Ergod. Th. & Dynam. Sys., (2024), **44**, 1058–1101
doi:10.1017/etds.2023.43

Automorphisms of $\mathcal{B}$-free and other Toeplitz shifts

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(*Received* 12 *December* 2022 *and accepted in revised form* 1 *May* 2023)

Abstract. We present sufficient conditions for the triviality of the automorphism group of regular Toeplitz subshifts and give a broad class of examples from the class of $\mathcal{B}$-free subshifts satisfying them, extending the work of Dymek [Automorphisms of Toeplitz $\mathcal{B}$-free systems. *Bull. Pol. Acad. Sci. Math.* **65**(2) (2017), 139–152]. Additionally, we provide an example of a $\mathcal{B}$-free Toeplitz subshift whose automorphism group has elements of arbitrarily large finite order, answering Question 11 of S. Ferenczi *et al* [Sarnak's conjecture: what's new. *Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics (Lecture Notes in Mathematics, 2213)*. Eds. S. Ferenczi, J. Kułaga-Przymus and M. Lemańczyk. Springer, Cham, 2018, pp. 163–235].

Key words: $\mathcal{B}$-free dynamics, Toeplitz dynamical system, sets of multiples, automorphisms group, trivial centralizer
2020 Mathematics Subject Classification: 37B05, 37B10 (Primary); 37A44 (Secondary)

1. *Introduction*

1.1. *Toeplitz subshifts.* Let $\eta \in \{0, 1\}^{\mathbb{Z}}$ be a non-periodic Toeplitz sequence [17] with period structure $p_1 \mid p_2 \mid p_3 \ldots \to \infty$. That means for each $k \in \mathbb{Z}$, there exists $n \geq 1$ such that $\eta_{|k+p_n\mathbb{Z}}$ is constant, but η is not periodic (the latter excludes trivial cases). Denote the orbit closure of η under the left shift σ on $\{0, 1\}^{\mathbb{Z}}$ by X_η. Each such subshift is called a *Toeplitz shift*. It follows from the work of Williams [25] that such systems are in fact almost 1-1 extensions of their *maximal equicontinuous factor* (MEF), in this case of an associated odometer (G, T), where G is the compact topological group $\varprojlim \mathbb{Z}/p_n\mathbb{Z}$ built from the period structure $(p_n)_{n\geq 1}$, and T is the translation by $(1, 1, \ldots)$. Recall that odometers are minimal, equicontinuous and zero-dimensional dynamical systems, and the conjunction of these three properties characterizes them among all topological dynamical systems, see [8].

1.2. *The centralizer.* For any subshift (Y, σ), that is, a shift invariant closed subset $Y \subseteq \{0, 1\}^{\mathbb{Z}}$, the *automorphism group* (or *centralizer*) is the group of all homeomorphisms $U : Y \to Y$ which commute with σ. Its elements are sliding block codes [16]. Therefore, the automorphism group is countable. Since all powers of the shift are elements of the centralizer, it contains a copy of $\mathbb{Z}$ as a normal subgroup. We say that the automorphism group is *trivial* if it consists solely of powers of the shift.

Centralizers are studied for various classes of systems. Bułatek and Kwiatkowski [2] do it for Toeplitz subshifts with separated holes (Sh) using elements of the associated odometer (G, T). Moreover, in [3], they deliver examples of Toeplitz subshifts with positive topological entropy and trivial automorphism group. More recently, Cyr and Kra study automorphism groups of subshifts of subquadratic and linear growth in [4, 5]. In [5], they prove that the cosets of powers of the shift in the centralizer of any topologically transitive subshift of subquadratic growth form a periodic group. In [4], they show that any minimal subshift with non-superlinear complexity has a virtually $\mathbb{Z}$ automorphism group, answering the question asked in [24]. In [6], the same result is shown independently with different methods by Donoso *et al*. However, the centralizer can be quite a complicated group. In [23], Salo gives an example of a Toeplitz subshift with not finitely generated automorphism group. We provide another example with this property, see equation (53) and Corollary 3.25.

1.3. *This paper's contributions to general Toeplitz shifts.* Let η be a Toeplitz sequence with period structure $(p_n)_{n\geq 1}$ and recall from [2] that a position $k \in \mathbb{Z}$ is called a *hole at level N* if $\eta_{|k+p_N\mathbb{Z}}$ is not constant. We refine this concept and call such a hole *essential* if the residue class $k + p_N\mathbb{Z}$ contains holes of each level $n \geq N$, see Definition 2.2. The minimal period $\tilde{\tau}_N$ of the set of essential holes at level N divides p_N and, under a (seemingly strong) additional assumption, Theorem 2.8 provides restrictions on the size of the centralizer in terms of the quotients $p_N/\tilde{\tau}_N$. A direct application of this result to a variant of the Garcia–Hedlund sequence is discussed in Example 2.19. After that, using a mixture of topological and arithmetic arguments, we show that the additional assumption is satisfied more often than one may expect—the main tool is Theorem 2.31. Along this way, we exploit suitable topological variants of the separated holes condition (Sh) from [2] in Proposition 2.16, and verify these conditions under arithmetic assumptions tailor-made for the $\mathcal{B}$-free case in Proposition 2.30. We note that our techniques generalize the setting from [2], but are kind of transverse to the setting from [3], see Remark 2.10.

1.4. *The centralizer of $\mathcal{B}$-free Toeplitz shifts.* For any set $\mathcal{B} \subseteq \{2, 3, \dots\}$, let

$$\mathcal{M}_{\mathcal{B}} := \bigcup_{b\in\mathcal{B}} b\mathbb{Z}$$

be the set of *multiples of* $\mathcal{B}$ and

$$\mathcal{F}_{\mathcal{B}} := \mathbb{Z} \setminus \mathcal{M}_{\mathcal{B}}$$

the set of *$\mathcal{B}$-free numbers*. One can easily modify a set $\mathcal{B}$ to have the same set of multiples and to be *primitive*, that is, $b \nmid b'$ for different $b, b' \in \mathcal{B}$. So, we will tacitly assume that

$\mathcal{B}$ is primitive. The investigation of sets of multiples and $\mathcal{B}$-free numbers has a quite long history, see [15]. Recently, the subshifts associated with $\mathcal{B}$-free numbers are under intensive study, see e.g. [12, 18] and references therein. Namely, let $\eta \in \{0, 1\}^{\mathbb{Z}}$ be the characteristic function of the $\mathcal{B}$-free numbers. Minimality of (X_η, σ) is equivalent to η being Toeplitz is shown in [18, Theorem B], whenever $\mathcal{B}$ is taut. The tautness assumption was removed in [11, Theorem 3.7]. So, in the $\mathcal{B}$-free context, (X_η, σ) is minimal if and only if η is a Toeplitz sequence.

In the case of $\mathcal{B}$-free Toeplitz subshifts, the previously cited results for low complexity systems are not useful, because even very simple $\mathcal{B}$-free Toeplitz systems may have superpolynomial complexity, see §3.6, where this is shown for $\mathcal{B} = \mathcal{B}_1 := \{2^n c_n : n \in \mathbb{N}\}$ with pairwise coprime odd c_n. Although the separated holes condition (Sh) is satisfied for this simple example, we were not able to use the results from [2] to determine the centralizer. However, its triviality was shown with more direct methods in [10]. Nevertheless, there are many $\mathcal{B}$-free Toeplitz systems which do not satisfy (Sh) anyway, see the example discussed at the end of this introduction, so that there is need for techniques relying neither on low complexity nor on (Sh).

We mention briefly that Mentzen [22] proves the triviality of the automorphism group for any Erdős $\mathcal{B}$-free subshift, that is, when $\mathcal{B}$ is infinite, pairwise coprime and $\sum_{b\in\mathcal{B}}(1/b) < \infty$. This is extended to taut $\mathcal{B}$ containing an infinite pairwise coprime subset in [19, 20]. This class of $\mathcal{B}$-free subshifts is kind of opposite to the $\mathcal{B}$-free Toeplitz shifts.

1.5. *This paper's contributions to $\mathcal{B}$-free Toeplitz shifts.* In §3, the results from §2 are applied to $\mathcal{B}$-free examples. This is possible because Theorem 3.17 provides an arithmetic characterization of the sets of essential holes in terms of sets of multiples derived explicitly from the set $\mathcal{B}$. Then we can use the general results to produce examples of minimal $\mathcal{B}$-free systems, including not only the case $\mathcal{B} = \mathcal{B}_1$ treated in [10] but also many systems violating the separated holes condition (Sh), which have trivial centralizers, see §3.3. (The reader will notice that a very broad class of examples can be treated along the same lines.) The fact that the general results fail to guarantee triviality of the centralizer for some examples, to which even the basic Theorem 2.8 applies, is not a shortcoming of our approach. This is illustrated in §3.4, where we consider simple variants of $\mathcal{B} = \mathcal{B}_1$, still satisfying condition (Sh), but having non-trivial centralizers—just as big as Theorem 2.8 allows them to be. This provides a negative answer to [13, Question 11]. Indeed, a slight generalization of this construction provides examples for which the centralizer contains elements of arbitrarily large finite order, see Remark 3.24. It should be noticed that our examples have superpolynomial complexity, see Proposition 3.30, so that the complexity based results from the literature discussed above do not apply.

1.6. *The formal setting.* We recall some notation and results from [1], where a cut-and-project scheme is associated with a Toeplitz sequence $\eta \in \{0, 1\}^{\mathbb{Z}}$.

(i) $p_1 \mid p_2 \mid p_3 \ldots \to \infty$ is a period structure of η.

(ii) $\mathcal{P}_n^i := \{k \in \mathbb{Z} : \eta_{|k+p_n\mathbb{Z}} = i\}$ denotes p_n-periodic positions of $i = 0, 1$ on η and $\mathcal{H}_n := \mathbb{Z} \setminus (\mathcal{P}_n^0 \cup \mathcal{P}_n^1)$ denotes the set of *holes at level n*.

(iii) $G := \varprojlim \mathbb{Z}/p_n\mathbb{Z}$ and $\Delta\colon \mathbb{Z} \to G$, where $\Delta(n) = (n + p_1\mathbb{Z}, n + p_2\mathbb{Z}, \dots)$ denotes the diagonal embedding.

(iv) $T\colon G \to G$ denotes the rotation by $\Delta(1)$, that is, $(Tg)_n = g_n + 1 + p_n\mathbb{Z}$ for all $n \in \mathbb{N}$.

(v) The topology on G is generated by the (open and closed) cylinder sets

$$U_n(h) := \{g \in G : g_n = h_n\}, \quad h \in G.$$

(vi) $V^i := \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathcal{P}_n^i\cap[0,p_n)} U_n(\Delta(k))$ for $i = 0, 1$.

(vii) The *window* $W := \overline{V^1} = G \setminus V^0$ is topologically regular, i.e. $\overline{\mathrm{int}(W)} = W$.

(viii) $\phi\colon G \to \{0, 1\}^{\mathbb{Z}}$ is the coding function: $(\phi(g))_n = \mathbf{1}_W(g + \Delta(n))$. Observe that $\phi(\Delta(0)) = \eta$.

1.7. *More background material and an outline of the paper.* The odometer (G, T) is the MEF of (X_η, σ), see [25]. Let $F\colon X_\eta \to X_\eta$ be an automorphism commuting with σ, and denote by $\pi : (X_\eta, \sigma) \to (G, T)$ that factor map onto the MEF which is uniquely determined by $\pi(\eta) = \Delta(0)$. Then there is $y_F \in G$ such that the rotation $f : y \mapsto y + y_F$ on G represents F in the sense that $\pi \circ F = f \circ \pi$ (see e.g. [6, Lemma 2.4]). Observe that $y_F = \pi(F(\eta))$.

Denote by C_ϕ the set of continuity points of $\phi : G \to \{0, 1\}^{\mathbb{Z}}$. Then, $|\pi^{-1}\{\pi(x)\}| = 1$ if and only if $\pi(x) \in C_\phi$ and, in this case, $x = \phi(\pi(x))$. This is folklore knowledge, but the reader may consult [21, Remark 4.2] and also [8, §§5–7] for a related and more general perspective on this point.

Since F is a bijection respecting the fibre structure $X_\eta = \bigcup_{h\in G} \pi^{-1}\{h\}$, we have $|\pi^{-1}\{\pi(F(x))\}| = |\pi^{-1}\{\pi(x)\}|$ for all $x \in X_\eta$. In particular, $f(C_\phi) = C_\phi$, that is, $C_\phi + y_F = C_\phi$, see also [8, Lemma 4.2]. As the set of discontinuities of the indicator function $\mathbf{1}_W$ is precisely the boundary ∂W, a moment's reflection shows that $X_\eta \setminus C_\phi = \bigcup_{k\in\mathbb{Z}} \partial W + \Delta(k)$, so

$$\partial W + y_F \subseteq \bigcup_{k\in\mathbb{Z}} \partial W + \Delta(k). \tag{1}$$

We will use only this property of y_F to investigate the nature of possible automorphisms F. It is quite natural to expect that this will be much facilitated if the union on the right-hand side of equation (1) is disjoint. Indeed, for general Toeplitz sequences, Bułatek and Kwiatkowski [2] studied the centralizer problem under this assumption, because their condition (Sh) is equivalent to the *disjointness condition*

$$\partial W \cap (\partial W + \Delta(k)) = \varnothing \quad \text{for all } k \in \mathbb{Z} \setminus \{0\}. \tag{D}$$

Namely, condition (Sh) is satisfied if and only if each T-orbit in G hits the set of discontinuities of ϕ at most once, which is clearly equivalent to condition (D), see also [9, remark after Definition 1].

As in [2, Proposition 3], it follows that:

(1) each fibre over a point in the MEF contains either exactly one or exactly two points; and

(2) there exists $N \in \mathbb{N}$ such that $\partial W + y_F \subseteq \bigcup_{|k|\leqslant N} \partial W + \Delta(k)$.

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We will introduce a weaker disjointness condition which also implies condition (2), see Proposition 2.12:

$$\text{for all } k \in \mathbb{Z} \setminus \{0\} : \partial W \cap (\partial W - \Delta(k)) \text{ is nowhere dense with respect to the subspace topology of } \partial W. \tag{D$'$}$$

Moreover, we need a strengthened version of this condition which, in many examples, will help to show that $\partial W + y_F \subseteq \partial W + \Delta(k_0)$ for a single $k_0 \in \mathbb{Z}$:

$$\text{for all } k \in \mathbb{Z} \setminus \{0\} \text{ for all } \beta \in G : \partial W \cap (\partial W - \beta) \cap (\partial W - 2\beta - \Delta(k)) \text{ is nowhere dense with respect to the subspace topology of } \partial W. \tag{DD$'$}$$

Conditions (D$'$) and (DD$'$) and an additional growth restriction on the arithmetic structure, see equation (42), play essential roles for proving that the assumption of our basic Theorem 2.8 is satisfied. To prove conditions (D$'$) and (DD$'$), we assume in equation (AS) below that the sets of essential holes have some particular arithmetic structure motivated by the intended applications to $\mathcal{B}$-free Toeplitz shifts, see Propositions 2.20, 2.28 and 2.30. We note here that the additional growth restriction excludes irregular Toeplitz shifts, see Remark 2.29. In Theorem 3.17, we show that $\mathcal{B}$-free Toeplitz subshifts indeed satisfy the structural assumption in equation (AS).

