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DIVISORS OF INTEGERS IN ARITHMETIC PROGRESSION

BY

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ABSTRACT. Let $d(n;\, l,k)$ be the number of positive divisors of n which lie in the arithmetic progression $l \bmod k$. Using the complex integration technique the formula

$$\sum_{n\leqq x} d(n;\, l,k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{l} + \frac{x}{k}a_0(l,k) + O\left(x^{\epsilon}\left(\frac{x}{k}\right)^{\alpha}\right)$$

is proved. This formula holds uniformly in l, k and x satisfying $1 \leqq l \leqq k$, $(lx)^{1/2} \leqq k \leqq x^{1-\epsilon}$; the exponent $\alpha \leqq 1/3$.

R. A. Smith and M. V. Subbarao [1] obtained the asymptotic formula for the sum

$$\sum_{n\leqq x} d(n;\, l,k),$$

where $d(n;\, l,k)$ is the number of positive divisors of n which lie in the arithmetic progression $l \bmod k$. Their result was improved by W. G. Nowak [2].

The purpose of the present paper is to give a completely different method which allows to obtain Nowak's result. Our method can be also used in the case of imaginary quadratic fields.

THEOREM. *Let $\epsilon > 0$ be an arbitrary fixed number and l, k be natural numbers such that $1 \leqq l < k$ and $(lx)^{1/2} \leqq k \leqq x^{1-\epsilon}$. Then for $x \longrightarrow \infty$*

$$\sum_{n\leqq x} d(n;\, l,k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{l} + \frac{x}{k}a_0(l,k) + O\left(x^{\epsilon}\left(\frac{x}{k}\right)^{\alpha}\right), \tag{1}$$

where $\alpha \leqq 1/3$, $a_0(l,k)$ is a bounded computable function and the constant in the O-symbol does not depend on l, k and x.

In the proof of Theorem we shall use the following.

LEMMA. *Let x be a real number > 1, and $\epsilon > 0$ be an arbitrary fixed number. Let l, k be natural numbers such that $l \leqq k$ and $(lx)^{1/2} \leqq k \leqq x$. If $n_0 = (l + km_0)r_0$ with $m_0, r_0 \in \mathbf{N}$ and $n_0 \leqq x$, then there exists no more than $O(x^{\epsilon})$ natural numbers of the form $n = (l + km)r$ with $m, r \in \mathbf{N}$ such that $|n - n_0| < k$.*

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PROOF OF LEMMA. Let us notice that $r_0k \leqq x$ and $rk \leqq 2x$. Therefore

$$r_0 \leqq x/k \leqq (x/l)^{1/2}, \quad lr_0 \leqq l(x/l)^{1/2} = (xl)^{1/2} \leqq k,$$
$$r \leqq 2x/k \leqq 2(x/l)^{1/2}, \; lr \leqq 2l(x/l)^{1/2} = 2(xl)^{1/2} \leqq 2k.$$

Thus $l|r - r_0| \leqq 2k$ and

$$k > |n - n_0| = |k(mr - m_0r_0) + l(r - r_0)| \geqq \big|k|mr - m_0r_0| - l|r - r_0|\big|$$

It follows from the last inequality that

$$mr - m_0r_0 = 0, \pm 1, \pm 2,$$

but for fixed m_0, r_0 each of these equations has no more than

$$\mathrm{O}\big((m_0r_0)^\epsilon\big) = \mathrm{O}(x^\epsilon)$$

solutions m, r. This completes the proof of Lemma. □

PROOF OF THEOREM. Let us consider a function

$$d^*(n;\, l, k) = \begin{cases} d(n;\, l, k) & \text{if } l \nmid n, \\ d(n;\, l, k) - 1 & \text{if } l \mid n. \end{cases}$$

Obviously

$$\sum_{n \leqq x} d(n;\, l, k) = \sum_{n \leqq x} d^*(n;\, l, k) + \left[\frac{x}{l}\right].$$

For $\mathrm{Re}(s) > 1$ we have

$$\sum_{n=1}^{\infty} \frac{d^*(n;\, l, k)}{n^s} = \frac{1}{k^s}\,\zeta(s)\left[\zeta\left(s, \frac{l}{k}\right) - \frac{1}{(l/k)^s}\right],$$

where

$$\zeta(s, u) = \sum_{n=0}^{\infty} 1/(n + u)^s$$

is the Hurwitz zeta-function, $0 < u \leqq 1$. It follows from the well known relation

