Can. J. Math., Vol. XXXI, No. 5, 1979, pp. 1084–1106

H-FINITE IRREDUCIBLE REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

F. LEMIRE AND M. PAP

Let L denote a simple Lie algebra over the complex number field $\mathbf{C}$ with H a fixed Cartan subalgebra and $C(L)$ the centralizer of H in the universal enveloping algebra U of L. It is known [cf. **2**, **5**] that one can construct from each algebra homomorphism $\phi : C(L) \to \mathbf{C}$ a unique algebraically irreducible representation of L which admits a weight space decomposition relative to H in which the weight space corresponding to $\phi \downarrow H \in H^*$ is one-dimensional. Conversely, if (ρ, V) is an algebraically irreducible representation of L admitting a one-dimensional weight space V_λ for some $\lambda \in H^*$, then there exists a unique algebra homomorphism $\phi : C(L) \to \mathbf{C}$ which extends λ such that (ρ, V) is equivalent to the representation constructed from ϕ. Any such representation will be said to be *pointed*. The collection of all pointed representations clearly includes all dominated irreducible representations and is included in the family of all Harish-Chandra modules which are H-finite [cf. **2**, **3**].

In this paper we present a detailed study of the family of pointed representations—in particular, we shall provide a complete description, up to equivalence, of all pointed representations of the simple Lie algebras $sl(n, \mathbf{C})$ for $n = 2$, 3 and 4. Our approach will be to label the equivalence classes of pointed representations of L by elements from the family of algebra homomorphisms $\phi : C(L) \to \mathbf{C}$ in analogy to the technique of labelling the dominant irreducible representations by their "highest weight function".

Section 1. Aut $(L : H)$. In order to simplify our study of the family F_L of all algebra homomorphisms $\phi : C(L) \to \mathbf{C}$ and their associated pointed representations we shall introduce an equivalence relation on F_L. Let $\text{Aut}(L:H)$ denote the group of all automorphisms σ of L such that $\sigma(H) \subseteq H$. If one considers the weight space decomposition of U relation to H, viewed as an L-module under the adjoint representation, we have

$$U = \textstyle\sum_{\xi \in H^*} \oplus \, U_\xi.$$

Then for any $\sigma \in \text{Aut}\ (L:H)$ we have $\sigma(U_\xi) \subseteq U_{\xi \circ \tilde{\sigma}^{-1}}$ where $\tilde{\sigma} \equiv \sigma \downarrow H$. In particular $U_0 = C(L)$ and $\sigma(U_0) = U_0$; *ie.* if $\phi \in F_L$ then $\phi \circ \sigma \downarrow C \in F_L$ for all $\sigma \in \text{Aut}(L:H)$. (Note that we also denote by σ the natural extension of σ to an automorphism of U).

Received May 30, 1978 and in revised form October 4, 1978.

Definition. If $\phi_1, \phi_2 \in F_L$ we say that ϕ_1 is *weakly equivalent to* ϕ_2 if and only if there exists $\sigma \in \text{Aut}(L:H)$ such that $\phi_1 = \phi_2 \circ \sigma$. This is clearly an equivalence relation on F_L.

Let M_ϕ denote the unique maximal left ideal of U containing $\ker \phi$ for $\phi \in F_L$. Then [cf. **1**] the left regular representation of L on U/M_ϕ is the pointed representation constructed from ϕ. If $\phi_1, \phi_2 \in F_L$ are weakly equivalent then their associated pointed representations are related in the following way:

PROPOSITION 1. *Let* $\phi_1, \phi_2 \in F_L$ *with* $\phi_1 = \phi_2 \circ \sigma$ *for some* $\sigma \in \text{Aut}(L:H)$; *then there exists a linear space isomorphism* $\sigma : U/M_{\phi_1} \rightarrow U/M_{\phi_2}$ *which preserves weight spaces in the sense that*

$$\hat{\sigma}((U/M_{\phi_1})_\lambda) = (U/M_{\phi_2})_{\lambda\circ\bar{\sigma}^{-1}}.$$

Proof. Recall that for any $\phi \in F_L$ we have

$$M_\phi = \sum_{\xi\in H} \oplus (U_\xi \cap M_\phi) \quad \text{and} \quad u \in U_\xi \cap M_\phi \text{ if and only if } U_{-\xi}\, u \subseteq \ker \phi.$$

Now we observe that $\sigma(M_{\phi_1}) \subseteq M_{\phi_2}$. This follows since for any $u \in U_\xi \cap M_{\phi_1}$, $\sigma(u) \in U_{\xi\circ\bar{\sigma}^{-1}}$ and

$$\phi_2(U_{-\xi\circ\bar{\sigma}^{-1}}\sigma(u)) = \phi_2(\sigma(U_{-\xi})\sigma(u)) = \phi_2 \circ \sigma(U_{-\xi}u) = \phi_1(U_{-\xi}u) = 0.$$

Thus we can define a map $\hat{\sigma} : U/M_{\phi_1} \rightarrow U/M_{\phi_2}$ by setting

$$\sigma(u + M_{\phi_1}) = \sigma(u) + M_{\phi_2}.$$

Since $\sigma(M_{\phi_1}) = M_{\phi_2}$ and σ is an automorphism of U, $\hat{\sigma}$ is a well-defined, linear isomorphism from U/M_{ϕ_1} onto U/M_{ϕ_2}.

Finally, if $u + M_{\phi_1} \in (U/M_{\phi_1})_\lambda$ then for each $h \in H$

$$h(\sigma(u) + M_{\phi_2}) = \hat{\sigma}(\sigma^{-1}(h)u + M_{\phi_1}) = \hat{\sigma}(\lambda \circ \sigma^{-1}(h)u + M_{\phi_1})$$
$$= \lambda \circ \sigma^{-1}(h)\hat{\sigma}(u + M_{\phi_1}) = \lambda \circ \sigma^{-1}(h)(\sigma(u) + M_{\phi_2}).$$

That is,

$$\hat{\sigma}((U/M_{\phi_1})_\lambda) = (U/M_{\phi_2})_{\lambda\circ\bar{\sigma}^{-1}}.$$

Remark. It should be emphasized that the representations of L on U/M_{ϕ_1} and U/M_{ϕ_2} are not, in general, equivalent. However, we do have the following result:

PROPOSITION 2. *If* $\phi_1, \phi_2 \in F_L$ *with* $U/M_{\phi_1} \cong U/M_{\phi_2}$ *then for any* $\sigma \in \text{Aut}(L:H)$ *we have* $U/M_{\phi_1\circ\sigma} \cong U/M_{\phi_2\circ\sigma}$.

Proof. As an intermediate step we first show that $U/M_{\phi_1} \cong U/M_{\phi_2}$ if and only if for $\xi = (\phi_1 - \phi_2) \downarrow H$ there exists $u_0 \in U_\xi \backslash M_{\phi_2}$ such that $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0)$ for all $c \in C(L)$ and all $w \in U_{-\xi}$.

In fact if $U/M_{\phi_1} \cong U/M_{\phi_2}$ then there exists an L-module homomorphism $\psi : U/M_{\phi_1} \to U/M_{\phi_2}$. If $\psi(1 + M_{\phi_1}) = u_0 + M_{\phi_2}$ then clearly $u_0 \in U_\xi \backslash M_{\phi_2}$ and for $w \in U_{-\xi}$, $c \in C(L)$ we have

$$\psi(wc + M_{\phi_1}) = wcu_0 + M_{\phi_2} = \phi_2(wcu_0)(1 + M_{\phi_2})$$

and also

$$\psi(wc + M_{\phi_1}) = \psi(\phi_1(c)(w + M_{\phi_1})) = \phi_1(c)\psi(w + M_{\phi_1})$$
$$= \phi_1(c)(wu_0 + M_{\phi_2}) = \phi_1(c)\phi_2(wu_0)(1 + M_{\phi_2}).$$

Comparing, we have $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0)$.

Conversely if $\phi_1, \phi_2 \in F$ and there exists $u_0 \in U_\xi \backslash M_{\phi_2}$ such that $\phi_1(c)\phi_2(wu_0) = \phi_2(wcu_0)$ for all $c \in C(L)$ and all $w \in U_{-\xi}$ we claim $U/M_{\phi_1} \cong U/M_{\phi_2}$. Let

$$M = \text{Ann } (u_0 + M_{\phi_2}) = \{u \in U | uu_0 \in M_{\phi_2}\}.$$

Clearly M is a maximal left ideal of U and $U/M \cong U/M_{\phi_2}$. It remains only to show that $M = M_{\phi_1}$. Since M_{ϕ_1} is the unique maximal left ideal of U containing $\ker \phi_1$ it suffices to show that $\ker \phi_1 \subset M$. Take $c \in C(L)$ with $\phi_1(c) = 0$. Then we have that $\phi_2(wcu_0) = 0$ for all $w \in U_{-\xi}$. This implies that $cu_0 \in M_{\phi_2}$. That is, $c \in M$ as required.

Returning now to the proposition we assume $U/M_{\phi_1} \cong U/M_{\phi_2}$ and fix $u_0 \in U_\xi$ with properties as noted above. Then for any $\sigma \in \text{Aut } (L:H)$ we have

$$\phi_1 \circ \sigma(\sigma^{-1}(c))\phi_2 \circ \sigma(\sigma^{-1}(w)\sigma^{-1}(u_0)) = \sigma_2 \circ \sigma(\sigma^{-1}(w)\sigma^{-1}(c)\sigma^{-1}(u_0)).$$

But $\sigma^{-1}(C(L)) = C(L)$, $\sigma^{-1}(u_0) \in U_{\xi\circ\sigma} \backslash M_{\phi_2\circ\sigma}$ and $\sigma^{-1}(U_{-\xi}) = U_{-\xi\circ\sigma}$. Therefore for $\phi_1 \circ \sigma$, $\phi_2 \circ \sigma \in F_L$ where $\phi_1 \circ \sigma - \phi_2 \circ \sigma = \xi \circ \sigma$ there exists an element $\sigma^{-1}(u_0) \in U_{\xi\circ\sigma} \backslash M_{\phi_2\circ\sigma}$ such that for all $c' \in C(L)$ and all $w' \in U_{-\xi\circ\sigma}$ we have

$$\phi_1 \circ \sigma(c')\phi_2 \circ \sigma(w'\sigma^{-1}(u_0)) = \phi_2 \circ \sigma(w'c\sigma^{-1}(u_0))$$

which implies that $U/M_{\phi_1\circ\sigma} \cong M_{\phi_2\circ\sigma}$.