A characterization of the triviality of the centralizer is provided in [**2**, Theorem 1]. The example $\mathcal{B}_1 = \{2^n c_n : n \in \mathbb{N}\}$ with coprime odd $c_n > 1$ from [**10**] satisfies condition (D), but we were unable to evaluate the criterion from [**2**, Theorem 1] for it. Instead, we will show that the example not only satisfies $\partial W + y_F \subseteq \bigcup_{|k|\leqslant N} \partial W + \Delta(k)$, but that there exists a single integer k_0 such that $\partial W + y_F \subseteq \partial W + \Delta(k_0)$, i.e. $\partial W + (y_F - \Delta(k_0)) \subseteq \partial W$, see Example 3.20. The same holds for the example $\mathcal{B}_1' = \mathcal{B}_1 \cup \{c_1^2\}$ and for further generalizations of this kind. We shall see that this is the key to control the centralizer with modest efforts in Corollary 2.21: in the case of $\mathcal{B}_1$, the centralizer is trivial (see also [**10**]), while for $\mathcal{B}_1'$, our approach only yields that the c_1th iterate of each centralizer element is trivial. Indeed, we will show for this example that the centralizer has an element of order c_1, see Proposition 3.23 and Remark 3.24. More generally, we will show that elements of the centralizer can be of arbitrarily large finite order, see Corollary 3.25.

Already, the example $\mathcal{B}_2 = \{2^n c_n, 3^n c_n : n \in \mathbb{N}\}$, with coprime $c_n > 1$ also coprime to 2 and 3 and satisfying $\prod_{n\in\mathbb{N}}(1 - 1/c_n) > \frac{1}{2}$, violates condition (Sh) but satisfies conditions (D$'$) and (DD$'$), see Example 3.21. In §3.6, we show that the shift determined by $\mathcal{B}_2$ has superpolynomial complexity (the same holds for $\mathcal{B}_1$), so that the results from [**4–6**] do not apply. We also deliver examples for which not all holes are essential, see Examples 3.27 and 3.29.

2. *The abstract regular Toeplitz case*

2.1. *A first basic theorem.* Let $\eta \in \{0, 1\}^{\mathbb{Z}}$ be a non-periodic Toeplitz sequence with period structure $p_1 \mid p_2 \mid p_3 \ldots \to \infty$. That means for each $k \in \mathbb{Z}$, there exists $n \geq 1$ such that $\eta_{|k+p_n\mathbb{Z}}$ is constant, but η is not periodic. For $i = 0, 1$, denote $\mathcal{P}_n^i = \{k \in \mathbb{Z} : \eta_{|k+p_n\mathbb{Z}} = i\}$ (we use this notation, because it is shorter than $\mathrm{Per}_{p_n}(\eta, i)$ established in the literature) and $\mathcal{H}_n = \mathbb{Z} \setminus (\mathcal{P}_n^0 \cup \mathcal{P}_n^1)$. Here, $\mathcal{H}_n$ is called the set of *holes at level n*.

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The odometer group $G = \varprojlim \mathbb{Z}/p_n\mathbb{Z}$ with the $\mathbb{Z}$-action 'addition of 1' ($T : G \to G$, $(Tg)_n = g_n + 1$) is the MEF of the subshift X_η, which is the orbit closure of η under the left shift σ on $\{0, 1\}^{\mathbb{Z}}$. In symbols, $\pi : (X_\eta, \sigma) \to (G, T)$. Observe that $\mathcal{H}_n \subseteq \mathcal{H}_N$ if $n > N$.

In [1], a cut and project scheme is associated with η by specifying a compact and topologically regular window $W \subseteq G$: for $h \in G$, let $U_n(h) = \{g \in G : g_n = h_n\}$ and define

$$V^i = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathcal{P}_n^i\cap[0,p_n)} U_n(\Delta(k))$$

for $i = 0, 1$. From [1, Theorem 1 and its proof], we see:

(1) $W := \overline{V^1} = G \setminus V^0$ is topologically regular, i.e. $\overline{\mathrm{int}(W)} = W$;

(2) $\partial W = G \setminus (V^0 \cup V^1)$;

(3) $\Delta(k) \in V^i$ if and only if $\eta_k = i$ for $i = 0, 1$ and all $k \in \mathbb{Z}$, in particular $\eta_k = 1$ if and only if $\Delta(k) \in W$.

LEMMA 2.1. *$h \in \partial W$ if and only if $h_N \in \mathcal{H}_N$ for all $N \in \mathbb{N}$.*

Proof. We have

$$\begin{aligned} h \notin \partial W \Leftrightarrow h \in \bigcup_{i\in\{0,1\}} V^i \Leftrightarrow h \in \bigcup_{i\in\{0,1\}} \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathcal{P}_n^i\cap[0,p_n)} U_n(\Delta(k)) \\ &\Leftrightarrow \text{there exists } i \in \{0,1\} \text{ there exists } n \in \mathbb{N} \text{ there exists} \\ &\quad k \in \mathcal{P}_n^i \cap [0, p_n) : h_n = k \\ &\Leftrightarrow \text{there exists } i \in \{0,1\} \text{ there exists } n \in \mathbb{N} : h_n \in \mathcal{P}_n^i \\ &\Leftrightarrow \text{there exists } n \in \mathbb{N} : h_n \notin \mathcal{H}_n \end{aligned}$$

□

Definition 2.2. (Essential holes) The set of *essential holes at level N* is defined as

$$\widetilde{\mathcal{H}}_N := \{k \in \mathcal{H}_N : \mathcal{H}_n \cap (k + p_N\mathbb{Z}) \neq \emptyset \text{ for all } n \geq N\}. \tag{2}$$

Definition 2.3. The minimal periods of $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$ are denoted by τ_n and $\tilde{\tau}_n$, respectively.

Remark 2.4.

(a) $\widetilde{\mathcal{H}}_n \subseteq \widetilde{\mathcal{H}}_N$ if $n > N$.

(b) $\mathcal{H}_N$ and $\widetilde{\mathcal{H}}_N$ are p_N-periodic by definition—although this need not be their minimal period. (For $\widetilde{\mathcal{H}}_N$, just observe that $k \in \widetilde{\mathcal{H}}_N$ if and only if $k + p_N \in \widetilde{\mathcal{H}}_N$.) Hence, $\tau_N \mid p_N$ and $\tilde{\tau}_N \mid p_N$, so that expressions like '$\tau_N \mid h_N$' or '$\gcd(\tilde{\tau}_N, h_N)$' are well defined when $h_N \in \mathbb{Z}/p_N\mathbb{Z}$. (This is consistent with the following general convention: an element z of an abelian group Z is divisible by $n \in \mathbb{N}$ if $z \in nZ$. If Z is a cyclic group and $z_1, z_2 \in Z$, then $\gcd(z_1, z_2)$ is a generator of the subgroup of Z generated by z_1 and z_2. If $Z = \mathbb{Z}$, then we choose $\gcd(z_1, z_2)$ to be positive by convention. Finally, if $n \in \mathbb{Z}$ and $z + p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z}$, then we understand by $\gcd(n, z + p\mathbb{Z})$ the (positive) generator of the group generated by n and $z + p\mathbb{Z}$; this is the greatest common divisor of the numbers n, z and p in the usual sense.)

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LEMMA 2.5.
(a) *$k \in \widetilde{\mathcal{H}}_N$ if and only if $U_N(\Delta(k)) \cap \partial W \neq \emptyset$.*
(b) *$h \in \partial W$ if and only if $h_N \in \widetilde{\mathcal{H}}_N$ for all $N \in \mathbb{N}$.*

Proof. (a) Let $h \in U_N(\Delta(k)) \cap \partial W$. Then $k \in h_N + p_N\mathbb{Z} \subseteq \mathcal{H}_N + p_N\mathbb{Z} = \mathcal{H}_N$ by Lemma 2.1, and for all $n \geq N$, we have in view of Lemma 2.1: $h_n \in \mathcal{H}_n \cap (h_N + p_N\mathbb{Z}) = \mathcal{H}_n \cap (k + p_N\mathbb{Z})$. For the reverse implication, let $k \in \widetilde{\mathcal{H}}_N$. We construct $h \in \partial W$ with $h_N = k \bmod p_N$: for $j > N$, there is $r_j \in \mathcal{H}_j \cap (k + p_N\mathbb{Z})$. Let $h^{(j)} = \Delta(r_j)$ and fix any accumulation point h of the sequence $(h^{(j)})_j$. Consider any $n \geq N$. For some sufficiently large $j_n \geq n$, we have $h_n = h_n^{(j_n)} = r_{j_n} \bmod p_n$. Hence, $h_N \in r_{j_N} + p_N\mathbb{Z} = k + p_N\mathbb{Z}$, so that $h \in U_N(\Delta(k))$, and $h_n \in r_{j_n} + p_n\mathbb{Z} \subseteq \mathcal{H}_{j_n} + p_n\mathbb{Z} \subseteq \mathcal{H}_n + p_n\mathbb{Z} = \mathcal{H}_n$, so that $h \in \partial W$ by Lemma 2.1.

(b) We have

$$\begin{aligned} h \in \partial W &\Leftrightarrow \text{for all } N \in \mathbb{N} : U_N(h) \cap \partial W \neq \emptyset \\ &\Leftrightarrow \text{for all } N \in \mathbb{N} : U_N(\Delta(h_N)) \cap \partial W \neq \emptyset \\ &\Leftrightarrow \text{for all } N \in \mathbb{N} : h_N \in \widetilde{\mathcal{H}}_N \quad \text{by part (a).} \end{aligned}$$

□

The following simple lemma is basic for our approach.

LEMMA 2.6.
(a) *If $h + \beta\mathbb{Z} \subseteq \partial W$ for some $h, \beta \in G$, then $h_n + \gcd(\beta_n, \tilde{\tau}_n)\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n$ for all $n > 0$.*
(b) *If $\partial W + \beta \subseteq \partial W$ for some $\beta \in G$, then $\tilde{\tau}_n \mid \beta_n$ for all $n > 0$.*

Proof. (a) $h_n + \beta_n\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n$ for all n by Lemma 2.5(b). As $\widetilde{\mathcal{H}}_n$ is $\tilde{\tau}_n$-periodic, this implies $h_n + \gcd(\beta_n, \tilde{\tau}_n)\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n$.

(b) Let $j \in \widetilde{\mathcal{H}}_n$. Then there is some $h \in U_n(\Delta(j)) \cap \partial W$ by Lemma 2.5(a). Now, $h + \beta\mathbb{Z} \subseteq \partial W$ by assumption. As $h_n = j$, this implies, in view of part (a) of the lemma, $j + \gcd(\beta_n, \tilde{\tau}_n)\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n$. As this holds for each $j \in \widetilde{\mathcal{H}}_n$, we have $\widetilde{\mathcal{H}}_n + \gcd(\beta_n, \tilde{\tau}_n)\mathbb{Z} = \widetilde{\mathcal{H}}_n$, and as $\tilde{\tau}_n$ is the minimal period of $\widetilde{\mathcal{H}}_n$, we conclude that $\tilde{\tau}_n \mid \beta_n$. □

Remark 2.7. If $h + \beta\mathbb{Z} \subseteq \partial W$, Lemma 2.6 implies

$$\boldsymbol{\delta}(\mathcal{H}_n) \geq \boldsymbol{\delta}(\widetilde{\mathcal{H}}_n) \geq \frac{1}{\gcd(\beta_n, \tilde{\tau}_n)}.$$

Therefore, the last lemma seems to be useful only for regular Toeplitz shifts, because for irregular ones, $\inf_n \boldsymbol{\delta}(\mathcal{H}_n) > 0$ so that no useful lower bound on $\gcd(\beta_n, \tilde{\tau}_n)$ can be expected.

Each automorphism F of (X_η, σ) determines an element $y_F \in G$ such that $\pi(F(x)) = \pi(x) + y_F$ for all $x \in X_\eta$. Observe that

$$\partial W + y_F \subseteq \bigcup_{k \in \mathbb{Z}} (\partial W + \Delta(k)), \tag{3}$$

because F leaves the set of non-one-point fibres over the MEF invariant. We first focus on a stronger property than equation (3).

THEOREM 2.8. *Recall that $\tilde{\tau}_n$ denotes the minimal period of $\widetilde{\mathcal{H}}_n$.*

(a) *If an automorphism F of (X_η, σ) satisfies $\partial W + y_F \subseteq \partial W + \Delta(k)$ for some $k = k_F \in \mathbb{Z}$, then $\tilde{\tau}_n \mid (y_F)_n - k_F$. In particular, if infinitely many $\widetilde{\mathcal{H}}_n$ have minimal period p_n, then $y_F = \Delta(k_F)$.*

(b) *Suppose that for each automorphism $F \in \mathrm{Aut}_\sigma(X_\eta)$, there exists a unique $k_F \in \mathbb{Z}$ such that*

$$\partial W + y_F \subseteq \partial W + \Delta(k_F). \tag{4}$$

If $M := \liminf_{n\to\infty} p_n/\tilde{\tau}_n < \infty$, then

$$\mathrm{Aut}_\sigma(X_\eta) = \langle\sigma\rangle \oplus \mathrm{Tor},$$

where Tor *denotes the torsion group of* $\mathrm{Aut}_\sigma(X_\eta)$*. Moreover,* Tor *is a cyclic group (possibly trivial), whose order divides M. In particular, if infinitely many $\widetilde{\mathcal{H}}_n$ have minimal period p_n, then the centralizer of (X_η, σ) is trivial.*

Proof. (a) The first claim follows from Lemma 2.6(b), the second one is just a special case of this.

(b) In each residue class of $\mathrm{Aut}_\sigma(X_\eta)/\langle\sigma\rangle$, there is exactly one element F for which the associated integer k_F satisfying equation (4) equals 0. These elements F form a subgroup J of $\mathrm{Aut}_\sigma(X_\eta)$. Suppose for a contradiction that there are $M + 1$ different automorphisms $F_1, \ldots, F_{M+1} \in J$. In view of Lemma 2.6(b), they all satisfy

$$\tilde{\tau}_n \mid (y_{F_i})_{S_n} \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Hence, there exists arbitrarily large $n \in \mathbb{N}$ such that $M = p_n/\tilde{\tau}_n$ and

$$(y_{F_i})_{S_n}/\tilde{\tau}_n \in \{0, \ldots, M-1\} \quad \text{for all } i = 1, \ldots, M+1.$$

It follows that there exist two different $i, j \in \{1, \ldots, M+1\}$ for which $(y_{F_i})_{S_n} = (y_{F_j})_{S_n}$ for infinitely many n, which is only possible if $y_{F_i} = y_{F_j}$. In view of [**6**, Lemma 2.4], this implies $F_i = F_j \bmod \langle\sigma\rangle$, so that $i = j$, which is a contradiction. Hence, J is a finite group of order $m \leqslant M$, say $J = \{F_1, \ldots, F_m\}$. In particular, $J \subseteq \mathrm{Tor}$. However, if $F \in \mathrm{Tor}$, then $r y_F = \Delta(0) \in G$ for some positive integer r. Hence, with the integer k_F satisfying equation (4), we have

$$\partial W + y_{Id} - \Delta(rk_F) = \partial W - \Delta(rk_F) = \partial W + r y_F - \Delta(rk_F) \subseteq \partial W,$$

but $k_{Id} = 0$, so equation (4) implies $k_F = 0$. It follows that $F \in J$, and we proved that $J = \mathrm{Tor}$. Then [**7**, Theorem 3.2(2)] implies that Tor is cyclic, and $\mathrm{Aut}_\sigma(X_\eta) = J \oplus \langle\sigma\rangle = \mathrm{Tor} \oplus \langle\sigma\rangle$.