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} (y^s/s)ds = \begin{cases} 1 + 0\left((\log y)^{-1}\,\dfrac{y^c}{T}\right) & \text{if } y > 1, \\ 0\left(|\log y|^{-1}\,\dfrac{y^c}{T}\right) & \text{if } 0 < y < 1, \end{cases}$$

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that

$$\sum_{n\leqq x-k} d^*(n;\, l,k) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left[\frac{\zeta(s)}{k^s}\left(\zeta\left(s,\,\frac{l}{k}\right)-\left(\frac{k}{l}\right)^s\right) - \sum_{x-k<n<x+k}\frac{d^*(n;\, l,k)}{n^s}\right]\frac{x^s}{s}\,ds + \mathrm{O}\left(\left\{\sum_{n\leqq x-k}+\sum_{n\geqq x+k}\right\}\frac{d^*(n;\, l,k)x^c}{n^cT|\log(x/n)|}\right) \tag{2}$$

(here, x is equal to one-half of an odd integer). By Lemma we get

$$\sum_{x-k<n\leqq x} d^*(n;\, l,k) = \mathrm{O}(x^{2\epsilon}),\quad \sum_{x-k<n<x+k} d^*(n;\, l,k) = \mathrm{O}(x^{2\epsilon}), \tag{3}$$

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_{x-k<n<x+k}\frac{d^*(n;\, l,k)}{n^s}\,\frac{x^s}{s}\,ds = \mathrm{O}(x^{2\epsilon}\log T). \tag{4}$$

Therefore by (2), (3), (4)

$$\sum_{n\leqq x} d^*(n;\, l,k) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta(s)\left(\zeta\left(s,\,\frac{l}{k}\right)-\left(\frac{k}{l}\right)^s\right)\left(\frac{x}{k}\right)^s\frac{ds}{s} + \mathrm{O}(x^{2\epsilon}\log T) + \mathrm{O}\left(\left\{\sum_{n\leqq x-k}+\sum_{n\geqq x+k}\right\}\frac{x^c d^*(n;\, l,k)}{n^cT|\log(x/n)|}\right). \tag{5}$$

We split the last sum in the right hand side of (5) into three parts: $n \leqq x/2$, $x/2 < n < 2x$, $n \geqq 2x$. We get

$$\begin{aligned}\sum_{n\leqq x/2}+\sum_{n\geqq 2x} &= \mathrm{O}\left(\frac{x^c}{T}\sum_{n=1}\frac{d^*(n;\, l,k)}{n^c}\right)\\ &= \mathrm{O}\left(\frac{x^c}{T}\sum_{\substack{r=1\\ n=(l+km)r}}^{\infty}\sum_{m=1}^{\infty}\frac{d^*(n;\, l,k)}{[(l+km)r]^c}\right)\\ &= \mathrm{O}\left(\frac{x^c}{T}\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}1/(kmr)^{c-\epsilon}\right) = \mathrm{O}(x^{c+\epsilon}/kT).\end{aligned} \tag{6}$$

Let us consider the interval of summation

$$J = \{n : x/2 < n < 2x,\ n \notin (x-k, x+k)\}.$$

Let n_1 be the least number from J such that $d^*(n;\, l,k) \neq 0$. By n_2 we denote the least number $n \in J$ such that $|n-n_1| > k$ and $d^*(n;\, l,k) \neq 0$. Analogously we define

$n_3, \ldots, n_N$. It is clear that $N = 0(x/k)$. For each n_j there exist at most $O(x^\epsilon)$ integers n from J such that $d^*(n;\, l,k) \neq 0$. If now $n_j \leqq n \leqq n_{j+1}$ and $d^*(n;\, l,k) \neq 0$, then

$$c_1/|\log(x/n_j)| \leqq 1/|\log(x/n)| \leqq c_2/|\log(x/n_{j+1})|.$$

Therefore

$$(7) \qquad \sum_{n\in J} = O\left(\frac{x^{c+\epsilon}}{Tx^c}\sum_{n_j\in J}\frac{1}{|\log(x/n_j)|}\right) = O\left(\frac{x^\epsilon}{T}\sum_{j=1}^{x/k}\frac{x}{jk}\right) = O\left(\frac{x^{2\epsilon+1}}{kT}\right)$$