We now single out a finite subgroup of Aut $(L:H)$ which will be of importance in this paper. Calling liberally on the results of chapters 14 and 25 of [**4**] we let $\Delta \subset H^*$ be a root system of L with basis Δ_{++} and select a Chevalley basis

$$\{X_\beta, h_\alpha | B \in \Delta, \alpha \in \Delta_{++}\}$$

of L. To each $\alpha \in \Delta_{++}$ we define a map $S_\alpha : H^* \to H^*$ by setting

$$S_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha$$

where (,) denotes the symmetric, non-degenerate Killing form on H^*. The maps S_α are automorphisms sending Δ into itself and one can induce, via the Killing form, an automorphism (again denoted by S_α) of the Cartan subalgebra H. By Theorem 14.2 [**4**] there exists a unique automorphism, denoted by σ_α, of L such that σ_α extends S_α and

$$\sigma_\alpha(s_{\alpha\alpha'}) = X_{S_\alpha(\alpha')}$$

for all $\alpha' \in \Delta_{++}$. Let $A(L)$ denote the subgroup of Aut $(L:H)$ generated by $\{\sigma_\alpha | \alpha \in \Delta_{++}\}$. From the definition of the maps σ_α we can show that

$$\sigma_\alpha(X_\gamma) = \pm X_{\sigma_\alpha(\gamma)}$$

for all $\gamma \in \Delta$. Since $\{\sigma_\alpha \downarrow H | \alpha \in \Delta_{++}\}$ generates a group isomorphic to the Weyl group we can conclude that $A(L)$ is a finite group. In the particular case of $L = A_n$ the group $A(L)$ is isomorphic to the Weyl group of A_n.

Section 2. The family F_L**.** By combining the results of two previous papers [**6, 7**] we construct a family of algebra homomorphisms $\phi : C(L) \to \mathbf{C}$ as follows. In [**7**] we constructed for each fixed $s \in \mathbf{C}$ and each fixed linear functional λ in the dual of the Cartan subalgebra of A_n an explicit representation $(\rho, V_{s,\lambda})$ of A_n. The representation space $V_{s,\lambda}$ is the complex linear space having basis

$$\{v(\mathbf{k}) | \mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{Z} \times \ldots \times \mathbf{Z}\}$$

and the representatives of elements $x_{\alpha_i} = e_{i,i+1}$ and $y_{\alpha_i} = e_{i+1,i}$ in A_n are given by the formulas

$$\rho(x_{\alpha_i})v(\mathbf{k}) = (s - \lambda(h_1 + \ldots + h_{i-1}) - k_{i-1} + k_i)v(\mathbf{k} + \xi_i)$$
$$\rho(y_{\alpha_i})v(\mathbf{k}) = (s - \lambda(h_1 + \ldots + h_i) - k_i + k_{i+1})v(\mathbf{k} - \xi_i)$$

where ξ_i is the n-tuple having 1 in its i^{th} component and zeroes elsewhere. By convention $h_0 = 0$ and $k_0 = k_{n+1} = 0$. Since $\{x_{\alpha_i}, y_{\alpha_i} | i = 1, 2, \ldots, n\}$ generates A_n these formulas completely specify the representation $(\rho, V_{s,\lambda})$. For any such representation we obtain an algebra homomorphism $\phi : C(A_n) \to \mathbf{C}$ by setting

$$\phi(c)v(\mathbf{0}) = \rho(c)v(\mathbf{0}) \quad (\forall c \in C(A_n)).$$

Any algebra homomorphism defined as above will be called *standard*. As is easily checked for $n \geqq 2$ the parameters s and λ of a standard algebra homomorphism are uniquely determined.

To construct algebra homomorphisms $\phi : C(L) \to \mathbf{C}$ for an arbitrary simple Lie algebra L we first require some notation. Let $\Delta \subset H^*$ be the root system of L with basis Δ_{++} and set Δ_+ as the positive roots of L relative to Δ_{++}. Let $\{\Gamma_i\}_{i=1,2,\ldots,l}$ be a collection of disconnected complete subsets of Δ relative to Δ_{++}. Recall [cf. **6**] that this means:

1) $-\Gamma_i \subseteq \Gamma_i \quad (\forall i)$

2) $\alpha, \beta \in \Gamma_i, \quad \alpha + B \in \Delta \Rightarrow \alpha + \beta \in \Gamma_i \quad (\forall i)$

3) $\alpha, \beta \in \Delta_+, \quad \alpha + \beta \in \Gamma_i \Rightarrow \alpha, \beta \in \Gamma_i \quad (\forall i)$

4) $\Delta_{++} \cap \Gamma_i$ is a basis of $\Gamma_i \quad (\forall i)$

5) $\alpha \in \Gamma_i, \quad \beta \in \Gamma_j, \quad i \neq j \Rightarrow \alpha + \beta \notin \Delta$.

Note that such a collection can be constructed by selecting any subset of Δ_{++} and forming the closure in Δ of this set under $\pm$.

Select a Chevalley basis of L say $\{y_\beta, x_\beta, h_\alpha | \beta \in \Delta_+, \alpha \in \Delta_{++}\}$ and apply the Poincarré-Birkhoff-Witt Theorem to obtain a linear basis of $U(L)$ consisting of all monomials

$$\prod_{\beta\in\Delta_+} y_\beta{}^{t_\beta} \prod_{\beta\in\Delta_+} x_\beta{}^{r_\beta} \prod_{\alpha\in\Delta_{++}} h_\alpha{}^{l_\alpha} \quad (*)$$

where the exponents are non-negative integers and each product preserves a fixed order. A linear basis of $C(L)$ then consists of all monomials of the form (*) where

$$\sum_{\beta\in\Delta_+} (r_\beta - t_\beta)\beta = 0.$$

Denote by $C(\bigcup_i \Gamma_i)$ (resp. $C(\Gamma_i)$) the linear subspace of $C(L)$ generated by all basis elements of $C(L)$ for which $t_\beta = r_\beta = 0$ for all $\beta \in \Delta_+ \backslash \bigcup_i \Gamma_i$ (resp. $\beta \in \Delta_+ \backslash \Gamma_i$). Also set $\bar{C}(\bigcup_i \Gamma_i)$ (resp. $\bar{C}(\Gamma_i)$) equal to the linear subspace of $C(L)$ generated by all basis elements of $C(L)$ not in $C(\bigcup \Gamma_i)$ (resp. $C(L_i)$). By the properties of the Γ_i's one can readily see that $C(\bigcup_i \Gamma_i)$ and $C(\Gamma_i)$ are subalgebras of $C(L)$ and $\bar{C}(\bigcup_i \Gamma_i)$ and $\bar{C}(\Gamma_i)$ are two-sided ideals of $C(L)$ with

$$C(L) = C(\bigcup \Gamma_i) \oplus \bar{C}(\bigcup \Gamma_i) = C(\Gamma_i) \oplus \bar{C}(\Gamma_i)$$

as linear spaces.

From now on we assume that the Γ_i's are isomorphic to root systems of algebras A_{n_i} (for positive integers n_i). Then the subalgebra $U(\Gamma_i)$ of U generated by

$$\{1, h_\alpha, x_\beta, y_\beta | \alpha \in \Delta_{++} \cap \Gamma_i, \beta \in \Delta_+ \cap \Gamma_i\}$$

is isomorphic to the universal enveloping algebra of A_{n_i} and $C(L) \cap U(\Gamma_i) \cong C(A_{n_i})$. Identifying $C(A_{n_i})$ with $C(L) \cap U(\Gamma_i)$ and observing that

$$C(\Gamma_i) = \{C(L) \cap U(\Gamma_i)\} \cdot U(H),$$

any algebra homomorphism $\phi : C(A_{n_i}) \to \mathbf{C}$ can be extended to an algebra homomorphism $\bar{\phi} : C(\Gamma_i) \to \mathbf{C}$ by setting $\bar{\phi}(h_\alpha)$ to an arbitrary value for $\alpha \in \Delta_{++} \backslash \Gamma_i$.

Finally if $\bar{\phi}_i : C(\Gamma_i) \to \mathbf{C}$ are constructed as above starting from standard algebra homomorphisms ϕ_i: $C(A_{n_i}) \to \mathbf{C}$ such that $\bar{\phi}_i \downarrow U(H) = \bar{\phi}_j \downarrow U(H)$ for all i, j then by Theorem 6 [**6**] there exists an algebra homomorphism ϕ: $C(L) \to \mathbf{C}$ such that

1) $\phi \downarrow C(\Gamma_i) = \bar{\phi}_i$ for all i and

2) $\phi \downarrow \bar{C}(\bigcup_i \Gamma_i) = 0.$

Any such algebra homomorphism will be called a *generalized* (or *g*-) *standard algebra* homomorphism relative to $\bigcup_i \Gamma_i$.

CONJECTURE I. *Every algebra homomorphism* $\phi: C(L) \to \mathbf{C}$ *is weakly equivalent to a g-standard one. More precisely, there exists* $\sigma \in A(L)$ *such that* $\phi \circ \sigma$ *is g-standard.*

We now proceed to verify this conjecture for the algebras A_1, A_2 and A_3.

Case 1. *The Algebra* $A_1 = sl(2, \mathbf{C})$. A Chevalley basis of A_1 is given by $h = e_{11} - e_{22}$, $x = e_{12}$ and $y = e_{21}$ (where e_{ij} denotes the 2×2 matrix with $(i, j)^{th}$ component 1 and zero elsewhere). Fix $\mathbf{C} \cdot h$ as the Cartan subalgebra and observe that $C(A_1)$ is generated, as an algebra, by $\{1, h, yx\}$. Clearly $C(A_1)$ is commutative and has a linear basis given by

$$\{(yx)^{q_1}h^{q_2} | q_1, q_2 \in \mathbf{Z}^+\}.$$

Any algebra homomorphism $\phi \in F_{A_1}$ is then completely determined by specifying arbitrary values for $\phi(h)$ and $\phi(yx)$ and extending. In particular, we may select arbitrary scalars $s, \lambda \in \mathbf{C}$ and set $\phi(h) = \lambda$ and $\phi(yx) = s(s - \lambda - 1)$. Hence any algebra homomorphism $\phi \in F_{A_1}$ is standard.