It remains to determine the order m of Tor: fix some $F \in \mathrm{Tor}$. In view of equation (5),

$$p_n = M \cdot \tilde{\tau}_n \mid M \cdot (y_F)_{S_n} \equiv (y_{F^M})_{S_n} \bmod p_n$$

for all $n \in \mathbb{N}$. Hence, $F^M = \mathrm{id}_{X_\eta}$, so that the order of F is a divisor of M. $\square$

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To apply this theorem, we need to verify the assumption in equation (4) and to determine the periods $\tilde{\tau}_n$. So we focus next on finding sufficient conditions that imply $\partial W + (y_F - \Delta(k))\mathbb{Z} \subseteq \partial W$.

2.2. *The separated holes conditions and its variants.* The *separated holes condition* (Sh) was introduced in [2]. It requires

$$\text{for all } k \in \mathbb{Z} \setminus \{0\}, \quad \text{there exists } N \in \mathbb{N} \quad \text{for all } n \geq N : \mathcal{H}_n \cap (\mathcal{H}_n - k) = \varnothing. \quad \text{(Sh)}$$

As mentioned in the introduction, it is equivalent to the *disjointness condition*

$$\text{for all } \ k \in \mathbb{Z} \setminus \{0\} : \partial W \cap (\partial W - \Delta(k)) = \varnothing. \quad \text{(D)}$$

Indeed, it is even equivalent to the *separated essential holes condition*

$$\text{for all } k \in \mathbb{Z} \setminus \{0\}, \quad \text{there exists } N \in \mathbb{N} \quad \text{for all } n \geq N : \widetilde{\mathcal{H}}_n \cap (\widetilde{\mathcal{H}}_n - k) = \varnothing. \quad \text{(Seh)}$$

As we do not make use of this equivalence, we leave it as an exercise.

The following variants of the separated essential holes condition, which allow to study Toeplitz subshifts that violate condition (Sh), will play important roles; however, so we provide proofs for the corresponding equivalences:

(i) *weak disjointness condition* (D′), equivalent to *weak separated essential holes condition* (Seh′):

$$\text{for all } k \in \mathbb{Z} \setminus \{0\} : \partial W \cap (\partial W - \Delta(k)) \\ \text{is nowhere dense with respect to the subspace topology of } \partial W. \quad \text{(D}'\text{)}$$

$$\text{for all } k \in \mathbb{Z} \setminus \{0\} : \text{there is no arithmetic progression } r + p_N\mathbb{Z} \text{ such that} \\ \text{for all } n \geq N : \varnothing \neq (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n \subseteq \widetilde{\mathcal{H}}_n - k; \quad \text{(Seh}'\text{)}$$

(ii) *weak double disjointness condition* (DD′), equivalent to *weak double separated essential holes condition* (DSeh′):

$$\text{for all } k \in \mathbb{Z} \setminus \{0\} \quad \text{for all } \beta \in G : \partial W \cap (\partial W - \beta) \cap (\partial W - 2\beta - \Delta(k)) \\ \text{is nowhere dense with respect to the subspace topology of } \partial W, \quad \text{(DD}'\text{)}$$

$$\text{for all } k \in \mathbb{Z} \setminus \{0\} \ \text{ for all } \beta \in G : \text{there is no arithmetic progression } r + p_N\mathbb{Z} \\ \text{such that for all } n \geq N : \varnothing \neq (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n \subseteq (\widetilde{\mathcal{H}}_n - \beta_n) \cap (\widetilde{\mathcal{H}}_n - 2\beta_n - k). \\ \text{(DSeh}'\text{)}$$

Observe that for $\beta = 0$ conditions (DD′) and (DSeh′) reduce to conditions (D′) and (Seh′), respectively. Moreover, conditions (D) and (Sh) clearly imply conditions (D′) and (Seh′), respectively.

In the following, we will assume condition (DD′)—indeed, for some results, only the weaker condition (D′) is needed. In the $\mathcal{B}$-free setting, condition (DD′) will be verified under suitable assumptions in Proposition 2.30.

Both equivalences above, namely $\neg$ condition (D′) $\Leftrightarrow \neg$ condition (Seh′) and $\neg$ condition (DD′) $\Leftrightarrow \neg$ condition (DSeh′), follow immediately from the next lemma.

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LEMMA 2.9. *Let $k \in \mathbb{Z} \setminus \{0\}$, $\beta \in G$, $N > 0$ and $r \in \mathbb{Z}$. Then,*

$$\emptyset \neq U_N(\Delta(r)) \cap \partial W \subseteq (\partial W - \beta) \cap (\partial W - 2\beta - \Delta(k)) \tag{6}$$

if and only if

$$\text{for all } n \geq N : \emptyset \neq (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n \subseteq (\widetilde{\mathcal{H}}_n - \beta_n) \cap (\widetilde{\mathcal{H}}_n - 2\beta_n - k). \tag{7}$$

Proof. Suppose that equation (6) holds. Then, for all $n \geq N$, there is $r_n \in r + p_N\mathbb{Z}$ such that $U_n(\Delta(r_n)) \cap \partial W \neq \emptyset$, whence $r_n \in (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n$ in view of Lemma 2.5(a).

Now consider any $r_n \in (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n$. By the inclusion in equation (6),

$$\begin{aligned} U_n(\Delta(r_n)) \cap \partial W &\subseteq U_N(\Delta(r_n)) \cap \partial W \\ &= U_N(\Delta(r)) \cap \partial W \subseteq (\partial W - \beta) \cap (\partial W - 2\beta - \Delta(k)), \end{aligned}$$

so that, by Lemma 2.5(a) again, $r_n + \beta_n \in \widetilde{\mathcal{H}}_n$ and $r_n + 2\beta_n + k \in \widetilde{\mathcal{H}}_n$, which proves equation (7).

Conversely, suppose that equation (7) holds. For each $n \geq N$, there is some $r_n \in (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n$, and we find a subsequence $\Delta(r_{n_i})$ that converges to some $h \in G$. Hence, for each $m > 0$, there is $n_i \geq m$ such that $h_m \in r_{n_i} + p_m\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_{n_i} + p_m\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_m + p_m\mathbb{Z} = \widetilde{\mathcal{H}}_m$, so that $h \in \partial W$ in view of Lemma 2.5(b). As $h_N \in r_{n_i} + p_N\mathbb{Z} = r + p_N\mathbb{Z}$ for some n_i, this shows that $h \in U_N(\Delta(r)) \cap \partial W$.

Now consider any $h \in U_N(\Delta(r)) \cap \partial W$. Then, for all $n \geq N$, $U_n(h) \cap \partial W \neq \emptyset$, so that $h_n \in (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n$, where we used Lemma 2.5(a) once more. The inclusion in equation (7) then implies $h_n \in (\widetilde{\mathcal{H}}_n - \beta_n) \cap (\widetilde{\mathcal{H}}_n - 2\beta_n - k)$ for all $n \geq N$. Now Lemma 2.5(b) shows that $h + \beta \in \partial W$ and $h + 2\beta + \Delta(k) \in \partial W$, which proves equation (6). □

Remark 2.10. Suppose that the condition (*) from [3] holds, that is, for any $n \in \mathbb{N}$ and $s \in \mathbb{Z}$,

$$[sp_n, (s+1)p_n) \cap \mathcal{H}_n \subseteq \mathcal{H}_{n+1} \text{ or } [sp_n, (s+1)p_n) \cap \mathcal{H}_{n+1} = \emptyset, \tag{$*$}$$

and recall that it implies triviality of the centralizer [3, Theorem 1]. Here we show that it implies $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$ for all n, but mostly precludes property (Seh′): note first that

$$\text{for all } n \in \mathbb{N} \quad \text{for all } r \in \mathcal{H}_n, \quad \text{there exists } s \in \mathbb{Z} : r + sp_n \in \mathcal{H}_{n+1}. \tag{8}$$

Indeed, otherwise there are $n \in \mathbb{N}$ and $r \in \mathcal{H}_n$ such that $r + sp_n \in \mathcal{H}_n \setminus \mathcal{H}_{n+1}$ for all $s \in \mathbb{Z}$. However, then $\mathcal{H}_{n+1} = \emptyset$ in view of condition (*), which is excluded because we study only non-periodic Toeplitz sequences. A straightforward inductive application of condition (8) shows that for all $n \in \mathbb{N}$, $r \in \mathcal{H}_n$ and $m > n$, there are integers $s_n, \ldots, s_{m-1}$ such that $r + s_n p_n + \cdots + s_{m-1}p_{m-1} \in (r + p_n\mathbb{Z}) \cap \mathcal{H}_m$. Hence, $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$ for all $n \in \mathbb{N}$. We claim

$$(r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n \subseteq \widetilde{\mathcal{H}}_n - (r' - r) \quad \text{for any } r, r' \in \mathcal{H}_N \cap [0, p_N). \tag{9}$$

Indeed, let $r, r' \in \mathcal{H}_N \cap [0, p_N)$ and $s \in \mathbb{Z}$. Suppose that $r + sp_N \in \mathcal{H}_n$ for some $n > N$. Since for any $m \geq N$ we have $r + sp_N \in [s'_m p_m, (s'_m + 1)p_m)$, where $s'_m = [sp_N/p_m]$,

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and $\mathcal{H}_n \subseteq \mathcal{H}_{n-1} \subseteq \cdots \subseteq \mathcal{H}_N$, by condition (*), we obtain $[s'_m p_m, (s'_m + 1)p_m) \cap \mathcal{H}_m \subseteq \mathcal{H}_{m+1}$ for any $N \leq m < n$. In particular, since $[sp_N, (s+1)p_N) \subseteq [s'_m p_m, (s'_m + 1)p_m)$, we have $[sp_N, (s+1)p_N) \cap \mathcal{H}_m \subseteq \mathcal{H}_{m+1}$ for any $N \leq m < n$. Hence,

$$[sp_N, (s+1)p_N) \cap \mathcal{H}_N = [sp_N, (s+1)p_N) \cap \mathcal{H}_{n-1} \subseteq \mathcal{H}_n.$$

Of course, $r' + sp_N \in [sp_N, (s+1)p_N) \cap \mathcal{H}_N$. So $r' + sp_N \in \mathcal{H}_n = \widetilde{\mathcal{H}}_n$. The same arguments as above, with roles of r and r' interchanged, show that

$$r + sp_N \in \widetilde{\mathcal{H}}_n \Leftrightarrow r' + sp_N \in \widetilde{\mathcal{H}}_n \Leftrightarrow r + sp_N \in \widetilde{\mathcal{H}}_n - (r' - r).$$

So equation (9) follows. Hence, if $[0, p_N)$ contains at least two holes at level N, then condition (Seh′) does not hold. If, however, $\mathcal{H}_n \cap [0, p_n)$ is a singleton for any $n \geq N$, then the distance between consecutive holes at level n is p_n, so even condition (Sh) holds.

Remark 2.11. Given $k \in \mathcal{H}_N \setminus \widetilde{\mathcal{H}}_N$, let $n_k \geq N$ be minimal such that $(k + p_N\mathbb{Z}) \cap \mathcal{H}_{n_k} = \varnothing$. Clearly, n_k depends only on the residue of k modulo p_N, so the n_k are bounded, say, by m_N. Then, for $k \in \mathcal{H}_N$, $k \in \widetilde{\mathcal{H}}_N$ if and only if $(k + p_N\mathbb{Z}) \cap \mathcal{H}_{m_N} \neq \varnothing$. More generally, for every $n \geq m_N$: $k \in \widetilde{\mathcal{H}}_N$ if and only if $(k + p_N\mathbb{Z}) \cap \mathcal{H}_n \neq \varnothing$. Hence, for every $n \geq m_N$, $\widetilde{\mathcal{H}}_N = \mathcal{H}_n + p_N\mathbb{Z}$. It follows that the minimal period $\tilde{\tau}_N$ of $\widetilde{\mathcal{H}}_N$ divides $\gcd(\tau_n, p_N)$ for $n \geq m_N$.

2.3. *Consequences of the weak disjointness condition (D′).* Consider any automorphism F of (X_η, σ). Recall from the introduction that $\pi \circ F = f \circ \pi$, where $\pi : (X_\eta, \sigma) \to (G, T)$ is the MEF-map and $f : G \to G$, $y \mapsto y + y_F$ for some $y_F \in G$, and that $\partial W + y_F \subseteq \bigcup_{k \in \mathbb{Z}} \partial W + \Delta(k)$. Denote

$$V_k = \partial W \cap (\partial W + \Delta(k) - y_F) \quad (k \in \mathbb{Z}) \tag{10}$$

and

$$K = \{k \in \mathbb{Z} : \mathrm{int}_{\partial W}(V_k) \neq \varnothing\}. \tag{11}$$

PROPOSITION 2.12. *Assume that the weak disjointness condition (D′) holds. Let the automorphism F of (X_η, σ) be described by a block code $\{0,1\}^{[-m:m]} \to \{0,1\}$. Then,*

$$\partial W + y_F \subseteq \bigcup_{k=-m}^{m} \partial W + \Delta(k). \tag{12}$$

Proof. Recall from equation (3) that $\partial W + y_F \subseteq \bigcup_{k \in \mathbb{Z}} \partial W + \Delta(k)$.

Let $y \in G$ and recall that $\pi : X_\eta \to G$ denotes the factor map onto the MEF. At the end of the proof, we show

$$|\pi^{-1}\{y\}| = 2 \Leftrightarrow \text{there exists } j \in \mathbb{Z} : y + \Delta(j) \in R := \partial W \setminus \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \partial W + \Delta(k). \tag{13}$$

As $F : X_\eta \to X_\eta$ is a bijection that maps π-fibres to π-fibres and as $\pi \circ F = f \circ \pi$, it follows that when $\pi^{-1}\{y\} = \{x_1, x_2\}$ with $x_1 \neq x_2$, then $\pi^{-1}\{f(y)\} = \{x'_1 = F(x_1), x'_2 = F(x_2)\}$, and there are exactly one index $j \in \mathbb{Z}$ such that $y + \Delta(j) \in \partial W$ and $(x_1)_j \neq (x_2)_j$, and exactly one index $k \in \mathbb{Z}$ such that $f(y) + \Delta(k) \in \partial W$ and $(x'_1)_k \neq (x'_2)_k$.

As F is described by a block code $\{0,1\}^{[-m:m]} \to \{0,1\}$, it follows that $|j-k| \leqslant m$. (This argument is taken from the proof of [2, Corollary 1].)

Consider any $y \in R$. Then the index j in equation (13) equals 0 and $y + y_F \in \partial W - \Delta(k)$ for some k with $|k| \leqslant m$. In other words: $f(R)$ is contained in the closed set $\bigcup_{k=-m}^{m} T^k(\partial W)$. Because of condition (D′), the set R defined in equation (13) is residual with respect to the subspace topology of ∂W. Hence,

$$\partial W + y_F = f(\partial W) = f(\overline{R}) \subseteq \overline{f(R)} \subseteq \bigcup_{k=-m}^{m} T^k(\partial W) = \bigcup_{k=-m}^{m} \partial W + \Delta(k).$$

It remains to prove equation (13). Recall first that $|\pi^{-1}\{y\}| = 1$ if and only if $y \in C_\phi$, so that $|\pi^{-1}\{y\}| > 1$ if and only if $y \in \bigcup_{k\in\mathbb{Z}} \partial W + \Delta(k)$. So all we must show is that (A) $|\pi^{-1}\{y\}| > 2$ if and only if (B) $y + \Delta(j) \in \partial W \cap (\partial W + \Delta(k))$ for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z} \setminus \{0\}$.

Suppose first that part (A) holds, i.e. that there are at least three different points in $\pi^{-1}\{y\}$. As points in the same π-fibre can differ only at positions k where $y + \Delta(k) \in \partial W$, there must be at least two such positions, and that is part (B).