(as usually, we put $n_j = x + \vartheta kj$, where $1/2 \leqq |\vartheta| \leqq 2$). Now, it follows from (5), (6) and (7) that

$$(8) \qquad \sum_{n\leqq x} d^*(n;\, l,k) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta(s)\left[\zeta\left(s,\, \frac{l}{k}\right) - \left(\frac{k}{l}\right)^s\right]\left(\frac{x}{k}\right)^s\frac{ds}{s}$$
$$+ 0(x^\epsilon \log T) + 0(x^{1+\epsilon}/kT).$$

We shall calculate the integral in (8) by moving the contour of integration to $\mathrm{Re}(s) = -\epsilon$. The integrand function has two singular points: double pole in a point $s = 1$ and a simple pole in the point $s = 0$. Hence

$$\operatorname*{Res}_{s=1}\left\{\zeta(s)\left[\zeta\left(s,\, \frac{l}{k}\right) - \left(\frac{k}{l}\right)^s\right]\left(\frac{x}{k}\right)^s\frac{1}{s}\right\}$$
$$= \frac{x}{k}\log\frac{x}{k} + \frac{x}{k}\left[\gamma - 1 - \frac{\Gamma'(l/k)}{\Gamma(l/k)} - \frac{k}{l}\right]$$
$$= \frac{x}{k}\log\frac{x}{k} + \frac{x}{k}a_0(l,k),$$

where $|a_0(l,k)| \leqq A_0$ and A_0 is an absolute constant,

$$\operatorname*{Res}_{s=0}\left\{\zeta(s)\left[\zeta\left(s,\, \frac{l}{k}\right) - \left(\frac{k}{l}\right)^s\right]\left(\frac{x}{k}\right)^s\frac{1}{s}\right\} = \frac{1}{2}\left(\frac{l}{k} + \frac{1}{2}\right).$$

Note that

$$\left|\zeta\left(1+\delta+it,\, \frac{l}{k}\right) - \left(\frac{k}{l}\right)^{1+\delta+it}\right| = O\left(\frac{1}{\delta}\right), \quad \delta > 0,$$
$$\left|\zeta\left(-\epsilon+it,\, \frac{l}{k}\right) - \left(\frac{k}{l}\right)^{-\epsilon+it}\right| = O\big((|t|+1)^{1/2+\epsilon}\big).$$

Therefore, applying the Cauchy theorem on residues, the functional equation for $\zeta(s)$ and the Hurwitz relation for $\zeta(s, l/k)$, after simple calculations we get

$$(9) \qquad \sum_{n\leqq x} d^*(n;\, l,k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{k}a_0(l,k) + \frac{i}{\pi^2}\sum_{n=1}^{\infty}\frac{a_n}{n}\frac{1}{2\pi i}$$
$$\times \int_{-\epsilon-iT}^{-\epsilon+iT}\frac{\Gamma\big((1-s)/2\big)\Gamma(1-s)}{\Gamma(s/2)}\left(\frac{4\pi^2 nx}{k}\right)^s\frac{e^{-\pi is}}{s}\,ds$$
$$- \frac{i}{\pi^2}\sum_{n=1}^{\infty}\frac{b_n}{n}\frac{1}{2\pi i}\int_{-\epsilon-iT}^{-\epsilon+iT}\frac{\Gamma\big((1-s)/2\big)\Gamma(1-s)}{s/2}$$
$$\times \left(\frac{4\pi^2 nx}{k}\right)^s\frac{e^{\pi is}}{s}\,ds + O(x^\epsilon \log T) + O(x^{1+\epsilon}/kT),$$

where

$$a_n = \sum_{d|n} \exp\left(-2\pi i \frac{dl}{k}\right), \quad b_n = \sum_{d|n} \exp\left(2\pi i \frac{dl}{k}\right).$$

We denote $X = T^2k/4\pi^2 x$. For computation of the integrals in (9) we use the Stirling formula for $\Gamma(s)$. We obtain

$$(10) \quad \int_{-\epsilon-iT}^{-\epsilon+iT} \frac{\Gamma((1-s)/2)\Gamma(1-s)}{\Gamma(s/2)} \left(\frac{4\pi^2 nx}{k}\right)^s e^{\pm\pi i s} \frac{ds}{s}$$
$$= \int_{-T}^{T} |t|^{2\epsilon} \left(\frac{4\pi^2 nx}{k}\right)^{-\epsilon} \exp\left(-2it\left(\log|t| - 1 - \log\sqrt{\frac{4\pi^2 nx}{k}}\right)\right) dt$$
$$+ \mathrm{O}\left((T^2k/nx)^\epsilon\right).$$