Case 2. *The algebra* $A_2 = sl(3, \mathbf{C})$. A Chevalley basis for A_2 is given by the elements

$$\{h_\alpha = e_{11} - e_{22}, h_\beta = e_{22} - e_{33}, \quad x_\alpha = e_{12}, \quad x_\beta = e_{23}, \quad x_{\alpha+\beta} = e_{13}, \quad y_\alpha = e_{21}, \quad y_\beta = e_{32}, \quad y_{\alpha+\beta} = e_{31}\}$$

where e_{ij} denotes the 3×3 matrix with 1 in the $(i, j)^{th}$ component and zeroes elsewhere. Let $H = \mathbf{C}h_\alpha + \mathbf{C}h_\beta$ be the fixed Cartan subalgebra. As in [**1**] we observe that $C(A_2)$ is generated, as an algebra, by

$$\{1, h_\alpha, h_\beta, c_1 = y_\alpha x_\alpha, c_2 = y_\beta x_\beta, c_3 = y_{\alpha+\beta}x_{\alpha+\beta}, c_4 = y_{\alpha+\beta}x_\alpha x_\beta, c_5 = y_\beta y_\alpha x_{\alpha+\beta}\}$$

and has a linear basis given by

$$\{(c_5 \text{ or } c_4)^{q_1}c_3^{q_2}c_2^{q_3}c_1^{q_4}h_\alpha^{q_5}h_\beta^{q_6} | q_i \text{ are non-negative integers}\}.$$

If one sets $\phi(h_\alpha) = \lambda_1$, $\phi(h_\beta) = \lambda_2$ and $\phi(c_i) = z_i$ for $i = 1, 2, \ldots, 6$ then ϕ can be extended to a linear map on $C(A_2)$ using the above linear basis. This linear map ϕ is an algebra homomorphisms if and only if ϕ preserves the multiplication of the generators. This gives rise to the following four equations:

1. Since $c_1c_2 = c_2c_1 + c_5 - c_4$ we must have

$$z_4 = z_5.$$

2. Since $c_1c_4 = c_4c_1 + c_3c_1 - c_2c_1 + c_5 - c_3 - (c_4 - c_3)(h_\alpha + 1)$ we must have

$$\lambda_1(z_4 - z_3) = z_1(z_3 - z_2).$$

3. Since $c_2c_4 = c_4c_2 + c_2c_1 + c_5 - c_3c_2 - c_4h_\beta - c_4$ we must have

$$\lambda_2 z_4 = z_2(z_1 - z_3).$$

4. Since $c_4c_5 = c_3c_2c_1 + c_3c_2h_\alpha + c_3c_1h_\beta + c_3h_\alpha h_\beta + c_5c_3 + 2c_3c_1 + 2c_3h + 2c_4 - 2c_3c_2 - c_5h_\alpha - c_5h_\beta - 2c_5 + c_4c_2 - c_4c_1 - c_4h_\alpha$ we must have

$$(z_4 - z_3)(z_2 - z_1 - \lambda_1 - z_4) + z_3(z_2 + \lambda_2)(z_1 + \lambda_1) = 0.$$

The conditions imposed by multiplication of all other pairs of generators yield equations which are dependent on those above. Provided $z_i \neq 0, -\lambda_i$ for $i = 1, 2$ any solution of this system of equations is also a solution of the following system:

1′. $z_4 = z_5$

2′. $Nz_4 = (z_1 + \lambda_1 - z_2)z_1z_2$

3′. $Nz_3 = (\lambda_1 + \lambda_2)z_1z_2$

4′. $N(\lambda_1 + \lambda_2) = (z_2 - z_1 + \lambda_2)(z_2 - z_1 - \lambda_1)$

where $N = z_1\lambda_2 + z_2\lambda_1 + \lambda_1\lambda_2$. This latter system of equations has been solved by Bouwer [**1**] under the tacit assumption that $\lambda_1 + \lambda_2 \neq 0$. Since every such solution of $1' - 4'$ is also a solution of $1 - 4$ in order to determine all solutions of $1 - 4$ it remains only to solve this system under each of the above mentioned restrictions separately. Solving we obtain the following complete list of solutions to $1 - 4$ and hence all algebra homomorphisms $\phi : C(A_2) \to \mathbf{C}$.

Table I. Algebra Homomorphisms $\phi : C(A_2) \to \mathbf{C}$.

	T_0	T_1	T_2	T_3	T_4	T_5	T_6
h_α	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1
h_β	λ_2	λ_2	λ_2	λ_2	λ_2	λ_2	λ_2
c_1	$s(s - \lambda_1 - 1)$	p	0	$-\lambda_1$	0	$-\lambda_1$	p
c_2	$(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$	0	$-\lambda_2$	p	p	0	$-\lambda_2$
c_3	$s(s - \lambda_1 - \lambda_2 - 1)$	0	p	$-\lambda_1-\lambda_2$	0	p	$-\lambda_1-\lambda_2$
c_4	$s(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$	0	p	p	0	0	$-\lambda_1-\lambda_2-p$
c_5	$s(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$	0	p	p	0	0	$-\lambda_1-\lambda_2-p$

(The symbols λ_1, λ_2, s and p denote fixed but arbitrary complex numbers).

Note that the solutions of type T_0, T_1 and T_4 are g-standard algebra homomorphisms relative to Δ, $\{\pm\alpha\}$ and $\{\pm\beta\}$ respectively. We claim that the other solutions are weakly equivalent to T_1 or T_4. In fact recall that $A(A_2)$ is generated by the two elements σ_α and σ_β where the explicit definition of these automorphisms is given by

	h_α	h_β	x_α	x_β	$x_{\alpha+\beta}$	y_α	y_β	$y_{\alpha+\beta}$
σ_α	$-h_\alpha$	$h_\alpha + h_\beta$	y_α	$x_{\alpha+\beta}$	x_β	x_α	$y_{\alpha+\beta}$	y_β
σ_β	$h_\alpha + h_\beta$	$-h_\beta$	$x_{\alpha+\beta}$	y_β	x_α	$y_{\alpha+\beta}$	x_β	y_α

Extending these maps to automorphisms of $C(A_2)$ a direct computation verifies that if ϕ is a solution of type T_2 then $\phi \circ \sigma_\beta$ is a solution of type T_1 and if ϕ is of type T_3 then $\phi \circ \sigma_\alpha \circ \sigma_\beta$ is of type T_1. In addition if ϕ is a solution of type T_5 (resp. type T_6) then $\phi \circ \sigma_\alpha$ (resp. $\sigma \circ \sigma_\beta \circ \sigma_\alpha$) is a solution of type T_4. Thus we have shown that conjecture I is valid for the algebra A_2.

Remark. Solutions of type T_1 and T_4 are also weakly equivalent using the automorphism Φ defined by $\Phi(h_\alpha) = h_\beta$, $\Phi(h_\beta) = h_\alpha$, $\Phi(x_\alpha) = -x_\beta$ and $\Phi(x_\beta) = -x_\alpha$. Note however that $\Phi \notin A(A_2)$.

Case 3. *The algebras* $A_n = sl(n+1, \mathbf{C})$ *for* $n \geqq 3$. A Chevalley basis for A_n is given by the following set of elements:

$$\begin{aligned} h_{\alpha_i} &= e_{ii} - e_{i+1,i+1} && \text{for} \quad i = 1, 2, \ldots, n \\ x_{\alpha_i+\alpha_{i+1}+\ldots+\alpha_j} &= e_{i,j+1} && \text{for} \quad 1 \leqq i \leqq j \leqq n \\ y_{\alpha_i+\alpha_{i+1}+\ldots+\alpha_j} &= e_{j+1,i} && \text{for} \quad 1 \leqq i \leqq j \leqq n \end{aligned}$$

where e_{ij} denotes an $(n+1) \times (n+1)$ matrix with 1 in the $(i,j)^{th}$ component and zeroes elsewhere. We fix

$$H = \sum_{i=1}^n \mathbf{C} h_{\alpha_i}$$

as a Cartan subalgebra. By the Poincaré-Birkhoff-Witt Theorem there exists a linear basis of $U(A_n)$ given by

$$\prod_{1 \leqq i \leqq j \leqq n} y_{\alpha_i+\ldots+\alpha_j}^{t_{i,j+1}} \prod_{1 \leqq i \leqq j \leqq n} x_{\alpha_i+\ldots+\alpha_j}^{r_{i,j+1}} \prod_{i=1}^n h_{\alpha_i}^{l_i}$$

where the products preserve a fixed order on the basis elements of A_n and the exponents are non-negative integers. By the degree of any such monomial we mean

$$\sum_{1 \leqq i \leqq j \leqq n} (t_{i,j+1} + r_{i,j+1}) + \sum_{i=1}^n l_i.$$

PROPOSITION 3. *The algebra* $C(A_n)$ *is generated by the set*

$$\{1, h_{\alpha_1}, \ldots, h_{\alpha_n}\} \cup \Big\{ C(M) = \prod_{1 \leqq i \leqq j \leqq n} y_{\alpha_i+\ldots+\alpha_j}^{m_{j+1,i}} \prod_{1 \leqq i \leqq j \leqq n} x_{\alpha_i+\ldots+\alpha_j}^{m_{i,j+1}}$$

$M = (m_{ij}) \neq 0$ *is an* $(n+1) \times (n+1)$ *matrix of* 0's *and* 1's *with* $m_{ii} = 0$

and

$$\sum_{i=1}^{n+1} m_{i,k} = \sum_{i=1}^{n+1} m_{k,i} = 0 \quad \text{or } 1 \text{ for each } k$$

and M *cannot be expressed as a nontrivial sum of two such matrices* $\Big\}$.

Proof. The automorphisms $\sigma_{\alpha_i} \in A(A_n)$ can be realized by setting $\sigma_{\alpha_i}(x) = P_i^{-1} X P_i$ for all $x \in A_n$ where P_i is the permutation matrix of the transposition $(i, i+1)$.

To prove this proposition it suffices to show that every basis monomial $c \in C(A_n)$ can be expressed as a linear combination of products of the given generators. We assume inductively that the theorem is true for A_{n-1} and that the above statement is valid for basis monomials of $C(A_n)$ of degree $<k$. Now if $c \in C(A_n)$ is a basis monomial of degree k and contains some h_α as a factor then we can express c as a product of two basis monomials of $C(A_n)$ of degree strictly less than k and then the result follows from the inductive hypothesis.

Thus without loss of generality we assume $c \in C(A_n)$ is a basis monomial of degree k where

$$c = \prod_{1 \leqq i \leqq j \leqq n} y_{\alpha_i + \ldots + \alpha_j}^{l_{j+1,i}} \prod_{1 \leqq i \leqq j \leqq n} x_{\alpha_i + \ldots + \alpha_j}^{l_{i,j+1}}$$

and we associate with c the matrix $\Lambda = (l_{ij})$ where $l_{ii} = 0$. If Λ is one of the matrices described in the statement of the proposition then c itself is a generator and we are finished. If not, we note that since $c \in C(A_n)$ we have

$$\sum_{i=1}^{n+1} l_{i,k} = \sum_{i=1}^{n+1} l_{k,i}$$

for all k and hence we must have for some k

$$\sum_{i=1}^{n+1} l_{i,k} = \sum_{i=1}^{n+1} l_{k,i} \geqq 2.$$

In fact we may assume that this is true for $k = n + 1$. (This follows since we have $\sigma_{\alpha_i}(c) = c' +$ terms of degree $<k$ where c' is a basis monomial of $C(A_n)$ with associated matrix $P_i^{-1}\Lambda P_i$).