Conversely, if part (B) holds and if $x = \phi(y)$, then $x_j = x_{j-k} = 1$.

(i) As $\partial W \cup (\partial W + \Delta(k))$ is nowhere dense in G, there are arbitrarily small perturbations y' of y such that $y' + \Delta(j) \notin \partial W \cup (\partial W + \Delta(k))$, resulting in points $x' = \phi(y')$ with $x'_j = x'_{j-k} = 0$.

(ii) As $\partial W \cap (\partial W + \Delta(k))$ is nowhere dense in ∂W with respect to the subspace topology on ∂W (because of condition (D′)), there are arbitrarily small perturbations y' of y such that $y' + \Delta(j) \in \partial W \setminus (\partial W + \Delta(k))$, resulting in $x' = \phi(y')$ with $x'_j = 1$ and $x'_{j-k} = 0$.

Hence, $|\pi^{-1}\{y\}| \geqslant 3$, and that is part (A). □

PROPOSITION 2.13. *We have $K \neq \emptyset$, and there is a countable collection $U_{N_j}(\Delta(r_j))$, $j \in \mathbb{N}$, of cylinder sets in G with the following properties: for each $j \in \mathbb{N}$, there exists some $k_j \in \mathbb{Z}$ such that*

$$\emptyset \neq (U_{N_j}(\Delta(r_j)) \cap \partial W) + (y_F - \Delta(k_j)) \subseteq \partial W, \tag{14}$$

and

$$\bigcup_{j\in\mathbb{N}} U_{N_j}(\Delta(r_j)) \cap \partial W \quad \textit{is dense in } \partial W. \tag{15}$$

Proof. We start from the observation of equation (3), namely $\partial W + y_F \subseteq \bigcup_{k\in\mathbb{Z}} \partial W + \Delta(k)$. This implies

$$\partial W = \bigcup_{k\in\mathbb{Z}} \partial W \cap (\partial W + \Delta(k) - y_F) = \bigcup_{k\in\mathbb{Z}} V_k.$$

Hence, $M := \bigcup_{k\in\mathbb{Z}} V_k \setminus \mathrm{int}_{\partial W}(V_k)$ is a meagre subset of the compact space ∂W and $\partial W = M \cup \bigcup_{k\in K} \mathrm{int}_{\partial W}(V_k)$. Now Baire's category theorem implies that $K \neq \emptyset$. As ∂W is separable, there is a countable collection $U_{N_j}(\Delta(r_j))$, $j \in \mathbb{N}$, of cylinder sets in G, for each of which there exists $k_j \in K$ such that $\emptyset \neq \partial W \cap U_{N_j}(\Delta(r_j)) \subseteq \mathrm{int}_{\partial W}(V_{k_j})$ and

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such that

$$\partial W = M \cup \bigcup_{j\in\mathbb{N}} (\partial W \cap U_{N_j}(\Delta(r_j))).$$

As M is meagre in ∂W, these cylinder sets satisfy equation (15), and as

$$(U_{N_j}(\Delta(r_j)) \cap \partial W) \subseteq \mathrm{int}_{\partial W}(V_{k_j}) \subseteq \partial W + \Delta(k_j) - y_F,$$

also equation (14) holds. □

COROLLARY 2.14. *Assume that the weak disjointness condition (D') holds. Let the automorphism F of (X_η, σ) be described by a block code $\{0,1\}^{[-m:m]} \to \{0,1\}$. Then the set K is contained in $[-m, m]$, $\mathrm{int}_{\partial W}(V_{k_i}) \cap \mathrm{int}_{\partial W}(V_{k_j}) = \varnothing$ for any different $k_i, k_j \in K$, and $\partial W = \bigcup_{k\in K} V'_k$, where $V'_k := \overline{\mathrm{int}_{\partial W}(V_k)}$.*

Proof. Let $K' := K \cap [-m, m]$, m as in Proposition 2.12. Because of that proposition and Proposition 2.13, $\partial W = \bigcup_{k\in K'} \overline{\mathrm{int}_{\partial W}(V_k)}$. Suppose there are $k_i, k_j \in K$ such that $\mathrm{int}_{\partial W}(V_{k_i}) \cap \mathrm{int}_{\partial W}(V_{k_j}) \neq \varnothing$. Then there is some cylinder set $U_N(\Delta(r)) \cap \partial W$ contained in this intersection. Let $\tilde{U} := (U_N(\Delta(r)) \cap \partial W) + y_F - \Delta(k_i)$. Then, $\tilde{U} \subseteq \partial W$ and $\tilde{U} + \Delta(k_i - k_j) \subseteq \partial W$, so that $\tilde{U} \subseteq \partial W \cap (\partial W + \Delta(k_j - k_i))$. As $\tilde{U}$ is non-empty and open in the relative topology on ∂W, the weak disjointness assumption implies $k_i = k_j$. This also proves that $K = K' \subseteq [-m, m]$. □

For later use, we note a further consequence of Proposition 2.13.

COROLLARY 2.15. *Assume that* $(\mathrm{int}_{\partial W}(V_{k_i}) + (y_F - \Delta(k_i)) \cap \mathrm{int}_{\partial W}(V_{k_j}) \neq \varnothing$ *for some $k_i, k_j \in K$. Then there exist $N \in \mathbb{N}$ (which can be chosen arbitrarily large) and $r \in \mathbb{Z}$ such that*

$$\begin{aligned} \varnothing \neq (U_N(\Delta(r)) \cap \partial W) + (y_F - \Delta(k_i)) \subseteq \partial W \quad \textit{and} \\ \varnothing \neq (U_N(\Delta(r)) \cap \partial W) + (y_F - \Delta(k_i)) + (y_F - \Delta(k_j)) \subseteq \partial W. \end{aligned} \tag{16}$$

Proof. Fix some cylinder set $U_{N'}(\Delta(r'))$ such that

$$\varnothing \neq U_{N'}(\Delta(r')) \cap \partial W \subseteq \mathrm{int}_{\partial W}(V_{k_i}) \cap (\mathrm{int}_{\partial W}(V_{k_j}) - (y_F - \Delta(k_i))).$$

In view of Proposition 2.13, there are $N'_i, N'_j \in \mathbb{N}$ and $r'_i, r'_j \in \mathbb{Z}$ such that

$$U_{N'}(\Delta(r')) \cap U_{N'_i}(\Delta(r'_i)) \cap (U_{N'_j}(\Delta(r'_j)) - (y_F - \Delta(k_i))) \cap \partial W \neq \varnothing \tag{17}$$

and

$$\begin{aligned} (U_{N'_i}(\Delta(r'_i)) \cap \partial W) + (y_F - \Delta(k_i)) \subseteq \partial W, \\ (U_{N'_j}(\Delta(r'_j)) \cap \partial W) + (y_F - \Delta(k_j)) \subseteq \partial W. \end{aligned} \tag{18}$$

Because of equations (17) and (18), the set $U_{N'}(\Delta(r')) \cap U_{N'_i}(\Delta(r'_i)) \cap (U_{N'_j}(\Delta(r'_j)) - (y_F - \Delta(k_i)))$ contains a cylinder set $U_N(\Delta(r))$ for which $U_N(\Delta(r)) \cap \partial W \neq \varnothing$ and

$$\begin{aligned} (U_N(\Delta(r)) \cap \partial W) + (y_F - \Delta(k_i)) \subseteq \partial W \quad \text{and} \\ (U_N(\Delta(r)) \cap \partial W) + (y_F - \Delta(k_i)) + (y_F - \Delta(k_j)) \subseteq \partial W. \end{aligned}$$

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Clearly, N can be chosen arbitrarily large. $\square$

2.4. *Consequences of the weak double disjointness condition (DD′).* The crucial step is now to show that $\mathrm{int}_{\partial W}(V_k) + (y_F - \Delta(k)) \subseteq V_k$ for all $k \in K$ under suitable assumptions.

PROPOSITION 2.16. *Assume that the weak double disjointness condition (DD′) holds. Let the automorphism F of (X_η, σ) be described by a block code $\{0, 1\}^{[-m:m]} \to \{0, 1\}$. Then, $V'_k + (y_F - \Delta(k)) \subseteq V'_k$ for all $k \in K$, the set K is contained in $[-m, m]$ and the sets V'_k have pairwise disjoint interiors.*

Proof. It suffices to prove that $\mathrm{int}_{\partial W}(V_k) + (y_F - \Delta(k)) \subseteq V'_k$ for all $k \in K$. Suppose for a contradiction that there is $k_i \in K$ such that $\mathrm{int}_{\partial W}(V_{k_i}) + (y_F - \Delta(k_i)) \not\subseteq V'_{k_i}$. By Proposition 2.13, there exists $k_j \in K \setminus \{k_i\}$ such that $(\mathrm{int}_{\partial W}(V_{k_i}) + (y_F - \Delta(k_i))) \cap \mathrm{int}_{\partial W}(V_{k_j}) \neq \varnothing$. So we can apply Corollary 2.15. Hence, there are $N \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that, setting $\beta = y_F - \Delta(k_i)$,

$$(U_N(\Delta(r)) \cap \partial W) \subseteq (\partial W - \beta) \cap (\partial W - 2\beta - \Delta(k_i - k_j)).$$

In view of condition (DD′), this implies $k_i = k_j$. The remaining assertions follow from Corollary 2.14. $\square$

Remark 2.17. Suppose that the conclusions of Proposition 2.16 are satisfied (not necessarily condition (DD′)).

(a) For $k \in K$, let $\widetilde{\mathcal{H}}_n^k := \{j \in \widetilde{\mathcal{H}}_n : U_n(\Delta(j)) \cap V'_k \neq \varnothing\}$. Then, $\widetilde{\mathcal{H}}_n = \bigcup_{k \in K} \widetilde{\mathcal{H}}_n^k$ by Lemma 2.5 and Corollary 2.14. (However, observe that this need not be a disjoint union, in general!)

(b) $j \in \widetilde{\mathcal{H}}_n^k$ if and only if $j + p_n \in \widetilde{\mathcal{H}}_n^k$, that is, all $\widetilde{\mathcal{H}}_n^k$ are p_n-periodic.

(c) For each $n > 0$ and $k \in K$, we have $\widetilde{\mathcal{H}}_n^k + \gcd((y_F)_n - k, p_n)\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n^k$.

Proof of (c). For each $j \in \widetilde{\mathcal{H}}_n^k$, there exists $h \in U_n(\Delta(j)) \cap V'_k$. By assumption, $h + (y_F - \Delta(k))\mathbb{Z} \subseteq V'_k$, so that for all $t \in \mathbb{Z}$,

$$U_n(\Delta(j + ((y_F)_n - k)t) \cap V'_k = U_n(\Delta(j) + (y_F - \Delta(k))t) \cap V'_k \neq \varnothing,$$

that is, $j + ((y_F)_n - k)\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n^k$. $\square$

Remark 2.18. Remark 2.17(c) can be used to show that $|K| = 1$ and hence $\partial W + (y_F - \Delta(k)) \subseteq \partial W$, whenever $\boldsymbol{\delta}(\widetilde{\mathcal{H}}_n) = o(1/\sqrt{p_n})$. This allows us to apply Theorem 2.8, which imposes restrictions on y_F in terms of the minimal periods $\tilde{\tau}_n$ of the sets $\widetilde{\mathcal{H}}_n$. (However, that is quite far from what holds in the $\mathcal{B}$-free setting.)

Indeed, let $k, k' \in K$ and denote $\beta = y_F - \Delta(k)$ and $\beta' = y_F - \Delta(k')$. Suppose for a contradiction that $k \neq k'$. Then,

$$|\widetilde{\mathcal{H}}_n^k \cap [0, p_n)| \geq |\langle \gcd(\beta_n, p_n)\rangle_{p_n}| \quad \text{and} \quad |\widetilde{\mathcal{H}}_n^{k'} \cap [0, p_n)| \geq |\langle \gcd(\beta'_n, p_n)\rangle_{p_n}|,$$

where $\langle s\rangle p_n$ denotes the subgroup generated by s in $\mathbb{Z}/p_n\mathbb{Z}$, so that

$$|\widetilde{\mathcal{H}}_n \cap [0, p_n)|^2 \geq |\widetilde{\mathcal{H}}_n^k \cap [0, p_n)| \cdot |\widetilde{\mathcal{H}}_n^{k'} \cap [0, p_n)|$$

$$\geq |\langle(\gcd(\beta_n, p_n), \gcd(\beta_n', p_n))\rangle_{p_n \times p_n}|,$$

where $\langle(s,t)\rangle p_n \times p_n$ denotes the subgroup generated by (s,t) in $(\mathbb{Z}/p_n\mathbb{Z})^2$. Hence,

$$|\widetilde{\mathcal{H}}_n \cap [0, p_n)|^2 \geq \operatorname{lcm}\left(\frac{p_n}{\gcd(\beta_n, p_n)}, \frac{p_n}{\gcd(\beta_n', p_n)}\right) = \frac{p_n}{\gcd(\beta_n, \beta_n', p_n)}$$
$$\geq \frac{p_n}{\gcd(\beta_n - \beta_n', p_n)} = \frac{p_n}{\gcd(k-k', p_n)}.$$

However, the last denominator is at most $2m$, so

$$\frac{1}{2m \cdot p_n} \leq \left(\frac{|\widetilde{\mathcal{H}}_n \cap [0, p_n)|}{p_n}\right)^2 = \delta(\widetilde{\mathcal{H}}_n)^2,$$

which is in contradiction to the assumption.

Here is an example of a Toeplitz sequence for which Remark 2.18 applies and for which τ_n and $\tilde{\tau}_n$, the smallest periods of $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$, are different.

Example 2.19. Garcia and Hedlund [**14**] gave the first example of a 0–1 non-periodic Toeplitz sequence. At each level n of their construction, there is exactly one hole in each interval of length p_n, so all holes are essential, and the centralizer is trivial because condition (*) is satisfied, see Remark 2.10. Our example is a modification of this construction: (to be precise, the example from [**14**] is not really a Toeplitz sequence, because it is not periodic at position 0. However, its orbit closure is minimal and contains many Toeplitz sequences. Our modification takes this into account.)

let $r_n := \sum_{j=0}^{n-1} 2^{2j} = (2^{2n}-1)/3$. Define a Toeplitz sequence in such a way that

$$\mathcal{H}_n = 2^{2n}\mathbb{Z} - r_n.$$

Observe that $\mathcal{H}_n = 2^{2(n-1)}(4\mathbb{Z}-1) - r_{n-1} \subseteq \mathcal{H}_{n-1}$, in particular, $\bigcap_{n\geq 1} \mathcal{H}_n = \varnothing$. Here, $\mathcal{H}_{n-1} \setminus \mathcal{H}_n$ is the disjoint union of the residue classes $2^{2(n-1)}(4\mathbb{Z}-k) - r_{n-1}$, $k \in \{0, 2, 3\}$, and the positions in each of these residue classes should be filled alternatingly with 0 and 1. Then all these positions have minimal period 2^{2n+1}, and $(p_n)_{n\geq 1} = (2^{2n+1})_{n\geq 1}$ is a period structure for the resulting Toeplitz sequence.

