde{M}_{(n-1)}(F) \otimes F^*_{\mathbb{Q}}
\end{array}$$

(resp.

$$\begin{array}{ccc} <\{x\}_2, x \in F^* \setminus \{1\}> & \longrightarrow & \bigwedge^2 F^*_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \widetilde{M}_{(2)}(F) & \longrightarrow & \bigwedge^2 F^*_{\mathbb{Q}} \end{array}$$

for $n = 2$) commutes. Then we have to check that for any relation $\sum c_j\{f_j(x_1, \cdots, x_m)\}_n = 0$, the relation $\sum c_j[f_j(x_1, \ldots, x_m)]_n = 0$ holds in $\widetilde{M}_{(n)}(F)$. The element $\sum c_j[f_j(x_1, \ldots, x_m)]_n$ defines an element α in $H^1(\widetilde{\mathcal{M}}^{\bullet}_{(n)}(F))$, injecting into $K^{(n)}_{2n-1}(F)$. Using the spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} K^{(n-p)}_{-p-q}(k(x)) \Rightarrow K^{(n)}_{-p-q}(Z)$$

(see [9, Théorème 4 (iii)]) the Beilinson-Soulé conjecture implies that then $K^{(n)}_{2n-1}(F) \cong K^{(n)}_{2n-1}(Z)$, and the Beilinson conjectures imply $K^{(n)}_{2n-1}(Z) \cong K^{(n)}_{2n-1}(k)$ by pullback from the base. The remarks in Section 2 show that $\varphi^1_{(n)}$ is in fact defined over some Zariski open subset U of Z, and we have $K^{(n)}_{2n-1}(F) \cong K^{(n)}_{2n-1}(U) \cong K^{(n)}_{2n-1}(k)$ as well. We can select a point y in U such that $\varphi^1_{(n)}(\alpha)$ can be pulled back to y, mapping α to an element in $K^{(n)}_{2n-1}(k(y))$, namely the image from the corresponding element in $K^{(n)}_{2n-1}(k) \cong K^{(n)}_{2n-1}(U)$. Because the map $K^{(n)}_{2n-1}(k) \to K^{(n)}_{2n-1}(k(y))$ is injective, we can check that the image of α is zero by computing the regulator map, which according to Theorem 2.3 is given by computing $\sum_j c_j P_n^{\mathrm{mod}}(\sigma(f_j(y)))$ for all embeddings σ of $k(y)$ into $\mathbb{C}$. This vanishes by our assumptions.

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