We now factor c into generating elements of $C(A_{n-1}\{\alpha_1, \ldots, \alpha_{n-1}\})$ by suppressing the index α_n, say $c = c_1c_2 \ldots c_p +$ terms of lower degree. (Note that this factorization is not unique and whenever y_{α_n} or x_{α_n} occur as factors in c they are treated as separate factors in this product). Since each factor c_i is a generating element of $C(A_{n-1}\{\alpha_1, \ldots, \alpha_{n-1}\})$ or one of the terms y_{α_n} or x_{α_n} we have that it can contain at most one factor of the form $y_{\alpha_i + \ldots + \alpha_n}$. Thus for each i, $c_i \in C(A_n)$ or $U(A_n)_{\pm\alpha_n}$. By assumption

$$\sum_{i=1}^{n+1} l_{i,n+1} = \sum_{i=1}^{n+1} l_{n+1,i} \geqq 2$$

and hence the above factorization must contain at least two factors. If there are exactly two factors then each factor must contain exactly one term of the form $y_{\alpha_i + \ldots + \alpha_n}$ and one term of the form $x_{\alpha_j + \ldots + \alpha_n}$ and hence both factors are in $C(A_n)$ and we may apply out inductive hypothesis on each factor. If there are more than two factors, then either all are in $C(A_n)$ in which case we are finished or at least one, say c_1, is in $U(A_n)_{+\alpha_n}$ and at least one, say c_i, is in $U(A_n)_{-\alpha_n}$. Then $c = (c_1c_i)(c_2 \ldots) +$ terms of lower degree and $c_1c_i, c_2 \ldots \in C(A_n)$ and again we may apply our inductive hypothesis to complete the proof.

We now return to the problem of constructing the family of algebra homomorphisms F_{A_n} and prove the following reduction:

PROPOSITION 4. *Any algebra homomorphism* ϕ: $C(A_n) \rightarrow \mathbf{C}$ *is completely determined by its values on the generators of* $C(A_n)$ *of degree* $\leqq 3$. *In particular,* ϕ *is trivial on* $C(A_n)$ *if* $\phi = 0$ *on all generators of degrees* 1 *and* 2.

Proof. We proceed by induction on n, noting that the cases $n = 1$ and 2 are trivially true. For the inductive step we observe that every generator of $C(A_n)$ of degree $\leqq n$ is contained in a subalgebra isomorphic to $C(A_{n-1})$. Thus it suffices to verify that the value of ϕ on the generators of degree $n + 1$ are determined by the values of ϕ on the generators of degree $\leqq n$.

The problem is further reduced by observing that ϕ is completely determined on all generators of degree $n + 1$ provided ϕ is known on all generators of degree $\leqq n$ and one generator of degree $n + 1$. In fact consider the following identities in $C(A_n)$:

a) $$[y_{\alpha_n}x_{\alpha_n}, y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_{n-1}}] = y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_n} - y_{\alpha_n}y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}\ldots x_{\alpha_{n-1}+\alpha_n}$$

b) $$[y_{\alpha_1}x_{\alpha_1}, y_{\alpha_2+\ldots+\alpha_n}x_{\alpha_2}\ldots x_{\alpha_n}] = y_{\alpha_2+\ldots+\alpha_n}y_{\alpha_1}x_{\alpha_1+\alpha_2}x_{\alpha_3}\ldots x_{\alpha_n} - y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_n}$$

c) $$[y_{\alpha_i}x_{\alpha_i}, y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_{i-1}}x_{\alpha_i+\alpha_{i+1}}x_{\alpha_{i+2}}\ldots x_{\alpha_n}] = y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_n}$$
$$- y_{\alpha_1+\ldots+\alpha_n}y_{\alpha_i}x_{\alpha_1}\ldots x_{\alpha_{i-1}+\alpha_i}x_{\alpha_i+\alpha_{i+1}}x_{\alpha_{i+2}}\ldots x_{\alpha_n}$$
$$- y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_{i-1}}x_{\alpha_i+\alpha_{i+1}}x_{\alpha_{i+2}}\ldots x_{\alpha_n} \text{ for } i = 2, 3, \ldots, n-1.$$

Setting $M_0 = e_{n+1,1} + \sum_{i=1}^{n} e_{i,i+1}$ and applying the algebra homomorphism ϕ to the above identities we have

a) and b) $\Rightarrow \phi(c(M_0)) = \phi(c(P_n^{-1}M_0P_n)) = \phi(c(P_1^{-1}M_0P_1))$

c) $\Rightarrow \phi(c(M_0)) = \phi(c(P_i^{-1}M_0P_i)) + \phi(\text{a degree } n \text{ term})$
for $i = 2, 3, \ldots n - 1$.

If $c(M)$ is an arbitrary degree $n + 1$ generator of $C(A_n)$, we have $M = P^{-1}M_0P$ where P is a product of transposition matrices P_i. By sequentially applying the corresponding product of automorphisms $\sigma_{\alpha_i} \in A(A_n)$ to the above identities we may conclude that

$$\phi(c(M)) = \phi(c(M_0)) + \phi \text{ (terms of degree } \leqq n).$$

Thus ϕ is completely determined if one knows the image of ϕ on all generators of degree $\leqq n$ and on one generator of degree $n + 1$.

Assume now that ϕ is zero on all generators ($\neq 1$) of degree $\leqq n$. Considering the identity

$$(y_{\alpha_n}y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}\ldots x_{\alpha_{n-1}+\alpha_n})(y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_n})$$
$$= (y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}\ldots x_{\alpha_{n-1}+\alpha_n}y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_{n-1}})(y_{\alpha_n}x_{\alpha_n})$$
$$+ y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_{n-1}+\alpha_n} - y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}\ldots x_{\alpha_{n-1}})$$
$$\times (y_{\alpha_1+\ldots+\alpha_n}x_{\alpha_1}\ldots x_{\alpha_n})$$

and applying the map ϕ we obtain $\phi(c(P_n^{-1}M_0P_n))\phi(c(M_0)) = 0$. But by a) above this implies $\phi(c(M_0))^2 = 0$; ie. $\phi(c(M_0)) = 0$. Thus ϕ is identically

https://doi.org/10.4153/CJM-1979-100-5 Published online by Cambridge University Press

zero on all degree $n + 1$ generators. From Table I we note that any algebra homomorphism $\phi : C(A_n) \rightarrow C$ for which $\phi = 0$ on degree 1 and 2 generators is also zero on all degree 3 generators and hence the second statement of the proposition is verified.

We may now assume that ϕ is non-zero on some generator of degree $\leqq 2$; in fact, without loss of generality we may assume that $\phi \circ \sigma(y_{\alpha_1}x_{\alpha_1}) \neq 0$ for some $\sigma \in A(A_n)$. Now consider the identity

$$\begin{aligned}(y_{\alpha_2+\ldots+\alpha_n}x_{\alpha_2+\ldots+\alpha_n})(y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_{n-1}}) \\ = (y_{\alpha_2+\ldots+\alpha_n}y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1+\ldots+\alpha_n}x_{\alpha_2}\ldots x_{\alpha_{n-1}})(y_{\alpha_1}x_{\alpha_1}) \\ + (y_{\alpha_2+\ldots+\alpha_n}y_{\alpha_1}x_{\alpha_1+\ldots+\alpha_n})(y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1+\alpha_2}x_{\alpha_3}\ldots x_{\alpha_{n-1}}) \\ + (y_{\alpha_2+\ldots+\alpha_n}x_{\alpha_2+\ldots+\alpha_n})(y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_{n-1}}).\end{aligned}$$

Applying the homomorphism $\phi \circ \sigma$ to this identity we have that the value of ϕ on one generator of degree $n + 1$, namely the degree $n + 1$ generator associated with

$$\sigma(y_{\alpha_2+\ldots+\alpha_n}y_{\alpha_1+\ldots+\alpha_{n-1}}x_{\alpha_1+\ldots+\alpha_n}x_{\alpha_2}\ldots x_{\alpha_{n-1}}),$$

can be expressed as a rational function of the values of ϕ on generators of degree $\leqq n$.

We now particularize these results to the case of $n = 3$ where we construct, up to weak equivalence, all members of F_{A_3}. Take an arbitrary algebra homomorphsm $\phi \in F_{A_3}$ and assume first that ϕ, restricted to one of the four naturally embedded copies of $C(A_2)$, is of type T_i for $i = 1, 2, \ldots, 6$ (cf. Table I). Applying an appropriate automorphism from Aut (A_3) we may assume that ϕ restricted to $C(A_2\{\alpha, \beta + \gamma\})$ is of Type T_1. This places restrictions on the other values of ϕ as shown in the following table:

Table II

	1.	2*a*)	*b*)	*c*)	*d*)	3*a*)	*b*)	*c*)	*d*)
$\phi(h_\alpha)$	λ_1								
$\phi(h_\beta)$	λ_2								
$\phi(h_\gamma)$	λ_3								
$\phi(c_1)$	p								
$\phi(c_2)$		0	q	$-\lambda_2$	q				
$\phi(c_3)$		q	0	q	$-\lambda_3$	0	r	r	$-\lambda_3$
$\phi(c_4)$						r	0	$-\lambda_1-\lambda_2$	r
$\phi(c_5)$	0	0	0	0	0				
$\phi(c_6)$	0					0	0	0	0
$\phi(c_7) = \phi(c_9)$									
$\phi(c_8) = \phi(c_{10})$		0	0	q	$-q$				
$\phi(c_{11}) = \phi(c_{13})$						0	0	r	$-r$
$\phi(c_{12}) = \phi(c_{14})$	0								
$\phi(c_{15}) = \ldots = \phi(c_{20})$									

Remarks. 1. For convenience we have labelled the generators of $C(A_3)$ by setting

$$c_1 = y_\alpha x_\alpha; \quad c_2 = y_\beta x_\beta; \quad c_3 = y_\gamma x_\gamma; \quad c_4 = y_{\alpha+\beta} x_{\alpha+\beta}; \quad c_5 = y_{\beta+\gamma} x_{\beta+\gamma};$$
$$c_6 = y_{\alpha+\beta+\gamma} x_{\alpha+\beta+\gamma}; \quad c_7 = y_{\alpha+\beta} x_\alpha x_\beta; \quad c_8 = y_{\beta+\gamma} x_\beta x_\gamma; \quad c_9 = y_\beta y_\alpha x_{\alpha+\beta};$$
$$c_{10} = y_\gamma y_\beta x_{\beta+\gamma}; \quad c_{11} = y_{\alpha+\beta+\gamma} x_{\alpha+\beta} x_\gamma; \quad c_{12} = y_{\alpha+\beta+\gamma} x_\alpha x_{\beta+\gamma};$$
$$c_{13} = y_\gamma y_{\alpha+\beta} x_{\alpha+\beta+\gamma}; \quad c_{14} = y_{\beta+\gamma} y_\alpha x_{\alpha+\beta+\gamma}; \quad c_{15} = y_{\alpha+\beta+\gamma} x_\alpha x_\beta x_\gamma;$$
$$c_{16} = y_\gamma y_\beta y_\alpha x_{\alpha+\beta+\gamma}; \quad c_{17} = y_{\alpha+\beta} y_{\beta+\gamma} x_{\alpha+\beta+\gamma} x_\beta;$$
$$c_{18} = y_\beta y_{\alpha+\beta+\gamma} x_{\alpha+\beta} x_{\beta+\gamma}; \quad c_{19} = y_\gamma y_{\alpha+\beta} x_\alpha x_{\beta+\gamma}; \quad c_{20} = y_{\beta+\gamma} y_\alpha x_{\alpha+\beta} x_\gamma.$$

2. The values in column 1 result from the assumption that $\phi \downarrow C(A_2\{\alpha, \beta + \gamma\})$ is of type T_1.