If $2^{2N}t - r_N \in \mathcal{H}_N$ and $n > N$, then

$$(2^{2N}t - r_N + p_N\mathbb{Z}) \cap \mathcal{H}_n = ((2^{2N}t + (r_n - r_N) + p_N\mathbb{Z}) \cap 2^{2n}\mathbb{Z}) - r_n$$
$$= 2^{2N}\left(\left(t + \frac{2^{2(n-N)}-1}{3} + 2\mathbb{Z}\right) \cap 2^{2(n-N)}\mathbb{Z}\right) - r_n$$

is non-empty if and only if t is odd. Hence, $\widetilde{\mathcal{H}}_N = 2^{2N}(2\mathbb{Z}+1) - r_N$ and $\tilde{\tau}_N = 2^{2N+1} = p_N = 2\tau_N$. Notice that each interval of length p_n contains exactly two holes and the distance between them is $p_n/2$. However, $p_{n+1} = 4p_n$, so condition (*) is not satisfied. Nevertheless, the centralizer is trivial by Theorem 2.8 and Remark 2.18.

2.5. *Additional arithmetic structure (motivated by the $\mathcal{B}$-free case).* Throughout this subsection, F is again an automorphism of (X_η, σ) and $\pi(F(x)) = \pi(x) + y_F$ for $x \in X_\eta$. We start with a particularly simple situation based on the following (very strong) trivial

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intersection property: there are $A_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$ such that

$$\bigcap_{n\in\mathbb{N}} \langle A_n\rangle = \{0\} \quad \text{and} \quad \widetilde{\mathcal{H}}_n \subseteq \mathcal{M}_{A_n},\ n \in \mathbb{N}. \tag{TI}$$

We will check this property for some non-trivial $\mathcal{B}$-free examples, see Examples 3.20, 3.27 and 3.28, and also §3.4.

PROPOSITION 2.20. *Suppose property (TI) is satisfied.*

(a) *Then the disjointness condition (D) holds, and there exists a unique $k \in \mathbb{Z}$ such that $\partial W + y_F \subseteq \partial W + \Delta(k)$.*
(b) *$\tilde{\tau}_n \mid (y_F)_n - k$, where $\tilde{\tau}_n$ is the minimal period of $\widetilde{\mathcal{H}}_n$. In particular, if infinitely many $\widetilde{\mathcal{H}}_n$ have minimal period p_n then $y_F = \Delta(k)$.*

Proof. (a) If there is some $y \in \partial W \cap (\partial W - \Delta(k))$, then $U_N(\Delta(y_N)) \cap \partial W = U_N(y) \cap \partial W \neq \varnothing$ and $U_N(\Delta(k + y_N)) \cap \partial W = U_N(\Delta(k) + y) \cap \partial W \neq \varnothing$, so that $k = (k + y_N) - y_N \in \widetilde{\mathcal{H}}_N - \widetilde{\mathcal{H}}_N \subseteq \mathcal{M}_{A_N} - \mathcal{M}_{A_N} \subseteq \langle \mathcal{M}_{A_N}\rangle = \langle A_N\rangle$ for all $N > 0$ by Lemma 2.5a). Hence, $k = 0$ in view of property (TI). If there are $y_1, y_2 \in \partial W$ and $k_1, k_2 \in \mathbb{Z}$ such that $y_i + y_F \in \partial W - \Delta(k_i)$ $(i = 1, 2)$, then $U_N(\Delta(k_i + (y_i + y_F)_N)) \cap \partial W \neq \varnothing$ $(i = 1, 2)$, so that $k_2 - k_1 \in \langle A_N\rangle$ for all $N > 0$ as before. Hence, $k_2 = k_1$ because of property (TI).

(b) This follows from part (a) of the lemma and from Theorem 2.8a). □

Together with Theorem 2.8, this proposition yields the following corollary.

COROLLARY 2.21. *Suppose that property (TI) is satisfied. If $M := \liminf_{n\to\infty} p_n/\tilde{\tau}_n < \infty$, then*

$$\mathrm{Aut}_\sigma(X_\eta) = \langle\sigma\rangle \oplus \mathrm{Tor},$$

where Tor *denotes the torsion group of* $\mathrm{Aut}_\sigma(X_\eta)$. *It is a cyclic group (possibly trivial), whose order divides M. In particular, if infinitely many $\widetilde{\mathcal{H}}_n$ have minimal period p_n, then the centralizer of (X_η, σ) is trivial.*

If property (TI) is not satisfied, as is the case for more complex $\mathcal{B}$-free examples, we need additional tools to verify the assumption in equation (4) of Theorem 2.8. The weak double disjointness condition (DD′) turns out to be instrumental along this way, and hence its verification under mild arithmetic assumptions in Proposition 2.30 will be an important step.

We continue with some arithmetic preparations. For the sake of brevity, we sometimes write $u \vee v$ instead of $\mathrm{lcm}(u, v)$. The following notation will be used repeatedly for positive integers a and k:

$$a^{\div k} := \frac{a}{\gcd(a, k)} = \frac{a \vee k}{k}. \tag{19}$$

For $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, denote

$$A^{\div k} := \{a^{\div k} : a \in A\}, \tag{20}$$

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$$A^{\perp k} := \{a \in A : \gcd(a, k) = 1\} \tag{21}$$

and

$$A^{\mathrm{prim}} := \{a \in A : a' \mid a \Rightarrow a' = a \text{ for all } a' \in A\}. \tag{22}$$

If $A = A^{\mathrm{prim}}$, then A is called *primitive*.

Remark 2.22. Let $r, \ell, s, m \in \mathbb{Z}$ and assume that $\gcd(m, \ell) \mid s - r$. Then,

$$(r + \ell\mathbb{Z}) \cap (s + m\mathbb{Z}) = x + (\ell \vee m)\mathbb{Z} = g \cdot (\tilde{r} + \tilde{\ell}\mathbb{Z}), \tag{23}$$

where $x \in \{0, \ldots, (\ell \vee m) - 1\}$ is defined uniquely by the first identity, $g = \gcd(x, \ell \vee m)$, $\tilde{r} = x/g$, and $\tilde{\ell} = \ell \vee m/g$. Observe that $\gcd(\tilde{r}, \tilde{\ell}) = 1$. This formula will be applied in several settings, so that one should keep in mind that g, $\tilde{r}$ and $\tilde{\ell}$ depend on r, ℓ, s and m. For later use, observe also that

$$g = \gcd(r, \ell) \vee \gcd(s, m). \tag{24}$$

Here is the proof of equation (24): as $x - r \in \ell\mathbb{Z}$ and $x - s \in m\mathbb{Z}$, we have $\gcd(x, \ell) = \gcd(r, \ell)$ and $\gcd(x, m) = \gcd(s, m)$. Hence,

$$g = \gcd(x, \ell \vee m) = \gcd(x, \ell) \vee \gcd(x, m) = \gcd(r, \ell) \vee \gcd(s, m).$$

We list some further consequences:

$$\mathrm{lcm}(A^{\div g}) = \mathrm{lcm}(A)^{\div g}, \quad g \cdot \mathcal{M}_{A^{\div g}} = \mathcal{M}_A \cap g\mathbb{Z}, \quad \text{and} \quad \mathcal{M}_{A^{\div g}} \subseteq \mathcal{M}_{A^{\div \ell \vee m}}, \tag{25}$$

where we used $\gcd(a, g) \mid \gcd(a, \ell \vee m)$ for the last inclusion. Each subset $Z \subseteq \mathbb{Z}$ holds

$$(\tilde{r} + \tilde{\ell}Z) \cap \mathcal{M}_A = (\tilde{r} + \tilde{\ell}Z) \cap \mathcal{M}_{A^{\perp \tilde{\ell}}}, \tag{26}$$

because $\gcd(\tilde{r}, \tilde{\ell}) = 1$. (Indeed, if $a \in A$, $z \in Z$ and $x = \tilde{r} + z\tilde{\ell} \in a\mathbb{Z}$, then $\gcd(a, \tilde{\ell}) \mid \tilde{r}$, so that $\gcd(a, \tilde{\ell}) \mid \gcd(\tilde{r}, \tilde{\ell}) = 1$, i.e. $a \in A^{\perp \tilde{\ell}}$.) Combining equations (23), (25) and (26) yields

$$(r + \ell\mathbb{Z}) \cap (s + a\mathbb{Z}) \cap \mathcal{M}_A = g \cdot ((\tilde{r} + \tilde{\ell}\mathbb{Z}) \cap \mathcal{M}_{(A^{\div g})^{\perp \tilde{\ell}}}). \tag{27}$$

Given a set $A \subseteq \mathbb{Z}$, we denote by $\boldsymbol{\delta}(A) := \lim_{N\to\infty}(1/\log N) \sum_{k=1}^{N}(1/k)1_A(k)$ the logarithmic density of A (provided the limit exists).

LEMMA 2.23. *Let $r, \ell, s, m \in \mathbb{Z}$ and assume that $\gcd(m, \ell) \mid s - r$. Recall that $\tilde{\ell} = \ell \vee m/g$. Then,*

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$$\boldsymbol{\delta}((r+\ell\mathbb{Z})\cap(s+m\mathbb{Z})\cap\mathcal{M}_A)=\frac{1}{\ell\vee m}\cdot\boldsymbol{\delta}(\mathcal{M}_{(A^{\div g})^{\perp\tilde{\ell}}})\leqslant\frac{1}{\ell\vee m}\cdot\boldsymbol{\delta}(\mathcal{M}_{A^{\div\ell\vee m}}),\tag{28}$$

$$\boldsymbol{\delta}((r+\ell\mathbb{Z})\cap(s+m\mathbb{Z})\setminus\mathcal{M}_A)=\frac{1}{\ell\vee m}\cdot(1-\boldsymbol{\delta}(\mathcal{M}_{(A^{\div g})^{\perp\tilde{\ell}}})).\tag{29}$$

Proof. As in [**15**, Lemma 1.17], we have

$$\boldsymbol{\delta}((\tilde{r}+\tilde{\ell}\mathbb{Z})\cap\mathcal{M}_{(A^{\div g})^{\perp\tilde{\ell}}})=\frac{1}{\tilde{\ell}}\cdot\boldsymbol{\delta}(\mathcal{M}_{(A^{\div g})^{\perp\tilde{\ell}}}).$$

As $g\cdot\tilde{\ell}=\ell\vee m$, this together with equation (23) proves the identity in equation (28). As $\boldsymbol{\delta}((r+\ell\mathbb{Z})\cap(s+m\mathbb{Z}))=1/\ell\vee m$, equation (29) follows at once. For the inequality in equation (28), observe that $\mathcal{M}_{(A^{\div g})^{\perp\tilde{\ell}}}\subseteq\mathcal{M}_{A^{\div g}}\subseteq\mathcal{M}_{A^{\div\ell\vee m}}$ by equation (25). □

LEMMA 2.24. *Assume that $(r+\ell\mathbb{Z})\cap(s+m\mathbb{Z})\cap[N,\infty)\subseteq\mathcal{M}_C$ for some $r,\ell,s,m\in\mathbb{Z}$ satisfying $\gcd(m,\ell)\mid s-r$, some $N\in\mathbb{N}$ and a finite set $C\subset\mathbb{N}$. Then, $c\mid\gcd(r,\ell)\vee\gcd(s,m)$ for some $c\in C$.*

Proof. Let $A:=(C^{\div g})^{\perp\tilde{\ell}}$. In view of equation (27), our assumption implies $(\tilde{r}+\tilde{\ell}\mathbb{Z})\cap[N,\infty)\subseteq\mathcal{M}_A=\mathcal{M}_{A^{\text{prim}}}$. As A^{prim} is taut, [**12**, Proposition 4.31] shows that there is $a\in A^{\text{prim}}$ such that $a\mid\gcd(\tilde{r},\tilde{\ell})=1$, that is, $1\in A$. Hence, there is $c\in C$ such that $c\mid g=\gcd(r,\ell)\vee\gcd(s,m)$. □

We will need a more detailed arithmetic characterization of the inclusion from Lemma 2.24 when C is a singleton and $s=0$.

LEMMA 2.25. *Let $r,\ell,a,c\in\mathbb{Z}$ satisfying $\gcd(a,\ell)\mid r$. Then the following conditions are equivalent:*

(a) $(r+\ell\mathbb{Z})\cap a\mathbb{Z}\subseteq c\mathbb{Z}$;

(b) $c\mid\gcd(r,\ell)\vee a$;

(c) $c\mid\ell\vee a$ *and* $\gcd(c,\ell)\mid r$.

Proof. By Lemma 2.24, condition (a) implies condition (b). Conversely, condition (a) follows from condition (b), because $(r+\ell\mathbb{Z})\cap a\mathbb{Z}\subseteq(\gcd(r,\ell)\vee a)\mathbb{Z}$.

Suppose that conditions (a) and (b) hold. By condition (b), we have $c\mid\gcd(r,\ell)\vee a\mid\ell\vee a$. Since $\gcd(a,\ell)\mid r$, we have $(r+\ell\mathbb{Z})\cap a\mathbb{Z}\neq\varnothing$. So by condition (a), we get $r\in c\mathbb{Z}+\ell\mathbb{Z}=\gcd(c,\ell)\mathbb{Z}$. Hence, condition (c) holds.

Finally suppose that condition (c) holds. Then,

$$\begin{aligned}\gcd(c,\gcd(r,\ell)\vee a)&=\gcd(c,r,\ell)\vee\gcd(c,a)\\&=\gcd(c,\ell)\vee\gcd(c,a)=\gcd(c,\ell\vee a)=c,\end{aligned}$$

and condition (b) follows at once. □

We will need to know the smallest periods of the difference of the sets of multiples.

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LEMMA 2.26. *Assume that A and C are finite subsets of* $\mathbb{N}$ *and that the set A is primitive.*

(a) *If* $A = \{a\}$ *and* $a \notin \mathcal{M}_C$*, then* $a \cdot \mathrm{lcm}((C^{\div a})^{\mathrm{prim}})$ *is the minimal period of* $a\mathbb{Z} \setminus \mathcal{M}_C$.

(b) *If the set* $C^{\div a} = \{c/\gcd(a, c) : c \in C\} \subset \mathbb{N} \setminus \{1\}$ *is primitive for every* $a \in A$*, then* $\mathrm{lcm}(A \cup C)$ *is the minimal period of* $\mathcal{M}_A \setminus \mathcal{M}_C$.

Proof. (a) Let $a \in \mathbb{N} \setminus \mathcal{M}_C$. Then, $\varnothing \neq a\mathbb{Z} \setminus \mathcal{M}_C = a \cdot (\mathbb{Z} \setminus \mathcal{M}_{C^{\div a}})$, and the claim follows from the observation that $\mathrm{lcm}((C^{\div a})^{\mathrm{prim}})$ is the minimal period of $\mathcal{M}_{C^{\div a}}$, see [**18**, Lemma 5.1b)].

(b) Let T be the minimal period of $\mathcal{M}_A \setminus \mathcal{M}_C$. Clearly, $T \mid \mathrm{lcm}(A \cup C)$. Now let $a \in A$. Since $1 \notin C^{\div a}$, $a \in \mathcal{M}_A \setminus \mathcal{M}_C$, so $a + T\mathbb{Z} \subset \mathcal{M}_A$. By Lemma 2.24 and primitivity of A, we get $a|T$, so that $T + (a\mathbb{Z} \setminus \mathcal{M}_C) \subset a\mathbb{Z} \setminus \mathcal{M}_C$ for every $a \in A$ and therefore $a \cdot \mathrm{lcm}((C^{\div a})^{\mathrm{prim}}) \mid T$ in view of part (a). Hence, $\mathrm{lcm}(\{a\} \cup C) = a\,\mathrm{lcm}(C^{\div a}) = a\,\mathrm{lcm}((C^{\div a})^{\mathrm{prim}}) \mid T$ for every $a \in A$, so that $\mathrm{lcm}(A \cup C)|T$. $\square$

The following proposition is the 'multi-tool' of this section.