The integral in the right hand side of (10) we denote by $I(n)$. For $n \geqq X + (X/T)$ by integrating $I(n)$ by parts we get

$$I(n) = \mathrm{O}\left(\left(\frac{kT^2}{nx}\right)^\epsilon \frac{n}{n-X}\right).$$

For $X - (X/T) < n < X + (X/T)$ we use the trivial bounds

$$I(n) = \mathrm{O}\left(T\left(\frac{kT^2}{nx}\right)^\epsilon\right).$$

For $n \leqq X - (X/T)$ the stationary phase method (see [3], chapter III, theorem 1.4) gives

$$I(n) = \frac{\pi}{\sqrt{2}} \left(\frac{nx}{k}\right)^{1/4} \cos\left(4\pi\sqrt{\frac{nx}{k}} - \frac{\pi}{4}\right)\left(1 + \mathrm{O}\left(\left(\frac{k}{nx}\right)^{1/2}\right)\right).$$

Therefore by (9)

$$(11) \quad \sum_{n \leqq x} d^*(n;\, l, k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{k} a_0(l,k) + \mathrm{O}\left(\frac{x^{1+\epsilon}}{kT}\right) + \mathrm{O}(x^\epsilon \log T)$$
$$+ \mathrm{O}\left(X^{1-\epsilon}\left(\frac{kT^2}{x}\right)^\epsilon\right) + A_0'\left(\frac{x}{k}\right)^{1/4}$$
$$\times \sum_{n \leqq X_1} \frac{a_n}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{k}} - \frac{\pi}{4}\right)$$
$$+ B_0\left(\frac{x}{k}\right)^{1/4} \sum_{n \leqq X_1} \frac{b_n}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{k}} - \frac{\pi}{4}\right)$$

(A_0', B_0 are computable constants, $X_1 = X - (X/T)$). Now, the trivial bounds of the sums in the right hand side of (11) (note that $|a_n| \leqq d(n)$, $|b_n| \leqq d(n)$) give for $T = (x/k)^{2/3}$, $X = (1/4\pi^2)(x/k)^{1/3}$

$$(12) \quad \sum_{n \leqq x} d^*(n;\, l, k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{k} a_0(l,k) + \mathrm{O}\left(x^\epsilon\left(\frac{x}{k}\right)^{1/3}\right).$$

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The exponent 1/3 in the remainder term of the formula (12) can be improved. In fact, the last sums in the right hand side of (11) are similar to the sums in the remainder term in the Divisors Problem of Dirichlet (see [4], XII, §4, formula 4). Let us note that estimation of the sum

$$\sum_{n\leqq X_1} \frac{a_n}{n^{3/4}} \exp i\left(4\pi\sqrt{\frac{nx}{k}} - \frac{\pi}{4}\right) = \sum_{mn\leqq X_1} \frac{1}{mn^{3/4}} \exp\left(2\pi i\left(\frac{ml}{k} + 2\sqrt{\frac{mn}{k}} - \frac{1}{8}\right)\right)$$

can be obtained by the van der Corput method ([4], XII, §4) or by refinement of this method (see [5], [6]). Therefore

$$\sum_{n\leqq x} d^*(n;\, l, k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{k} a_0(l,k) + 0\left(x^\epsilon\left(\frac{x}{k}\right)^\alpha\right)$$

and

$$\sum_{n\leqq x} d(n;\, l, k) = \frac{x}{k}\log\frac{x}{k} + \frac{x}{l} + \frac{x}{k} a_0(l,k) + \mathrm{O}\left(x^\epsilon\left(\frac{x}{k}\right)^\alpha\right),$$

where $\alpha < 1/3$ and the constants in the O-terms do not depend on x, l, k, but can depend on ϵ. This completes the proof of Theorem. □

Remarks.

1. The case $l = k$ has no interest for us, as

$$d(n;\, k, k) = \begin{cases} d(n/k) & \text{if } k \nmid n, \\ 0 & \text{if } k \mid n \end{cases}$$

and therefore

$$\sum_{n\leqq x} d(n;\, k, k) = \sum_{n\leqq x/k} d(n).$$

2. Let us note that if l is fixed, then from the Smith-Subbarao result and from (1) we get the asymptotic formula for the sum $\sum_{n\leqq x} d(n;\, l, k)$ in the whole range of variation of k.

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