3. The values in columns 2a)-d) represent the four possible solutions for $\phi \downarrow C(A_2\{\beta, \gamma\})$ consistent with $\phi(c_5) = 0$. In columns 2*c*) and *d*) we also must have $\phi(h_\beta) + \phi(h_\gamma) = \lambda_2 + \lambda_3 = 0$.

4. The values in columns 3a)-d) represent the four possible solutions for $\phi \downarrow C(A_2\{\alpha + \beta, \gamma\})$ consistent with $\phi(c_6) = 0$. In columns 3c) and d) we also must have $\phi(h_\alpha + h_\beta) + \phi(h_\gamma) = \lambda_1 + \lambda_2 + \lambda_3 = 0$.

If ϕ satisfies conditions 2a) and 3a) then $\phi = 0$ on all generators of $C(A_3)$ in $\bar{C}\{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ of degree $\leqq 3$. Thus ϕ must coincide with the trivial extension of an algebra homomorphism $\phi : C(A_2\{\alpha, \beta\}) \to \mathbf{C}$. By the previous analysis of F_{A_2}, there exists $\sigma \in A(A_2\{\alpha, \beta\})$ such that $\phi \circ \sigma : C(A_2\{\alpha, \beta\}) \to \mathbf{C}$ is g-standard. Since any $\sigma \in A(A_2\{\alpha, \beta\})$ has a natural extension to a map $\bar{\sigma} \in A(A_3)$ with the property that

$$\bar{\sigma}(\bar{C}\{\pm\alpha, \pm\beta, \pm(\alpha, \beta)\}) \subseteq \bar{C}\{\pm\alpha, \pm\beta, \pm(\alpha, \beta)\}$$

we conclude that $\phi \circ \bar{\sigma}$ agrees with a g-standard algebra homomorphism of F_{A_3} on all generators of degree $\leqq 3$ and hence by Proposition 3, $\phi \circ \bar{\sigma}$ is itself g-standard.

If ϕ satisfies conditions 2a) and 3b) then $\phi = 0$ on all generators of $C(A_3)$ in $\bar{C}\{\pm\alpha, \pm\gamma\}$ of degree $\leqq 3$. Thus ϕ is a trivial extension of algebra homomorphisms ϕ_1: $C(\pm\alpha) \to \mathbf{C}$ and $\phi_2 : C(\pm\gamma) \to \mathbf{C}$ and hence is g-standard relative to $\{\pm\alpha\} \cup \{\pm\gamma\}$.

In each of the other cases, by using identities from $C(A_3)$, and automorphisms from $A(A_3)$ we can show that ϕ is weakly equivalent to a g-standard algebra homomorphism.

It remains now to consider those algebra homomorphisms $\phi \in F_{A_3}$ such that the restrictions of ϕ to each of the four copies of $C(A_2)$ in $C(A_3)$ are standard; ie. of type T_0 from Table I. We parametrize ϕ separately on each restriction as follows:

Table III

	$C(A_2\{\alpha, \beta\})$	$C(A_2\{\beta, \gamma\})$	$C(A_2\{\alpha + \beta, \gamma\})$	$C(A_2\{\alpha, \beta + \gamma\})$
$\phi(h_\alpha)$	λ_1	λ_1	λ_1	λ_1
$\phi(h_\beta)$	λ_2	λ_2	λ_2	λ_2
$\phi(h_\gamma)$	λ_3	λ_3	λ_3	λ_3
$\phi(c_1)$	$s(s - \lambda_1 - 1)$			$u(u - \lambda_1 - 1)$
$\phi(c_2)$	$(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$	$t(t - \lambda_2 - 1)$		
$\phi(c_3)$		$(t - \lambda_2)(t - \lambda_2 - \lambda_3 - 1)$	$(v - \lambda_1 - \lambda_2)(v - \lambda_1 - \lambda_2 - \lambda_3 - 1)$	
$\phi(c_4)$	$s(s - \lambda_1 - \lambda_2 - 1)$		$v(v - \lambda_1 - \lambda_2 - 1)$	
$\phi(c_5)$		$t(t - \lambda_2 - \lambda_3 - 1)$		$(u - \lambda_1)(u - \lambda_1 - \lambda_2 - \lambda_3 - 1)$
$\phi(c_6)$				$u(u - \lambda_1 - \lambda_2 - \lambda_3 - 1)$
$\phi(c_7) = \phi(c_9)$	$s(s - \lambda_1)(s - \lambda_1 - \lambda_2 - 1)$			
$\phi(c_8) = \phi(c_{10})$		$t(t - \lambda_2)(t - \lambda_2 - \lambda_3 - 1)$		
$\phi(c_{11}) = \phi(c_{13})$			$v(v - \lambda_1 - \lambda_2)(v - \lambda_1 - \lambda_2 - \lambda_3 - 1)$	
$\phi(c_{12}) = \phi(c_{14})$				$u(u - \lambda_1)(u - \lambda_1 - \lambda_2 - \lambda_3 - 1)$

In order that ϕ be well-defined we must have certain relations among the parameters; in fact, we must have

1. $s = u$ or $s = 1 + \lambda_1 - u$
2. $s = t + \lambda_1$ or $s = 1 + \lambda_1 + \lambda_2 - t$
3. $t = v - \lambda_1$ or $t = 1 + \lambda_1 + 2\lambda_2 + \lambda_3 - v$
4. $s = v$ or $s = 1 + \lambda_1 + \lambda_2 - v$
5. $t = u - \lambda_1$ or $t = 1 + \lambda_1 + \lambda_2 + \lambda_3 - u$
6. $v = u$ or $v = 1 + \lambda_1 + \lambda_2 + \lambda_3 - u$.

By analyzing each of the distinct combinations of relations and applying Proposition 3, we may conclude that either ϕ is a standard algebra homomorphism in F_{A_3} or ϕ is weakly equivalent under $A(A_3)$ to one of the previously described algebra homomorphisms. Thus to summarize we have that Conjecture I is valid for the algebra A_3.

Although we are as yet unable to verify this conjecture for the algebra A_n with $n \geqq 4$ we do have the following first step in this direction:

PROPOSITION 5. *If $\phi : C(A_n) \to \mathbf{C}$ is an algebra homomorphism such that ϕ restricted to each copy of $C(A_3)$ in $C(A_n)$ is standard then ϕ itself is standard.*

Proof. We proceed by induction on n, noting that the case $n = 3$ is trivially true. Assume that the proposition is true for $n - 1 \geqq 3$ and consider $\phi : C(A_n) \to \mathbf{C}$ as given. By our inductive hypothesis ϕ restricted to the subalgebras

$$C(A_{n-1}\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}), \quad C(A_{n-1}\{\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_n\}), \ldots, \\ C(A_{n-1}\{\alpha_2, \ldots, \alpha_n\})$$

is standard with parameters $s_1, s_2, \ldots, s_{n+1}$ respectively. In order that ϕ be well-defined we must have $s_1 = s_2 = \ldots = s_n = s_{n+1} + \phi(h_{\alpha_1})$. Since every degree $\leqq 3$ generator of $C(A_n)$ is in at least one of these subalgebras we have that ϕ agrees on all generators of degree $\leqq 3$ with a standard algebra homomorphism of F_{A_n} paramerized by s_1 and $\phi \downarrow H$. By Proposition 4 we have that ϕ itself is then standard.

Section 3. Pointed representations. In this section we shall "label" the pointed representations of a simple Lie algebra L in the following sense. We wish to specify a set $\hat{F}_L \subseteq F_L$ having the following properties:

1) If $\phi_1, \phi_2 \in \hat{F}_L$ with $\phi_1 \neq \phi_2$ then $U/M_{\phi_1} \ncong U/M_{\phi_2}$ as L-modules.

2) If V is a pointed representation of L then there exists $\phi \in F_L$ such that $V \cong U/M_\phi$ and ϕ is weakly equivalent modulo $A(L)$ to an element in $\hat{F}_L$.

Since the group $A(L)$ is finite we would thus associate with each $\phi \in \hat{F}_L$ a finite number of non-equivalent pointed representations of L.

Definition. A standard algebra homomorphism $\phi : C(A_n) \rightarrow \mathbf{C}$ with parameters $s \in \mathbf{C}$ and $\lambda \in H^*$ is said to be *complete* if and only if

$$s - \phi(\sum_{i=0}^{p} h_{\alpha i} \notin \mathbf{Z} \quad \text{for } p = 0, 1, \ldots, n \quad \text{and} \quad 0 \leqq \operatorname{Re} \phi(h_{\alpha_i}) < 1 \quad \text{for} \quad i = 1, 2, \ldots, n$$

where $\{\check{\alpha}_i\}$ is the dual basis of $\{\alpha_i\}$ relative to the Killing form.

(Note that if $\phi \downarrow H = \sum_{j=1}^{n} S_j \alpha_j$ then $\phi(h_{\check{\alpha}_i}) = S_i$).

Definition. A g-standard algebra homomorphism ϕ: $C(L) \rightarrow \mathbf{C}$ defined relative to $\bigcup_{i=1}^{l} \Gamma_i$ is said to be *extreme* if and only if $\phi \downarrow \{C(L) \cap U(\Gamma_i)\}$ is complete for each i.

Remark. In particular any algebra homomorphism $\phi : C(L) \rightarrow \mathbf{C}$ which is identically zero on the ideal $\bar{C}(\emptyset)$ is an extreme g-standard algebra homomorphism.

CONJECTURE II. *The family of all extreme g-standard algebra homomorphisms $\phi \in F_L$ labels the pointed representations of L.*

Our aim in this section will be to prove that any two distinct extreme g-standard algebra homomorphisms give rise to non-equivalent pointed representations and that if ϕ is a g-standard algebra homomorphism then there exists an extreme g-standard algebra homomorphism $\bar{\phi}$ such that $U/M_{\bar{\phi}} \cong U/M_{\phi \circ \sigma}$ for some $\sigma \in A(L)$. This will imply that for any algebra L satisfying Conjecture I, Conjecture II is also valid.