PROPOSITION 2.27. *Let* A_n *and* S_n *be sets of positive integers. Let* $b \in \mathbb{Z}$, $n \geqslant N \geqslant 0$, $a \in A_n$ *and* $r \in \mathbb{Z}$ *be such that*

$$(r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n} \neq \varnothing \tag{30}$$

and

$$((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) + b \subseteq \mathcal{M}_{A_n}. \tag{31}$$

Assume that there is a subset $E_{n,a}$ *of* $A_n \setminus \{a\}$ *for which*

$$\sum_{a' \in E_{n,a}} \frac{1}{\varphi(a'^{\div a})} < \frac{1}{p_N}, \tag{32}$$

where φ *is Euler's totient function. Then there is* $a' \in A_n \setminus E_{n,a}$ *such that*

$$b \in \gcd(a', \gcd(r, p_N) \vee a)\mathbb{Z} \quad \text{and} \quad \varnothing \neq ((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \cap (-b + a'\mathbb{Z}). \tag{33}$$

If $b = \beta_n$ *for some* $\beta = \Delta(k)$ *with* $|k| < \gcd(a', a)$ *for all* $a' \in A_n \setminus E_{n,a}$*, then* $\beta = 0$.

Proof. In view of the assumption in equation (31), we have

$$((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \subseteq (-b + \mathcal{M}_{E_{n,a}}) \cup (-b + \mathcal{M}_{D_{n,a}}), \tag{34}$$

where $D_{n,a} = A_n \setminus E_{n,a}$. We prove below that equation (32) implies

$$((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) + b \not\subseteq \mathcal{M}_{E_{n,a}}. \tag{35}$$

Hence and by equation (34), there is $a' \in D_{n,a}$ such that equation (33) holds. Hence, $b = 0$ or $|b| \geqslant \gcd(a', \gcd(r, p_N) \vee a) \geqslant \gcd(a', a)$. It remains to show that equation (32) implies the assumption in equation (35).

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Consider any $a \in A_n$ for which $((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \neq \varnothing$. Then $\gcd(a, p_N) \mid r$, and Lemma 2.23 implies

$$0 < \boldsymbol{\delta}((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) = \frac{1}{p_N \vee a}(1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div g_n})^{\perp(p_N^{\div g_n})}})), \tag{36}$$

where

$$g_n := \gcd(r, p_N) \vee a \quad \text{and} \quad p_N^{\div g_n} = \frac{p_N \vee g_n}{g_n} = \frac{p_N \vee a}{g_n} \mid p_N^{\div a}.$$

Suppose for a contradiction that there is inclusion in equation (35). Then,

$$(r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n} \subseteq \bigcup_{a' \in E_{n,a}} (-b + a'\mathbb{Z}),$$

so that

$$\begin{aligned}
&\boldsymbol{\delta}((r + p_N\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \\
&\quad \leqslant \sum_{a' \in E_{n,a}} \boldsymbol{\delta}((r + p_N\mathbb{Z}) \cap (-b + a'\mathbb{Z}) \cap (a\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \\
&\quad \leqslant \sum_{a' \in E_{n,a}} \boldsymbol{\delta}((g_n\mathbb{Z}) \cap (-b + a'\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \\
&\quad = \sum_{a' \in E_{n,a},\ \gcd(g_n, a') \mid b} \boldsymbol{\delta}((g_n\mathbb{Z}) \cap (-b + a'\mathbb{Z}) \setminus \mathcal{M}_{S_n}) \\
&\quad = \sum_{a' \in E_{n,a},\ \gcd(g_n, a') \mid b} \frac{1}{g_n \vee a'}(1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div(g_n \vee \gcd(b,a'))})^{\perp g_n \vee a'/g_n \vee \gcd(b,a')}})).
\end{aligned} \tag{37}$$

The last equality follows from Lemma 2.23. Notice that $\gcd(g_n, a') \mid b$ implies

$$\frac{g_n \vee a'}{g_n \vee \gcd(b, a')} = \frac{(g_n \vee a') \cdot \gcd(g_n, a')}{g_n \cdot \gcd(b, a')} = \frac{a'}{\gcd(b, a')} = a'^{\div b}. \tag{38}$$

As

$$\frac{p_N \vee a}{a' \vee g_n} \leqslant \frac{p_N \vee a}{a' \vee a} = \frac{p_N^{\div a}}{a'^{\div a}}, \tag{39}$$

equations (36), (37) and (38) together yield

$$1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div g_n})^{\perp(p_N^{\div g_n})}}) \leqslant \sum_{a' \in E_{n,a}} \frac{p_N^{\div a}}{a'^{\div a}}(1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div(g_n \vee \gcd(b,a'))})^{\perp a'^{\div b}}})). \tag{40}$$

Denote

$$R_n(a') := \{b^{\div g_n} : b \in S_n, b^{\div g_n} \perp p_N^{\div g_n}, b^{\div g_n} \not\perp a'^{\div g_n}\}.$$

Then $R_n(a') \subseteq S_n^{\div g_n}$ trivially and we claim that

$$(S_n^{\div g_n})^{\perp(p_N^{\div g_n})} \setminus R_n(a') \subseteq \mathcal{M}_{(S_n^{\div(g_n \vee \gcd(b,a'))})^{\perp a'^{\div b}}}.$$

Indeed, each $b^{\div g_n} \in S_n^{\div g_n} \setminus R_n(a')$, which is also coprime to $p_N^{\div g_n}$, is coprime to $a'^{\div g_n}$ and, *a fortiori*, to $a'^{\div b}$ because $\gcd(g_n, a') \mid b$. Moreover, $b^{\div g_n} = b \vee g_n / g_n$ is a multiple

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of $b \vee g_n \vee \gcd(b, a')/g_n \vee \gcd(b, a')$, so that the latter is also coprime to $a'^{\div b}$. Therefore,

$$(S_n^{\div g_n})^{\perp (p_N^{\div g_n})} \subseteq R_n(a') \cup \mathcal{M}_{(S_n^{\div (g_n \vee \gcd(b,a'))})^{\perp a'^{\div b}}},$$

so that Behrend's inequality (see [15, Theorem 0.12] for a reference) yields

$$\begin{aligned} 1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div g_n})^{\perp (p_N^{\div g_n})}}) &\geqslant 1 - \boldsymbol{\delta}(\mathcal{M}_{R_n(a') \cup \mathcal{M}_{(S_n^{\div (g_n \vee \gcd(b,a'))})^{\perp a'^{\div b}}}}) \\ &= 1 - \boldsymbol{\delta}(\mathcal{M}_{R_n(a') \cup (S_n^{\div (g_n \vee \gcd(b,a'))})^{\perp a'^{\div b}}}) \\ &\geqslant (1 - \boldsymbol{\delta}(\mathcal{M}_{R_n(a')})) \cdot (1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div (g_n \vee \gcd(b,a'))})^{\perp a'^{\div b}}})). \end{aligned}$$

Therefore, equation (40) leads to

$$1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div g_n})^{\perp (p_N^{\div g_n})}}) \leqslant \sum_{a' \in E_{n,a}} \frac{p_N{}^{\div a}}{a'^{\div a}} \cdot \frac{1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div g_n})^{\perp (p_N^{\div g_n})}})}{1 - \boldsymbol{\delta}(\mathcal{M}_{R_n(a')})}.$$

As $1 - \boldsymbol{\delta}(\mathcal{M}_{(S_n^{\div g_n})^{\perp (p_N^{\div g_n})}}) > 0$, in view of equation (36), we can divide the last inequality by this expression, so that

$$\begin{aligned} 1 &\leqslant \sum_{a' \in E_{n,a}} \frac{p_N^{\div a}}{a'^{\div a}} \cdot \frac{1}{1 - \boldsymbol{\delta}(\mathcal{M}_{R_n(a')})} \leqslant \sum_{a' \in E_{n,a}} \frac{p_N^{\div a}}{a'^{\div a}} \cdot \frac{1}{1 - \boldsymbol{\delta}(\mathcal{M}_{\mathrm{Spec}(a'^{\div a})})} \\ &= \sum_{a' \in E_{n,a}} \frac{p_N{}^{\div a}}{a'^{\div a}} \cdot \prod_{p \mid a'^{\div a}} \frac{1}{1 - \frac{1}{p}} \leqslant p_N \cdot \sum_{a' \in E_{n,a}} \frac{1}{\varphi(a'^{\div a})}, \end{aligned} \tag{41}$$

where we used the fact that $R_n(a') \subseteq \mathcal{M}_{\mathrm{Spec}(a'^{\div g_n})} \subseteq \mathcal{M}_{\mathrm{Spec}(a'^{\div a})}$. However, the last estimate contradicts the assumption in equation (32). □

From now on, we assume that the sets $\widetilde{\mathcal{H}}_n$ have some particular *arithmetic structure*: there is a primitive set A_n of positive integers such that for each $a_n \in A_n$, there is a set $S_n = S_n(a_n)$ of positive integers satisfying

$$\widetilde{\mathcal{H}}_n = \bigcup_{a_n \in A_n} a_n\mathbb{Z} \setminus \mathcal{M}_{S_n(a_n)} \quad \text{and} \quad a_n\mathbb{Z} \setminus \mathcal{M}_{S_n(a_n)} \neq \varnothing \ (a_n \in A_n). \tag{AS}$$

Observe that $\min A_n \to \infty$ as $n \to \infty$, because $A_n \subseteq \widetilde{\mathcal{H}}_n \subseteq \mathcal{H}_n$ and $\min \mathcal{H}_n \to \infty$.

In the remaining part of this section, we prove the weak double disjointness condition (DD′) under the arithmetic structure assumption in equation (AS), which allows us to apply Proposition 2.16 in this situation. Later, in Theorem 3.17, we verify equation (AS) in the $\mathcal{B}$-free setting. Recall from equation (11) the definition of the set $K = \{k \in \mathbb{Z} : \mathrm{int}_{\partial W}(V_k) \neq \varnothing\}$.

PROPOSITION 2.28. *Assume the condition in equation (AS). If*

$$\lim_{n\to\infty} \sum_{a' \in A_n} \frac{1}{\varphi(a')} = 0, \tag{42}$$

then the weak disjointness condition (D′) is satisfied.

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Moreover, if the automorphism F of (X_η, σ) is described by a block code $\{0,1\}^{[-m:m]} \to \{0,1\}$, then the set K is contained in $[-m, m]$, $\mathrm{int}_{\partial W}(V_{k_i}) \cap \mathrm{int}_{\partial W}(V_{k_j}) = \varnothing$ *for any different $k_i, k_j \in K$, and $\partial W = \bigcup_{k \in K} V'_k$, where $V'_k := \overline{\mathrm{int}_{\partial W}(V_k)}$.*

Proof. Suppose for a contradiction that condition (D′) does not hold, equivalently that condition (Seh′) does not hold. Then there are $k \in \mathbb{Z} \setminus \{0\}$ and an arithmetic progression $r + p_N\mathbb{Z}$ such that for all $n \geq N$,

$$\varnothing \neq (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n \subseteq \widetilde{\mathcal{H}}_n - k.$$

Let $n \geq N$. In view of property (AS), there is $a \in A_n$ such that

$$\varnothing \neq (r + p_N\mathbb{Z}) \cap a\mathbb{Z} \setminus \mathcal{M}_{S_n} + k \subseteq \mathcal{M}_{A_n}. \tag{43}$$

Let $E_{n,a} = \{a' \in A_n : \gcd(a', a) \leqslant |k|\}$. Since

$$\varphi(a'^{\div a}) = a'^{\div a} \prod_{p | a'^{\div a}} \left(1 - \frac{1}{p}\right) \geqslant \frac{1}{|k|} a' \prod_{p | a'} \left(1 - \frac{1}{p}\right) = \frac{1}{|k|} \varphi(a')$$

for any $a' \in E_{n,a}$, the assumption in equation (42) above implies the assumption in equation (32) of Proposition 2.27. So this proposition applies to the inclusion in equation (43), and there is $a' \in A_n \setminus E_{n,a}$ such that $\gcd(a', a) \mid k$. As $|k| < \gcd(a', a)$ for all $a' \in A_n \setminus E_{n,a}$, this contradicts the assumption $k \in \mathbb{Z} \setminus \{0\}$.

The remaining conclusions follow from Corollary 2.14. □

Remark 2.29. Any $\mathcal{B}$-free Toeplitz subshift satisfying equation (42) is regular, because $\widetilde{\mathcal{H}}_n \subseteq \mathcal{M}_{\mathcal{A}^\infty_{S_n}}$ and $d(\mathcal{M}_{\mathcal{A}^\infty_{S_n}}) \leq \sum_{a \in \mathcal{A}^\infty_{S_n}} 1/a \leq \sum_{a \in \mathcal{A}^\infty_{S_n}} 1/\varphi(a)$.

PROPOSITION 2.30. *Assume the condition in equation (AS). If*

$$\lim_{n \to \infty} \sum_{a' \in A_n \setminus \{a\}} \frac{1}{\varphi(a'^{\div a})} = 0 \quad \text{for all choices of } a \in A_n \ (\text{where } a'^{\div a} = a'/\gcd(a', a)), \tag{44}$$

then the weak double disjointness condition (DD′)—and a fortiori condition (D′)—is satisfied.

Moreover, the conclusions of Proposition 2.28 can be complemented by $V'_k + (y_F - \Delta(k))\mathbb{Z} \subseteq V'_k$ for all $k \in K$.

Proof. Suppose for a contradiction that condition (DD′) does not hold, equivalently that condition (DSeh′) does not hold. Then there are $k \in \mathbb{Z} \setminus \{0\}$, $\beta \in G$, an arithmetic progression $r + p_N\mathbb{Z}$, such that for all $n \geq N$,

$$\varnothing \neq (r + p_N\mathbb{Z}) \cap \widetilde{\mathcal{H}}_n \subseteq (\widetilde{\mathcal{H}}_n - \beta_n) \cap (\widetilde{\mathcal{H}}_n - 2\beta_n - k).$$

Let $n \geq N$. In view of the property in equation (AS), there is $a \in A_n$ such that

$$\begin{gathered} \varnothing \neq (r + p_N\mathbb{Z}) \cap a\mathbb{Z} \setminus \mathcal{M}_{S_n} + \beta_n \subseteq \mathcal{M}_{A_n} \quad \text{and} \\ \varnothing \neq (r + p_N\mathbb{Z}) \cap a\mathbb{Z} \setminus \mathcal{M}_{S_n} + 2\beta_n + k \subseteq \mathcal{M}_{A_n}. \end{gathered} \tag{45}$$

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Let $E_{n,a} = A_n \setminus \{a\}$. In view of the assumption in equation (44), Proposition 2.27 applies to both inclusions in equation (45), and as $A_n \setminus E_{n,a} = \{a\}$, we can conclude that $a \mid \beta_n$ and $a \mid 2\beta_n + k$, so that $a \mid k$. As $a \in A_n$ and $\min A_n \to \infty$, this contradicts the assumption $k \in \mathbb{Z} \setminus \{0\}$.

The final conclusion follows from Proposition 2.16. □

THEOREM 2.31. *Assume the condition in equation (AS) and let the automorphism F of (X_η, σ) be described by a block code $\{0, 1\}^{[-m:m]} \to \{0, 1\}$. Under the assumption in equation (44) of Proposition 2.30, the following hold.*

(a) *For each $n > 0$ and each $k \in K$, there exists $a \in A_n$ such that*

$$a \mid (y_F)_n - k.$$

(b) *For each $n > 0$ and each $a \in A_n$, there exists some $k \in K$ such that*

$$a \mid (y_F)_n - k.$$

If $a > 2m$, then this $k \in K$ is unique. Denote it by $\kappa_n(a)$.