We first give an explicit description for the pointed representations associated with standard and g-standard algebra homomorphisms. Let $\phi : C(A_n) \rightarrow \mathbf{C}$ be the standard algebra homomorphism parametrized by $s \in \mathbf{C}$ and $\lambda \in H^*$. For each $u \in U_\xi$ where $\xi = \sum_{i=1}^{n} k_i \alpha_i$ $(k_i \in \mathbf{Z})$ we define a scalar $\mu(u)$ by setting

$$\rho(u)v(\mathbf{0}) = \mu(u)v(k_1, \ldots, k_n).$$

We claim that $\mu(u) = 0$ implies $\mu \in M_\phi$. In fact, it suffices to show that for any $w \in U_{-\xi}$ we have $\phi(wu) = 0$ and this follows since

$$\phi(wu)v(\mathbf{0}) = \rho(wu)v(\mathbf{0}) = \rho(w)\rho(u)v(\mathbf{0}) = \rho(w)\mu(u)v(k_1, \ldots, k_n) = 0.$$

By construction of U/M_ϕ every weight function must be of the form

$$\eta = (\phi + \sum_{i=1}^{n} l_i \alpha_i) \downarrow H$$

where the coefficients l_i's are integers. Setting $\xi = \sum_{i=1}^{n} l_i \alpha_i$ we know that $(U/M_\phi)_\eta \cong U_\xi/(U_\xi \cap M_\phi)$ as H-modules. Taking $u_1, u_2 \in U_\xi$ we claim that the set $\{u_1 + M_\phi, u_2 + M_\phi\}$ is always linearly dependent. Without loss of generality we may assume that $\mu(u_2) \neq 0$ and hence consider the element

$$u_1 = \frac{\mu(u_1)}{\mu(u_2)} u_2.$$

For all $w \in U_{-\xi}$ we have

$$\begin{aligned}\phi\left(w\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)\right)v(\mathbf{0}) &= \rho\left(w\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)\right)v(\mathbf{0})\\ &= \rho(w)\rho\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)v(\mathbf{0})\\ &= \rho(w)\left(\mu(u_1) - \frac{\mu(u_1)}{\mu(u_2)}\mu(u_2)\right)v(l_1, \ldots, l_n)\\ &= 0 \quad \text{or}\end{aligned}$$

$$\phi\left(U_{-\xi}\left(u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2\right)\right) = 0 \quad \text{and hence} \quad u_1 - \frac{\mu(u_1)}{\mu(u_2)} u_2 \in M_\phi.$$

Therefore $\dim(U/M_\phi)_\eta \leqq 1$ for all η.

To complete our description of the representation U/M_ϕ it remains only to indicate which weight spaces are one-dimensional. To this end we set

$$P_i = \begin{cases} s - \lambda(h_{\alpha_1} + \ldots + h_{\alpha_i}) & \text{if this is a positive integer}\\ +\infty & \text{otherwise}\end{cases}$$

and

$$q_i = \begin{cases} s - \lambda(h_{\alpha_1} + \ldots + h_{\alpha_i}) & \text{if this is a non-positive integer}\\ -\infty & \text{otherwise}\end{cases}$$

where $i = 0, 1, 2, \ldots, n$ and by convention $h_{\alpha_0} = 0$. Define

$$D_{s,\lambda} = \{(l_1, \ldots, l_n) \in \mathbf{Z}^n | \, q_i \leqq l_i - l_{i+1} < P_i \quad \text{for all} \quad i = 0, 1, \ldots, n\}$$

(note that $l_0 = l_{n+1} = 0$ by convention). We claim then that the linear functional $(\phi + \sum_{i=1}^n l_i\alpha_i) \downarrow H$ is a one-dimensional weight function of U/M_ϕ if and only if $(l_1, \ldots, l_n) \in D_{s,\lambda}$. Recall from [**7**] that if

$$s - \lambda(h_{\alpha_1} + \ldots + h_{\alpha_i}) = m \in \mathbf{Z}$$

then the subspace of $V_{s,\lambda}$ with basis $\{v(k_1, \ldots, k_n) | \, k_i - k_{i+1} \geqq m\}$ is a subrepresentation of $(\rho, V_{s,\lambda})$. Suppose now that $u \in U_\xi$ with $\xi = \sum_{i=1}^n l_i\alpha_i$ and $(l_1, \ldots, l_n) \notin D_{s,\lambda}$ then for any $w \in U_{-\xi}$ we must have

$$\phi(wu)v(\mathbf{0}) = \rho(wu)v(\mathbf{0}) = \rho(w)\rho(u)v(\mathbf{0}) = 0$$

since there exists a subrepresentation of $V_{s,\lambda}$ to which only one of the vectors $v(\mathbf{0})$ and $v(l_1, \ldots, l_n)$ belongs. If, on the other hand, $(l_1, \ldots, l_n) \in D_{s,\lambda}$ then one can select elements $u \in U_\xi$ and $w \in U_{-\xi}$ such that $\phi(wu) \neq 0$; ie. $u \notin M_\phi$. Summarizing we have

PROPOSITION 6. *With the notation introduced above, if* $\phi : C(A_n) \to \mathbf{C}$ *is a standard algebra homomorphism parametrized by* $s \in \mathbf{C}$ *and* $\lambda \in H^*$ *then the associated pointed representation of* A_n *is*

$$U/M_\phi = \sum_{(l_1,\ldots,l_n)\in D_{s,\lambda}} \oplus \ (U/M_\phi)_{\phi+\sum_{i=1}^{n} l_i\alpha_i}$$

where each weight space is one-dimensional.

We now consider a g-standard algebra homomorphism $\phi : C(L) \to \mathbf{C}$ relative to $\bigcup \Gamma_i$ and make the following observations:

1) For any $v = \Delta_+ \backslash \bigcup \Gamma_i$, $x_v \in M_\phi$. In fact, if $w \in U_{-v}$ then $wx_v \in \bar{C}(\bigcup \Gamma_i)$ and hence $\phi(wx_v) = 0$; ie. $x_v \in M_\phi$.

2) If u is a basis element of U of the form (*) for which $\exists\ \beta \in \Delta_+\backslash\bigcup \Gamma_i$ with $r_\beta \neq 0$ then $u \in M_\phi$. This follows from 1) using induction on the degree of u.

3) If $\xi = \sum_{\alpha\in\Delta_{++}\cap(\bigcup\Gamma_i)} k_\alpha \cdot \alpha$ where $(\forall\alpha)\ k_\alpha \in \mathbf{Z}$ then for any basis element $u \in U_\xi$ we have either $u \in M_\phi$ or $u = u_1u_2\ldots u_l z$ where $z \in U(H)$ and, if

$$\xi_i = \sum_{\alpha\in\Delta_{++}\cap\Gamma_i} k_\alpha \cdot \alpha,$$

$u_i \in U(\Gamma_i)_{\xi_i}$.
If $u \notin M_\phi$ then by 2) we may assume that $r_\beta = t_\beta = 0$ for all $\beta \in \Delta_+\backslash\bigcup \Gamma_i$. Then applying induction on the degree of u, we may reorder the terms of u into the required form.

4) For each i, $M_\phi \cap U(\Gamma_i)$ is a maximal left ideal of $U(\Gamma_i)$.

It is clear that $M_\phi \cap U(\Gamma_i)$ is a left ideal of $U(\Gamma_i)$ and since $\ker\phi \cap U(\Gamma_i) \subseteq M_\phi \cap U(\Gamma_i)$ it remains only to show that for any $u \in U(\Gamma_i)_\eta\backslash M_\phi$, where η is an integral linear combination of roots from $\Delta_{++} \cap \Gamma_i$, there exists $v \in U(\Gamma_i)_{-\eta}$ such that $\phi(vu) \neq 0$. Since M_ϕ is maximal in U there exists $w \in U_{-\eta}$ with $\phi(wu) \neq 0$. If w_0 is a basis element of U of minimal degree such that $\phi(w_0u) \neq 0$ then $w_0 \in U(\Gamma_i)_{-\eta}$. In fact w_0 does not contain any factors of type h_α since in this case we have $w_0 = w'h_\alpha$ + lower degree terms and hence a contradiction;

$$0 \neq \phi(w_0u) = \phi(w'h_\alpha u) = \phi(w'u)\phi(h_\alpha) + \eta(h_\alpha)\phi(w'u) = 0.$$

We also know that $w_0 \in U(\bigcup \Gamma_i)$ as otherwise $w_0u \in C(\bigcup \Gamma_i)$. Thus by 3) we have $w_0 = cv$ + lower degree terms where $c \in C(L)$ and $v \in U(\Gamma_i)_{-\eta}$. By the minimality of the degree of w_0 we must have c is a non-zero scalar and hence $w_0 \in U(\Gamma_i)_{-\eta}$, as required.

With the help of these observations we can now prove the following result:

PROPOSITION 7. *Let*

$$\xi = \sum_{\alpha\in\Delta_{++}\cap(\bigcup\Gamma_i)} k_\alpha \cdot \alpha \quad \textit{and} \quad \xi_i = \sum_{\alpha\in\Delta_{++}\cap\Gamma_i} k_\alpha \cdot \alpha$$

where $k_\alpha \in \mathbf{Z}$ *for all* α. *Then* $\dim (U/M_\phi)_\lambda \leqq 1$ *for* $\lambda = (\phi + \xi) \downarrow H$ *and moreover* $\dim (U/M_\phi)_\lambda = 1$ *if and only if*

$$\dim (U(\Gamma_i)/(M_\phi \cap U(\Gamma_i)))_{\phi+\xi_i} = 1 \text{ for all } i = 1, 2, \ldots, l.$$

Proof. Since for each i, $U(\Gamma_i) \cong U(A_{n_i})$ and $\phi \downarrow (C(L) \cap U(\Gamma_i))$ is a standard algebra homomorphism, Proposition 6 implies that

$$\dim (U(\Gamma_i)/(M_\phi \cap U(\Gamma_i)))_{\phi+\xi_i} \leqq 1$$

and gives explicit conditions when it is exactly 1.

Assume first that there exists i_0 such that

$$\dim (U(\Gamma_{i_0})/(M_\phi \cap U(\Gamma_{i_0})))_{\phi+\xi_{i_0}} = 0.$$

This implies that

$$U(\Gamma_{i_0})_{\xi_{i_0}} \subseteq M_\phi.$$

We claim that in this case $U_\xi \subseteq M_\phi$ and hence $\dim (U/M_\phi)_\lambda = 0$. In fact if $u \in U_\xi$ is a basis element we may assume by remark 3 that $u = u_1 u_2 \ldots u_l z$ where $z \in U(H)$ and $u_i \in U(\Gamma_i)_{\xi_i}$. Then

$$u = u_1 u_2 \ldots u_l z = u_1 \ldots \hat{u}_{i_0} \ldots u_l z u_{i_0} + \xi_{i_0}(z) u_1 \ldots \hat{u}_{i_0} \ldots u_l u_{i_0} \in M_\phi.$$

That is, $U_\xi \subseteq M_\phi$ as required.

Assume now that for all $i = 1, 2, \ldots, l$ we have

$$\dim(U(\Gamma_i)/(M_\phi \cap U(\Gamma_i)))_{\phi+\xi_i} = 1$$

and hence there exists $g_i \in U(\Gamma_i)_{\xi_i} \backslash M_\phi$ such that for any $u_i \in U(\Gamma_i)_{\xi_i}$, u_i is a non-zero scalar multiple of g_i modulo M_ϕ. Since $[U(\Gamma_i), U(\Gamma_j)] = \{0\}$ for $i \neq j$ we have that $g_1 \ldots g_l \in U_\xi \backslash M_\phi$ and for any $u \in U_\xi$, u is a scalar multiple of $g_1 \ldots g_l$ modulo M_ϕ. That is, $\dim (U/M_\phi)_\lambda = 1$.

We now make use of these descriptions of pointed representations to complete our labelling programme.

LEMMA. *If* $\phi : C(L) \to \mathbf{C}$ *is an extreme g-standard algebra homomorphism relative to* $\bigcup_i \Gamma_i$ *then the set of weight functions of* U/M_ϕ *is contained in the set*

$$\{\phi + \textstyle\sum_{\alpha \in \Delta_{++}} k_\alpha \cdot \alpha | (\forall \alpha) k_\alpha \in \mathbf{Z}; \quad (\forall \alpha \in \Delta_{++} \backslash \bigcup \Gamma_i) \quad k_\alpha \leqq 0\}.$$

Proof. Set $\lambda = \phi + \sum_{\alpha \in \Delta_{++}} k_\alpha \cdot \alpha$ and $\xi = \sum k_\alpha \cdot \alpha$ where $(\forall \alpha)$ $k_\alpha \in \mathbf{Z}$ and consider any basis element $u \in U_\xi$. If $k_\beta > 0$ for some $\beta \in \Delta_{++} \backslash \bigcup \Gamma_i$ then there must exist some $\beta' \in \Delta_{++} \backslash \bigcup \Gamma_i$ such that $r_{\beta'} \neq 0$ in u and hence by remark 2 we have $u \in M_\phi$. That is, $\dim (U/M_\phi)_\lambda = 0$. Thus in order for λ to be a weight function of U/M_ϕ we must have $k_\alpha \leqq 0$ for all $\alpha \in \Delta_{++} \backslash \Gamma_i$.

PROPOSITION 8. *If* $\phi_1, \phi_2 : C(L) \to \mathbf{C}$ *are two distinct extreme g-standard algebra homomorphisms then* $U/M_{\phi_i} \ncong U/M_{\phi_2}$ *as L-modules.*

Proof. Assume that ϕ_1 and ϕ_2 are as given and $U/M_{\phi_1} \cong U/M_{\phi_2}$. We claim that $\phi_1 = \phi_2$. Since equivalent representations have the same set of weight functions we must have that $\phi_1 \downarrow H$ is a weight function of U/M_{ϕ_2} and hence

$$\phi_1 \downarrow H = \phi_2 \downarrow H + \textstyle\sum_{\alpha \in \Delta_{++}} l_\alpha \cdot \alpha$$

where $(\forall \alpha)\ l_\alpha \in \mathbf{Z}$. We also note that if ϕ_1 is g-standard relative to $\bigcup \Gamma_i^{(1)}$ and ϕ_2 is g-standard relative to $\bigcup \Gamma_i^{(2)}$ then $\bigcup \Gamma_i^{(1)} = \bigcup \Gamma_i^{(2)}$. Indeed if $\beta \in \bigcup \Gamma_i^{(1)}$ and $\beta \notin \bigcup \Gamma_i^{(2)}$ then $\phi_1 \downarrow H + l \cdot \beta$ is a weight function of U/M_{ϕ_1} and therefore of U/M_{ϕ_2} for all $l \in \mathbf{Z}$. But then

$$\phi_1 \downarrow H + l\beta = \phi_2 \downarrow H + \textstyle\sum_{\alpha \in \Delta_{++}} l_\alpha \cdot \alpha + l \cdot \beta$$

is a weight function of U/M_{ϕ_2} for all $l \in \mathbf{Z}$ and since $\beta \notin \Gamma_i^{(2)}$ this contradicts the lemma above.

Now fix any $\beta_0 \in \Delta_{++} \backslash \bigcup \Gamma_i^{(1)} = \Delta_{++} \backslash \bigcup \Gamma_i^{(2)}$ and note that

$$\phi_1 \downarrow H = \phi_2 \downarrow H + \textstyle\sum_{\alpha \in \Delta_{++}} l_\alpha \cdot \alpha$$

is a weight function of U/M_{ϕ_2}. Therefore by the above lemma $l_{\beta_0} \leqq 0$. But we also have that

$$\phi_2 \downarrow H = \phi_1 \downarrow H + \textstyle\sum_{\alpha \in \Delta_{++}} (-l_\alpha) \cdot \alpha$$

is a weight function of U/M_{ϕ_1} and again applying the lemma we have $-l_{\beta_0} \leqq 0$. Therefore we have that $l_{\beta_0} = 0$ for all $\beta_0 \in \Delta_{++} \backslash \bigcup \Gamma_i^{(1)}$.

On the other hand assume $\beta_0 \in \bigcup \Gamma_i^{(1)} = \bigcup \Gamma_i^{(2)}$. Then by definition of extreme g-standard we have that $0 \leqq \operatorname{Re} \phi_i(h_{\check{\beta}_0}) < 1$ for $i = 1, 2$. But

$$\phi_1(h_{\check{\beta}_0}) = \phi_2(h_{\check{\beta}_0}) + l_{\beta_0}$$

and hence $l_{\beta_0} = 0$. Thus $\phi_1 \downarrow H = \phi_2 \downarrow H$ and since $U/M_{\phi_1} \cong U/M_{\phi_2}$ we have $\phi_1 = \phi_2$ as required.

It remains now only to show that for any g-standard algebra homomorphism $\phi : C(L) \to \mathbf{C}$ there exists an extreme g-standard $\bar{\phi}: C(L) \to \mathbf{C}$ such that $U/M_\phi \cong U/M_{\bar{\phi} \circ \sigma}$ for some $\sigma \circ A(L)$. We proceed through a series of lemmas.

LEMMA 9a. *If $\phi : C(A_n) \to \mathbf{C}$ is a standard algebra homomorphism parametrized by $s \in \mathbf{C}$ and $\lambda \in H^*$ then*

1) $\phi \circ \sigma_{\alpha_1}$ *is standard parametrized by $s - \lambda(h_{\alpha_1}) \in \mathbf{C}$ and $\lambda \circ \sigma_1 \in H^*$.*

2) $\phi \circ \sigma_{\alpha_i}$ *is standard parametrized by $s \in \mathbf{C}$ and $\lambda \circ \sigma_i \in H^*$ for $i = 2, 3, \ldots, n$.*

3) *If $\xi = \sum_{i=1}^n l_i \cdot \alpha_i$ where $l_i \in \mathbf{Z}$ and $(\phi + \xi) \downarrow H$ is a 1-dimensional weight function of U/M_ϕ then the algebra homomorphism $\phi' : C(A_n) \to \mathbf{C}$ associated with $(\phi + \xi) \downarrow H$ is standard parametrized by $s + l_1 \in \mathbf{C}$ and $(\phi + \xi) \downarrow H \in H^*$.*

Proof. 1) Define two representations

$$(\rho, V_{s,\lambda}) \text{ and } (\rho', V_{s-\lambda(h_{\alpha_i}),\lambda\circ\sigma_{\alpha_i}})$$

as in [7] where the underlying vector space is the same for both. Using the explicit description of these representations one can easily verify that

$$(\rho \cdot \sigma_{\alpha 1}, V_{s,\lambda}) \cong (\rho', V_{s-\lambda(h_{\alpha_i}),\lambda\circ\sigma_{\alpha_i}})$$

where the equivalence map is the identity. Then we have

$$\phi \circ \sigma_{\alpha 1}(c)v(\mathbf{0}) = \rho \circ \sigma_{\alpha 1}(c)v(\mathbf{0}) = \rho'(c)v(\mathbf{0}) \quad (\forall c \in C(A_n)).$$

That is, $\phi \circ \sigma_{\alpha 1}$ is standard, parametrized by $s - \lambda(h_{\alpha 1}) \in C$ and $y \circ \sigma_{\alpha 1} \in H^*$.

2) This follows in the same manner as 1) on noting that for $i \geqq 2$

$$(\rho \circ \sigma_{\alpha_i}, V_{s,\lambda}) \cong (\rho', V_{s,\lambda\circ\sigma_{\alpha_i}})$$

where the equivalence map is the identity.

3) Recall from [7, Proposition 2] that the representations $(\rho, V_{s,\lambda})$ and $(\rho', V_{t,\lambda'})$ where $\lambda' - \lambda = \sum_{i=1}^{n} l_i \cdot \alpha_i$ and $t = s + l_1$ are equivalent and the equivalence map $\psi : V_{s,\lambda} \to V_{t,\lambda'}$ is given by

$$\psi(v(k_1, \ldots, k_n)) = v(k_1 - l_1, \ldots, k_n - l_n).$$

By assumption we also have $U/M_\phi \cong U/M_{\phi'}$ and this equivalence can be realized by the map $\Phi : U/M_\phi \to U/M_{\phi'}$ where $\Phi(1 + M_\phi) = u_0 + M_{\phi'}$ with

$$u_0 \in U_\tau \backslash M_\phi \quad \text{where } \tau = \sum_{i=1}^{n} l_i \alpha_i.$$

We may also assume that u_0 has been selected in such a way that

$$\rho'(u_0)v(-l_1, -l_2, \ldots, -l_n) = v(\mathbf{0}).$$

In fact for any $u \in U_\tau \backslash M_\phi$ we have

$$\rho'(u)v(-l_1, \ldots, -l_n) = \rho'(u)\psi(v(\mathbf{0})) = \psi(\rho(u)v(\mathbf{0}))$$

and $\rho(u)v(\mathbf{0})$ is a non-zero scalar multiple of $v(l_1, \ldots, l_n)$ since $u \notin M_\phi$. That is,

$$\rho'(u)v(-l_1, \ldots, -l_n) = Kv(\mathbf{0})$$

with $K \neq 0$ and hence we may select $u_0 = u/K$. Also since $u_0 \notin M_\phi$ we can select an element $w_0 \in U_\tau$ such that $\phi(w_0 u_0) = 1$. Now by Proposition 2 we have $\phi'(c) = \phi(w_0 c u_0)$ for all $c \in C(A_n)$. Finally for all $c \in C(A_n)$ we have

$$\begin{aligned}\rho'(c)v(\mathbf{0}) &= \rho'(c)\rho'(u_0)v(-l_1, \ldots, -l_n) = \rho'(cu_0)\psi(v(\mathbf{0})) \\ &= \psi \circ \rho(cu_0)v(\mathbf{0}) = \rho(w_0 c u_0)v(\mathbf{0}) = \phi(w_0 c u_0)v(\mathbf{0}) = \rho'(c)v(\mathbf{0}).\end{aligned}$$

Thus ϕ' is standard, parametrized by $s + l_1 \in \mathbf{C}$ and $(\phi + \xi) \downarrow H \in H^*$.