(c) *Suppose n is so large that $\min A_n > 2m$, and denote by $\mathcal{G}_n$ the graph with vertices A_n and edges (a, a') whenever $\gcd(a, a') > 2m$. Then, $\kappa_n(a) = \kappa_n(a')$ for any two a, a' in the same connected component of $\mathcal{G}_n$. In particular, $|K| = 1$ if $\mathcal{G}_n$ is connected.*

(d) *If $|K| = 1$, say $K = \{k\}$, then, for each n, $(y_F)_n - k$ is a multiple of the minimal period $\tilde{\tau}_n$ of $\widetilde{\mathcal{H}}_n$.*

(e) *If $|K| = 1$ and $\tilde{\tau}_n = p_n$ for all n, then (X_η, σ) has a trivial centralizer.*

Proof. (a) Let $k \in K$. Then $V_k' + (y_F - \Delta(k))\mathbb{Z} \subseteq V_k' \subseteq \partial W$ by Proposition 2.30, and because of Lemma 2.5, $h_n + (y_F - \Delta(k))_n\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n \subseteq \mathcal{M}_{A_n}$ for each $h \in V_k'$. It follows that there exists $a' \in A_n$ such that $a' \mid \gcd(h_n, (y_F - \Delta(k))_n)$.

(b) Let $a \in A_n$. Notice that $a \notin \mathcal{M}_{S_n(a)}$. Otherwise, $a\mathbb{Z} \subseteq \mathcal{M}_{S_n(a)}$ which contradicts equation (AS). So, $a \in \widetilde{\mathcal{H}}_n$ and, by Lemma 2.5, there exists some $h \in U_n(\Delta(a)) \cap \partial W$. In particular, $h_n = a \in \widetilde{\mathcal{H}}_n$. Because of Corollary 2.14, there exists $k \in K$ such that $h \in V_k'$. As in the proof of part (a), it follows that there exists $a' \in A_n$ such that $a' \mid \gcd(h_n, (y_F - \Delta(k))_n)$. As a' and $a = h_n$ belong to the same primitive set A_n, this implies $a = a' \mid (y_F - \Delta(k))_n$.

Suppose there is another $k' \in K$ such that $a \mid (y_F - \Delta(k'))_n$. Then $a \mid k - k'$, so that $k = k'$ or $a \leqslant |k - k'| \leqslant 2m$.

(c) It suffices to prove that $\kappa_n(a) = \kappa_n(a')$ for every edge (a, a') of $\mathcal{G}_n$, i.e. whenever $\gcd(a, a') > 2m$. However, as part (b) implies

$$\gcd(a, a') \mid ((y_F)_n - \kappa_n(a)) - ((y_F)_n - \kappa_n(a')) = \kappa_n(a') - \kappa_n(a),$$

it follows that $\kappa_n(a) = \kappa_n(a')$ or $2m < \gcd(a, a') \leqslant |\kappa_n(a) - \kappa_n(a')| \leqslant 2m$.

(d) If $K = \{k\}$, then $\partial W = V_k'$, that is, $\partial W + (y_F - \Delta(k)) \subseteq \partial W$, and the claim follows from Lemma 2.6(b).

(e) It follows from part (d) that $y_F = \Delta(k)$ for some $k \in \mathbb{Z}$. □

3. *The $\mathcal{B}$-free case*

In this section, we will apply our results for general Toeplitz subshifts from §2 to minimal $\mathcal{B}$-free subshifts.

3.1. *Preparations.*

Let us start with the following notation and observations.

(i) For a finite subset $S \subset \mathcal{B}$, define as in [18]

$$\ell_S := \operatorname{lcm}(S) \quad \text{and} \quad \mathcal{A}_S := \{\gcd(b, \ell_S) : b \in \mathcal{B}\}.$$

As $\mathcal{B}$ is primitive, S is a proper subset of $\mathcal{A}_S$.

(ii) The set $\mathcal{B}$ is taut, if $\delta(\mathcal{M}_{\mathcal{B}\setminus\{b\}}) < \delta(\mathcal{M}_\mathcal{B})$ for each $b \in \mathcal{B}$.

So a set is primitive if removing any single point from it changes its set of multiples, and a set is taut if removing any single point from it changes the logarithmic density of its set of multiples.

(iii) Let $S \subseteq S' \subset \mathcal{B}$. From [18, equation (17)], we recall that

$$S \subseteq S' \subseteq \mathcal{A}_{S'} \subseteq \mathcal{M}_{\mathcal{A}_S} \quad \text{so that} \quad \mathcal{M}_S \subseteq \mathcal{M}_{S'} \subseteq \mathcal{M}_{\mathcal{A}_{S'}} \subseteq \mathcal{M}_{\mathcal{A}_S}. \tag{46}$$

(iv) A finite set $S \subset \mathcal{B}$ is *saturated* if $\mathcal{A}_S \cap \mathcal{B} = S$.

(v) For a finite set $S \subset \mathcal{B}$, define $S^{\text{sat}} = \mathcal{A}_S \cap \mathcal{B}$.

Then $S \subseteq S^{\text{sat}}$, S^{sat} is finite, $\operatorname{lcm}(S^{\text{sat}}) \mid \operatorname{lcm}(\mathcal{A}_S) = \operatorname{lcm}(S)$, so that $\operatorname{lcm}(S^{\text{sat}}) = \operatorname{lcm}(S)$, and $\mathcal{A}_{S^{\text{sat}}} = \mathcal{A}_S$, because $\gcd(b, \operatorname{lcm}(S^{\text{sat}})) = \gcd(b, \operatorname{lcm}(S))$ for each $b \in \mathcal{B}$. In particular, $\mathcal{A}_{S^{\text{sat}}} \cap \mathcal{B} = \mathcal{A}_S \cap \mathcal{B} = S^{\text{sat}}$, so that S^{sat} is saturated.

Any filtration $S_1 \subseteq S_2 \subseteq \cdots$ of $\mathcal{B}$ by finite sets yields a period structure $p_n = \operatorname{lcm}(S_n)$ for X_η. The definition of the group G depends on the period structure, but G is naturally isomorphic with the inverse limit $\varprojlim \mathbb{Z}/\operatorname{lcm}(S)\mathbb{Z}$ of the inverse system of cyclic groups $\mathbb{Z}/\operatorname{lcm}(S)\mathbb{Z}$ indexed by the finite subsets $S \subset \mathcal{B}$ ordered by the inclusion. Moreover, there is an injective group homomorphism $\varprojlim \mathbb{Z}/\operatorname{lcm}(S)\mathbb{Z} \to \prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$ given by $(n_S)_{S\subset\mathcal{B}} \mapsto (n_{\{b\}} + b\mathbb{Z})_{b\in\mathcal{B}}$. We can identify the group G with the image of this homomorphism, which consists of the elements $h = (h_b)_{b\in\mathcal{B}} \in \prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$ satisfying $h_b = h_{b'} \mod \gcd(b, b')$ for any $b, b' \in \mathcal{B}$. Under this identification, $\Delta : \mathbb{Z} \to \prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$ is given by $(\Delta(n))_b = n + b\mathbb{Z}$ for $b \in \mathcal{B}$ and $\overline{\Delta(\mathbb{Z})} \cong G$. Given a sequence $(n_S)_{S\subset\mathcal{B}}$ of integers belonging to the inverse limit (that is, satisfying $n_S \equiv n_{S'} \mod \operatorname{lcm}(S)$ whenever $S \subseteq S'$), we denote by $\lim \Delta(n_S)$ the element $h \in \overline{\Delta(\mathbb{Z})} \subseteq \prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$ such that $h_b = n_S \mod b$ for $b \in S \subset \mathcal{B}$.

Remark 3.1. The coding function $\phi : G \to \{0,1\}^\mathbb{Z}$ defined in the introduction can be written as $\phi(y) = \mathbf{1}_{\mathbb{Z}\setminus\bigcup_{b\in\mathcal{B}}(b\mathbb{Z}-y_b)}$ for any $y = (y_b)_{b\in\mathcal{B}} \in \overline{\Delta(\mathbb{Z})} \cong G$. It is injective. Indeed, for $y \in G$, denote $I_y := \{s \in \mathbb{Z} : (\phi(y))_s = 1\} = \{s \in \mathbb{Z} : y + \Delta(s) \in W\}$. As $\overline{\operatorname{int}(W)} = W \subseteq \overline{\Delta(\mathbb{Z})}$ (see the introduction), we have $W = \overline{\operatorname{int}(W)} \subseteq \overline{\{y + \Delta(s) : s \in I_y\}} \subseteq W$ for each $y \in G$. Hence, if $\phi(y) = \phi(y')$, then $I_y = I_{y'}$ and

$$W = \overline{\{y + \Delta(s) : s \in I_y\}} = \overline{\{y' + \Delta(s) : s \in I_{y'}\}} + (y - y') = W + (y - y').$$

However, W is aperiodic [18, Proposition 5.1], so $y = y'$.

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LEMMA 3.2. [18, Lemma 2.5] *Let $U = U_S(\Delta(n))$ for some $S \subset \mathcal{B}$ and $n \in \mathbb{Z}$.*

(a) *If $n \in \mathcal{M}_S$, then $U \cap W = \emptyset$.*

(b) *If $U \cap W = \emptyset$, then $n + \mathrm{lcm}(S) \cdot \mathbb{Z} \subseteq \mathcal{M}_{\mathcal{B}\cap\mathcal{A}_S}$.*

(c) *If S is saturated, then $n \in \mathcal{M}_S$ if and only if $U \cap W = \emptyset$ if and only if $n + lcm(S) \cdot \mathbb{Z} \subseteq \mathcal{M}_S$.*

LEMMA 3.3. [18, Lemma 3.1]

(a) *For all $S \subset \mathcal{B}$ and $n \in \mathbb{Z}$ we have: $U_S(\Delta(n)) \subseteq W \Leftrightarrow n \in \mathcal{F}_{\mathcal{A}_S}$.*

(b) *If $(S_k)_k$ is a filtration of $\mathcal{B}$ by finite sets and $\lim_k \Delta(n_{S_k}) = h$, then $h \in \mathrm{int}(W)$ if and only if $n_{S_k} \in \mathcal{F}_{\mathcal{A}_{S_k}}$ for some k.*

LEMMA 3.4. [18, Lemma 5.2] *Assume that $S \subseteq S'$ are finite subsets of $\mathcal{B}$, then $\mathcal{A}_S = \{\gcd(a, \mathrm{lcm}(S)) \ : a \in \mathcal{A}_{S'}\}$.*

PROPOSITION 3.5. [18, Theorem B] *The following are equivalent.*

(a) *W is topologically regular, that is $W = \overline{\mathrm{int}(W)}$.*

(b) *There are no $d \in \mathbb{N}$ and no infinite pairwise coprime set $\mathcal{A} \subset \mathbb{N} \setminus \{1\}$ such that $d\mathcal{A} \subset \mathcal{B}$.*

(c) *$\eta = \phi(0)$ is a Toeplitz sequence different from $(\ldots, 0, 0, 0, \ldots)$.*

(d) *$\{n \in \mathbb{N} \ : \forall_{S\subset\mathcal{B}}\, \exists_{S'\subset\mathcal{B}} \ : S \subseteq S'$ and $n \in \mathcal{A}_{S'} \setminus S'\} = \emptyset$.*

LEMMA 3.6. *Each filtration $S_1 \subset S_2 \subset \cdots \nearrow \mathcal{B}$ has a sub-filtration of sets S_{n_k} such that $S^{\mathrm{sat}}_{n_1} \subset S^{\mathrm{sat}}_{n_2} \subset \cdots \nearrow \mathcal{B}$ is a filtration.*

Proof. Let $n_1 = 1$. If $n_1 < n_2 < \cdots < n_k$ are chosen, let $n_{k+1} = \min\{j \in \mathbb{N} : S^{\mathrm{sat}}_{n_k} \subseteq S_j\}$. □

This lemma allows us in the following to assume that a filtration is saturated without loosing generality.

3.2. *Sets of holes.*

Now we describe the set of holes and the set of essential holes.

PROPOSITION 3.7. *Let $S \subset \mathcal{B}$ be saturated and taut and $s \in \mathbb{Z}$. Then:*

(a) *$s \in \mathcal{M}_S \Leftrightarrow s + \ell_S\mathbb{Z} \subseteq \mathcal{M}_S \Leftrightarrow s + \ell_S\mathbb{Z} \subseteq \mathcal{M}_{\mathcal{B}}$;*

(b) *$s \in \mathcal{F}_{\mathcal{A}_S} \Leftrightarrow s + \ell_S\mathbb{Z} \subseteq \mathcal{F}_{\mathcal{A}_S} \Leftrightarrow s + \ell_S\mathbb{Z} \subseteq \mathcal{F}_{\mathcal{B}}$.*

In particular, $s \in \mathbb{Z}$ is not ℓ_S-periodic if and only if $s \in \mathcal{M}_{\mathcal{A}_S} \setminus \mathcal{M}_S$. Hence, $\mathcal{M}_{\mathcal{A}_{S_n}} \setminus \mathcal{M}_{S_n}$ is the set of all holes (with respect to the period structure given by $p_n = \mathrm{lcm}(S_n)$) in η on level n. (Observe also that if $\mathcal{B}$ is primitive and η is a Toeplitz sequence, then $\mathcal{B}$ is taut, see [18, Lemma 3.7].*)*

Proof. (i) Let $s \in \mathcal{M}_S$. Then $b \mid s$ for some $b \in S$. Since $b \mid \ell_S$, $s + \ell_S\mathbb{Z} \subseteq b\mathbb{Z} \subseteq \mathcal{M}_S$.

(ii) Let $s + \ell_S\mathbb{Z} \subseteq \mathcal{M}_{\mathcal{B}}$. By [12, Proposition 4.31], the tautness of $\mathcal{B}$ implies $b \mid \gcd(s, \ell_S)$ for some $b \in \mathcal{B}$. Since S is saturated, $b \in S$. So $s \in \mathcal{M}_S$.

(iii) Let $s \in \mathcal{F}_{\mathcal{A}_S}$. Assume for a contradiction that $\gcd(b, \ell_S) \mid s + \ell_S k$ for some $k \in \mathbb{Z}$ and some $b \in \mathcal{B}$. Since $\gcd(b, \ell_S) \mid \ell_S$, $\gcd(b, \ell_S) \mid s$, which contradicts $s \in \mathcal{F}_{\mathcal{A}_S}$. So $s + \ell_S\mathbb{Z} \subseteq \mathcal{F}_{\mathcal{A}_S}$.

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(iv) Let $s + \ell_S\mathbb{Z} \subseteq \mathcal{F}_\mathcal{B}$. Assume for a contradiction that $\gcd(b, \ell_S) \mid s$ for some $b \in \mathcal{B}$. Then there is $x \in \mathbb{Z}$ such that $x \equiv 0 \bmod b$ and $x \equiv s \bmod \ell_S$, i.e. $x \in b\mathbb{Z} \cap (s + \ell_S\mathbb{Z})$, in contradiction to $s + \ell_S\mathbb{Z} \subseteq \mathcal{F}_\mathcal{B}$. Hence, $s \in \mathcal{F}_{\mathcal{A}_S}$.

Now part (a) follows from items (i) and (ii), while part (b) follows from items (iii) and (iv). □

To describe the set of essential holes, we extract special elements of $\mathcal{A}_S$.

Definition 3.8. Let $S \subset \mathcal{B}$.

(a) An element $b \in \mathcal{B} \setminus S$ is a *source* of an element $a \in \mathcal{A}_S$ if $a = \gcd(b, \operatorname{lcm}(S))$.