LEMMA 9b. *Assume* $\phi : C(A_n) \rightarrow \mathbf{C}$ *is a standard algebra homomorphism, parametrized by* $s \in \mathbf{C}$ *and* $\lambda \in H^*$ *such that for some* $p = 0, 1, \ldots, n$,

$$s - \lambda(\textstyle\sum_{i=0}^{p} h_{\alpha_i}) \in \mathbf{Z}.$$

Then there exists a g-standard algebra homomorphism $\phi' : C(A_n) \rightarrow \mathbf{C}$ *relative to the complete subset* Γ' *or* Γ'' *of* Δ *generated by* $\{\alpha_1, \ldots, \alpha_{n-1}\}$ *or* $\{\alpha_2, \ldots, \alpha_n\}$ *such that* $U/M_{\phi'} \cong U/M_{\phi\circ\sigma}$ *for some* $\sigma \in A(A_n)$.

Proof. Let m denote the minimum integer, by absolute value, among the integers in the set

$$\{s - \lambda(\textstyle\sum_{i=0}^{p} h_{\alpha_i}) | \, p = 0, 1, \ldots, n\}.$$

Assume first that

$$m = s - \lambda(\textstyle\sum_{i=0}^{r} h_{\alpha_i}) \leqq 0.$$

If $r \neq 0$ (ie. $s \neq m$) then applying parts 1) and 2) of Lemma 9a we have that if

$$\sigma = \sigma_{\alpha_r} \circ \ldots \circ \sigma_{\alpha_1} \in A(A_n)$$

then $\phi \circ \sigma$ is a standard algebra homomorphism parametrized by $s' \in C$ and $\lambda' \in H^*$ where

$$s' = s - \lambda(\textstyle\sum_{i=0}^{r} h_{\alpha_i}) = m.$$

By Proposition 6, $(\phi \circ \sigma - m\alpha_1) \downarrow H$ is a 1-dimensional weight space of $U/M_{\phi\circ\sigma}$. Applying part 3) of Lemma 9a, the algebra homomorphism ϕ': $C(A_n) \rightarrow \mathbf{C}$ associated with the 1-dimensional weight function $(\phi \circ \sigma - m\alpha_1) \downarrow H$ is also standard parametrized by $s'' \in C$ and $\lambda'' \in H^*$ where $s'' = s' - m_1 = 0$. It then follows that $\phi' \downarrow \bar{C}(\Gamma') \equiv 0$. That is, ϕ' is g-standard relative to Γ'. Finally we also have $U/M_{\phi_1} \cong U/M_{\phi\circ\sigma}$.

On the other hand, if we assume that $m > 0$ by a similar argument we can define an algebra homomorphism $\phi' : C(A_n) \rightarrow \mathbf{C}$ which is g-standard relative to Γ'' and $U/M_{\phi'} \cong U/M_{\phi\circ\sigma}$ for some $\sigma \in A(A_n)$.

LEMMA 9c. *Let* $\phi : C(L) \rightarrow \mathbf{C}$ *be a g-standard algebra homomorphism relative to* $\bigcup_{i=1}^{l} \Gamma_i$. *Then*:

1) *For any* $\alpha \in \Delta_{++} \cap \Gamma_{i_0}$ *we have* $\phi \circ \sigma_\alpha$ *is g-standard relative to* $\bigcup \Gamma_i$. *More precisely we have* $\phi \circ \sigma_\alpha \equiv \phi$ *on* $U(\Gamma_j) \cap C(L)$ *for* $j \neq i_0$ *and* $\phi \circ \sigma_\alpha \equiv 0$ *on* $\bar{C}(\bigcup \Gamma_i)$.

2) *If*

$$\xi = \textstyle\sum_{\alpha \in \Delta_{++} \cap \Gamma_{i_0}} l_\alpha \cdot \alpha$$

with $l_\alpha \in Z$ *for all* α *such that* $(\phi + \xi) \downarrow H$ *is a 1-dimensional weight function of* U/M_ϕ *then the algebra homomorphism* ϕ' *associated with* $(\phi + \xi) \downarrow H$ *is g-standard relative to* $\bigcup \Gamma_i$. *More precisely we have* $\phi' \equiv \phi$ *on* $U(\Gamma_j) \cap C(L)$ *for* $j \neq i_0$ *and* $\phi' \equiv 0$ *on* $\bar{C}(\bigcup \Gamma_i)$.

Proof. 1) For any $j \neq i_0$ and $\beta \in \Delta \cap \Gamma_j$ we have $\sigma_\alpha(\beta) = \beta$. That is, for any $c \in C(L) \cap U(\Gamma_j)$, $\sigma_\alpha(c) = c$. Hence $\phi \circ \sigma_\alpha(c) = \phi(c)$ for all $c \in C(L) \cap U(\Gamma_j)$.

For any $\beta \in \Delta \cup \Gamma_i$, $\sigma_\alpha(\beta) \in \Delta \cup \Gamma_i$ and hence for any $c \in \bar{C}(\cup \Gamma_i)$, $\sigma_\alpha(c) \in \bar{C}(\cup \Gamma_i)$. Therefore $\phi \circ \sigma_\alpha(c) = 0$ for all $c \in \bar{C}(\cup \Gamma_i)$.

Finally $\phi \circ \sigma_\alpha \downarrow (C(L) \cap U(\Gamma_{i_0}))$ is standard by Lemma 9c and hence $\phi \circ \sigma_\alpha$ is g-standard relative to $\cup \Gamma_i$.

2) Take $u \in U(\Gamma_{i_0})_\xi \backslash M_\phi$ and note that

$$(\forall c \in C(L))\ \phi'(c)(u + M_\phi) = c(u + M_\phi)$$

for any $c \in C(L) \cap U(\Gamma_j)$ with $j \neq i_0$ we have

$$\phi'(c)(u + M_\phi) = c(u + M_\phi) = uc + M_\phi = \phi(c)(u + M_\phi).$$

Hence $\phi'(c) = \phi(c)$.

Also for any $c \in \bar{C}(\cup_{i=1}^{l} \Gamma_i)$ we note that $U_{-\xi}\, cu \subseteq \bar{C}(\cup \Gamma_i) \subseteq M_\phi$ and hence $cu \in M_\phi$. Therefore

$$\phi'(c)(u + M_\phi) = cu + M_\phi = 0(u + M_\phi).$$

Thus $\phi'(c) = 0$.

Finally $\phi' \downarrow (C(L) \cap U(\Gamma_{i_0}))$ is standard by Lemma 9a and hence ϕ' is g-standard relative to $\cup \Gamma_i$.

Combining these lemmas we now have the main result of this section.

PROPOSITION 9. *Let $\phi : C(L) \to \mathbf{C}$ be a g-standard algebra homomorphism relative to $\cup_{i=1}^{l} \Gamma_i$. Then there exists an extreme g-standard algebra homomorphism $\bar{\phi} : C(L) \to \mathbf{C}$ such that $U/M_{\bar{\phi}} \cong U/M_{\phi\circ\sigma}$ for some $\sigma \in A(L)$.*

Proof. We define the order of a g-standard algebra homomorphism relative to $\cup_{i=1}^{l} \Gamma_i$ to be $\sum_{i=1}^{n} \#(\Delta_{++} \cap \Gamma_i)$. Every order 0 g-standard algebra homomorphism is by definition extreme hence we assume inductively that the proposition is true for g-standard algebra homomorphisms of order $<N$. Then consider a g-standard algebra homomorphism ϕ: $C(L) \to \mathbf{C}$ of order N.

If there exists $i_0 = 1, 2, \ldots, l$ such that $\phi \downarrow (C(L) \cap U(\Gamma_{i_0}))$ satisfies the conditions of Lemma 9b then by Lemmas 9b and 9c there exists $\sigma \in A(L)$ such that $\phi \circ \sigma$ is g-standard of order N-1 and $U/M_\phi \cong U/M_{\phi\circ\sigma}$. By the inductive hypothesis then there exists an extreme g-standard algebra homomorphism $\bar{\phi}$: $C(L) \to \mathbf{C}$ such that $U/M_{\phi\circ\sigma} \cong U/M_{\bar{\phi}\circ\sigma_1}$ for some $\sigma_1 \in A(L)$. Hence by Proposition 2 $U/M_{\bar{\phi}} \cong U/M_{\phi\circ\sigma\circ\sigma_1^{-1}}$ as required.

We may therefore assume that

$$(\phi + \textstyle\sum_{\alpha \in \Delta_{++} \cap (\cup \Gamma_i)} l_\alpha \cdot \alpha) \downarrow H$$

is a 1-dimensional weight function of U/M_ϕ for all $l_\alpha \in \mathbf{Z}$. Thus setting $k_\alpha = [\mathrm{Re}\ \phi(h_{\check{\alpha}})]$ for all $\alpha \in \Delta_{++} \cap (\cup \Gamma_i)$ (where $[\cdot]$ denote the greatest integer function),

$$(\phi - \textstyle\sum_{\alpha \in \Delta_{++} \cap (\cup \Gamma_i)} k_\alpha \cdot \alpha) \downarrow H$$

is a 1-dimensional weight function of U/M_ϕ. If $\bar{\phi}$ is the associated algebra homomorphism then $U/M_\phi \cong U/M_{\bar{\phi}}$, $\bar{\phi}$ is g-standard by Lemma 9c and is extreme since $0 \leqq \text{Re}\,(\phi(h_\alpha) - k_\alpha) - 1$.

References

1. I. Z. Bouwer, *Standard representations of simple Lie algebras*, Can. J. Math. *20* (1968), 344–361.

2. J. Dixmier, *Algèbres enveloppantes* (Gauthier-Villars, Paris, 1974).

3. S. G. Gindikin, A. A. Kirillov and D. B. Fuks, *The works of I. M. Gel'fand on functional analysis, algebra and topology*, Russian Math. Surveys *29* (1974), 3–61.

4. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate texts in Mathematics *9* (Springer-Verlag, New York, 1972).

5. F. W. Lemire, *Weight spaces and irreducible representations of simple Lie algebras*, Proc. Amer. Math. Soc. *22* (1969), 192–197.

6. ——— *One-dimensional representation of the cycle subalgebra of a semi-simple Lie algebra*, Can. Math. Bull. *13* (1970), 463–467.

7. ——— *A new family of irreducible representations of A_n*, Can. Math. Bull. *18* (1975), 543–546.

Université de Montréal,
Montréal, Québec;
University of Windsor,
Windsor, Ontario

https://doi.org/10.4153/CJM-1979-100-5 Published online by Cambridge University Press