(b) $\mathcal{A}_S^\infty := \{a \in \mathcal{A}_S : a \text{ has infinitely many sources}\}$ and $\mathcal{A}_S^{\infty,p} := (\mathcal{A}_S^\infty)^{\text{prim}}$.

We will use some basic properties of $\mathcal{A}_S^\infty$.

LEMMA 3.9.

(a) $\mathcal{A}_S^{\infty,p} \subseteq \mathcal{A}_S^\infty \subseteq \mathcal{A}_S \setminus \mathcal{M}_S$.

(b) *Let $S \subset S' \subset \mathcal{B}$ and $a \in \mathcal{A}_S^\infty$. There exists at least one $a' \in \mathcal{A}_{S'}^\infty$ such that $a = \gcd(a', \ell_S)$.*

(c) *Let $S \subset S' \subset \mathcal{B}$ and $a' \in \mathcal{A}_{S'}^\infty$. Then, $\gcd(\ell_S, a') \in \mathcal{A}_S^\infty$.*

(d) *Let $S \subset S' \subset \mathcal{B}$ and $a \in \mathcal{A}_S^{\infty,p}$. There exists at least one $a' \in \mathcal{A}_{S'}^{\infty,p}$ such that $a = \gcd(a', \ell_S)$. In particular, $|\mathcal{A}_{S'}^{\infty,p}| \geqslant |\mathcal{A}_S^{\infty,p}|$.*

(e) *Let $S \subset S' \subset \mathcal{B}$ with $|\mathcal{A}_{S'}^{\infty,p}| = |\mathcal{A}_S^{\infty,p}|$ and $a' \in \mathcal{A}_{S'}^{\infty,p}$. Then, $\gcd(a', \ell_S) \in \mathcal{A}_S^{\infty,p}$. (Without the extra assumption, this need not hold, see Example 3.29.)*

Proof. (a) Let $a \in \mathcal{A}_S^\infty$. If $a = \gcd(b, \ell_S) \in \mathcal{M}_S$ for some $b \in \mathcal{B}$, then there is $b' \in S$ such that $b' \mid a \mid b$, and the primitivity of $\mathcal{B}$ implies $b = a = b' \in S$. This contradicts to a having infinitely many sources.

(b) Let $a \in \mathcal{A}_S^\infty$. There are infinitely many $b \in \mathcal{B} \setminus S'$ such that $a = \gcd(b, \ell_S)$. Let $a'_b := \gcd(b, \ell_{S'})$ for these b. Then, $\gcd(a'_b, \ell_S) = \gcd(b, \ell_S) = a$ for all these b, and as $\mathcal{A}_{S'}$ is finite, there exists some $a' \in \mathcal{A}_{S'}$ such that $a' = a'_b$ for infinitely many of them. Hence, $a' \in \mathcal{A}_{S'}^\infty$.

(c) Clear.

(d) In the situation of item (b), suppose that $a \in \mathcal{A}_S^{\infty,p}$ and that there is $a'_0 \in \mathcal{A}_{S'}^{\infty,p}$ such that $a'_0 \mid a'$. Then, $a_0 := \gcd(a'_0, \ell_S) \in \mathcal{A}_S^\infty$ and $a_0 \mid \gcd(a', \ell_S) = a$. As $a \in \mathcal{A}_S^{\infty,p}$, this implies $a_0 = a$, so that $\gcd(a'_0, \ell_S) = a$.

(e) This follows from item (d). □

LEMMA 3.10. *Suppose that $S_1 \subseteq S_2 \subseteq \cdots \nearrow \mathcal{B}$ is a filtration by saturated sets. Then, for each $N \in \mathbb{N}$,*

$$r \in \mathbb{Z}\,,\ U_{S_N}(\Delta(r)) \cap \partial W \neq \varnothing \Rightarrow r \in \mathcal{M}_{\mathcal{A}_{S_N}^\infty} \setminus \mathcal{M}_{S_N} \tag{47}$$

and, for all $r \in \mathbb{Z}$,

$$U_{S_N}(\Delta(r)) \cap \partial W \neq \varnothing \Leftrightarrow \text{for all } n \geqslant N : (r + \ell_{S_N}\mathbb{Z}) \cap (\mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n}) \neq \varnothing. \tag{48}$$

More precisely, if $h \in U_{S_N}(\Delta(r)) \cap \partial W$, then $h_{S_n} \in (r + \ell_{S_N}\mathbb{Z}) \cap (\mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n})$ for all $n \geqslant N$.

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Proof. Suppose there exists some $h \in U_{S_N}(\Delta(r)) \cap \partial W$. Then, $h_{S_n} \in (r + \ell_{S_N}\mathbb{Z}) \cap (\mathcal{M}_{\mathcal{A}_{S_n}} \setminus \mathcal{M}_{S_n})$ for all $n \geqslant N$, see Lemma 2.1 and Proposition 3.7. Hence, $r \notin \mathcal{M}_{S_N}$, and there are numbers $k_n \in \mathbb{Z}$ and $b_n \in \mathcal{B} \setminus S_n (n \geqslant N)$ such that $\gcd(b_n, \ell_{S_N}) \mid \gcd(b_n, \ell_{S_n}) \mid h_{S_n} = r + k_n \ell_{S_N}$. In particular, $\gcd(b_n, \ell_{S_N}) \mid r$ for all $n \geqslant N$, and as $\mathcal{A}_{S_N}$ is a finite set, there exist $a \in \mathcal{A}_{S_N}$ and infinitely many b_{n_i} such that $\gcd(b_{n_i}, \ell_{S_N}) = a$. It follows that $a \in \mathcal{A}_{S_N}^\infty$ and $a \mid r$. This proves equation (47).

Observe that, trivially, $h \in U_{S_n}(\Delta(h_{S_n})) \cap \partial W$ for each $n \geqslant N$. Hence, we can apply equation (47) to n and h_{S_n} instead of N and r, respectively. It follows that $h_{S_n} \in \mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n}$. As $h_{S_n} = r + k_n \ell_{S_N}$, this proves the '$\Rightarrow$'-direction of equation (48) and also the final claim.

The '$\Leftarrow$'-implication of equation (48) follows from Lemma 2.5 and Proposition 3.7, because $\mathcal{A}_{S_n}^\infty \subseteq \mathcal{A}_{S_n}$ for all $n \in \mathbb{N}$. □

COROLLARY 3.11. *Suppose that $S_1 \subseteq S_2 \subseteq \cdots \nearrow \mathcal{B}$ is a filtration by saturated sets. Then, for each $n \in \mathbb{N}$,*

$$m_G(\partial W) \leq d(\mathcal{M}_{\mathcal{A}_{S_n}^{\infty,p}}) \leq 1 - \prod_{a \in \mathcal{A}_{S_n}^{\infty,p}} \left(1 - \frac{1}{a}\right).$$

Proof. For $n \in \mathbb{N}$, denote by $\mathcal{U}_n$ the family of all sets $U_{S_n}(\Delta(r))$ that have non-empty intersection with ∂W and by $\bigcup \mathcal{U}_n$ the union of these sets. Then, equation (47) implies $m_G(\partial W) \leq m_G(\bigcup \mathcal{U}_n) = |\mathcal{U}_n|/\ell_{S_n} \leq d(\mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n}) \leq d(\mathcal{M}_{\mathcal{A}_{S_n}^{\infty,p}})$ for all n, and the second inequality is the Heilbronn–Rohrbach inequality [**15**, Theorem 0.9]. □

Remark 3.12. Fix the period structure given by $p_n = \operatorname{lcm}(S_n)$. Then,

$$\widetilde{\mathcal{H}}_n \subseteq \mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n} \subseteq \mathcal{H}_n = \mathcal{M}_{\mathcal{A}_{S_n}} \setminus \mathcal{M}_{S_n}$$

by Proposition 3.7, Lemma 2.5(a) and equation (47) of Lemma 3.10. Moreover, equation (48) of Lemma 3.10 shows that $r \in \widetilde{\mathcal{H}}_N$ if and only if for all $n \geqslant N : (r + \ell_{S_N}\mathbb{Z}) \cap (\mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n}) \neq \varnothing$. This characterization is also the starting point for verifying the structural assumption in equation (AS) on $\widetilde{\mathcal{H}}_n$ from §2.5, see Proposition 3.16 and Theorem 3.17 below.

Remark 3.13. In the general Toeplitz case, one can easily construct a Toeplitz sequence for which not all holes are essential. We construct a $\mathcal{B}$-free Toeplitz subshift with this property in Example 3.27. Moreover, the property $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$ may depend on the choice of the period structure, as we show in Example 3.28, and a $\mathcal{B}$-free Toeplitz subshift for which $\widetilde{\mathcal{H}}_n \subsetneq \mathcal{M}_{\mathcal{A}_{S_n}^\infty} \setminus \mathcal{M}_{S_n}$ is provided in Example 3.29.

In the rest of this subsection, we show that the structural assumption in equation (AS) of Propositions 2.28 and 2.30 and of Theorem 2.31 is satisfied in the $\mathcal{B}$-free setting.

Definition 3.14. Let $(S_n)_n$ be a filtration of $\mathcal{B}$ by finite sets. An integer sequence $(a_n)_{n \geq N}$ is called an $(a, \mathcal{A})$-*sequence*, if $a_N = a$ and if $a_n \in \mathcal{A}_{S_n} \backslash S_n$ and $\gcd(a_{n+1}, \ell_{S_n}) = a_n$ for all $n \geq N$.

Remark 3.15. If $(a_n)_{n\geq N}$ is an $(a, \mathcal{A})$-sequence, then $a_n \in \mathcal{A}^{\infty}_{S_n}$ for all $n \geq N$. Indeed, suppose that a_m has only finitely many sources $b_1, \ldots, b_k$ for some $m \geq N$. Consider $n \geq m$ such that $b_1, \ldots, b_k \in S_n$. Since $a_n \notin S_n$, there exists $b \in \mathcal{B} \setminus S_n$ such that $a_n = \gcd(b, \ell_{S_n})$. Then $a_m = \gcd(a_n, \ell_{S_m}) = \gcd(b, \ell_{S_m})$. So b is a source of a_m different from $b_1, \ldots, b_k$. This yields a contradiction. Note also that by Lemma 3.9(b), for every $a \in \mathcal{A}^{\infty}_{S_N}$ there exists an $(a, \mathcal{A})$-sequence.

PROPOSITION 3.16. *Let $(S_n)_n$ be a filtration of $\mathcal{B}$ by finite sets. Then, for all $N \in \mathbb{N}$, the set $\widetilde{\mathcal{H}}_N$ is the union of sets $a\mathbb{Z} \setminus \mathcal{M}_{S_N((a_n)_{n\geq N})}$, where the union extends over all $(a, \mathcal{A})$-sequences $(a_n)_{n\geq N}$ and where*

$$S_N((a_n)_{n\geq N}) = \{\gcd(b, \ell_{S_N}) : b \in \mathcal{B} \text{ and } b \mid a_n \vee \ell_{S_N} \text{ for some } n \geq N\}. \tag{49}$$

Proof. Notice that in view of Remark 3.12,

$$r \in \widetilde{\mathcal{H}}_N \Leftrightarrow \text{for all } n \geqslant N : (r + \ell_{S_N}\mathbb{Z}) \cap 1(\mathcal{M}_{\mathcal{A}^{\infty}_{S_n}} \setminus \mathcal{M}_{S_n}) \neq \varnothing.$$

Hence,

$$\begin{aligned}
r \notin \widetilde{\mathcal{H}}_N &\Leftrightarrow \text{there exists } n \geq N : (r + \ell_{S_N}\mathbb{Z}) \cap (\mathcal{M}_{\mathcal{A}^{\infty}_{S_n}} \setminus \mathcal{M}_{S_n}) = \varnothing \\
&\Leftrightarrow \text{there exists } n \geq N \ \text{ for all } a_n \in \mathcal{A}^{\infty}_{S_n} : (r + \ell_{S_N}\mathbb{Z}) \cap a_n\mathbb{Z} \subseteq \mathcal{M}_{S_n} \\
&\Leftrightarrow (\text{there exists } n \geq N \ \text{ for all } a_n \in \mathcal{A}^{\infty}_{S_n} : (r + \ell_{S_N}\mathbb{Z}) \cap a_n\mathbb{Z} \neq \varnothing \\
&\qquad \Rightarrow \text{there exists } b \in S_n : (r + \ell_{S_N}\mathbb{Z}) \cap a_n\mathbb{Z} \subseteq b\mathbb{Z}) \\
&\Leftrightarrow (\text{there exists } n \geq N \ \text{ for all } a_n \in \mathcal{A}^{\infty}_{S_n} : \gcd(a_n, \ell_{S_N}) \mid r \\
&\qquad \Rightarrow \text{there exists } b \in S_n : b \mid a_n \vee \ell_{S_N} \text{ and } \gcd(b, \ell_{S_N}) \mid r).
\end{aligned}$$

The third equivalence follows from Lemma 2.24 and the last one from Lemma 2.25. Now, the first of the following two equivalences is immediate:

$$\begin{aligned}
&r \in \widetilde{\mathcal{H}}_N \\
&\Leftrightarrow \text{for all } n \geq N \quad \text{there exists } a_n \in \mathcal{A}^{\infty}_{S_n} : \gcd(a_n, \ell_{S_N}) \mid r \quad \text{and} \\
&\qquad [\text{for all } b \in S_n : b \mid a_n \vee \ell_{S_N} \Rightarrow \gcd(b, \ell_{S_N}) \nmid r] \\
&\Leftrightarrow \text{there exists } a \in \mathcal{A}^{\infty}_{S_N} \quad \text{there exists an } (a, \mathcal{A})\text{-sequence } (a_n)_{n\geq N} : a \mid r \quad \text{and} \\
&\qquad \text{for all } b \in \mathcal{B} \quad \text{for all } n \geq N : b \mid a_n \vee \ell_{S_N} \Rightarrow \gcd(b, \ell_{S_N}) \nmid r.
\end{aligned} \tag{50}$$

The '$\Leftarrow$'-direction of the second equivalence is obvious—just a matter of notation. For the '$\Rightarrow$'-direction, we construct a suitable $(a, \mathcal{A})$-sequence $(a'_m)_{m\geq N}$ from the given numbers a_n: there is $a \in \mathcal{A}^{\infty}_{S_N}$ such that $a = \gcd(a_n, \ell_{S_N})$ for infinitely many indices n. Obviously $a \mid r$, and we choose $a'_N = a$. Suppose inductively that suitable $a'_N, \ldots, a'_m$ are constructed in such a way that $a'_m = \gcd(a_n, \ell_{S_m})$ for infinitely many different indices $n \geq m$. Then there is an increasing subsequence $(a_{n_i})_i$ such that $\gcd(a_{n_i}, \ell_{S_{m+1}})$ is the same value for all $n_i \geq m + 1$. This common value is denoted by a'_{m+1}. It satisfies $\gcd(a'_{m+1}, \ell_{S_m}) = \gcd(a_{n_i}, \ell_{S_{m+1}}, \ell_{S_m}) = \gcd(a_{n_i}, \ell_{S_m}) = a'_m$ for all n_i. Suppose now that $b \in \mathcal{B}$, $n \geq N$ and $b \mid a_n \vee \ell_{S_N}$. Fix $n' \geq n$ such that $b \in S_{n'}$. Then $b \mid a_{n'} \vee \ell_{S_n}$, because $a_n \mid a_{n'}$, and we conclude that $\gcd(b, \ell_{S_N}) \nmid r$. The claim follows. □

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THEOREM 3.17. *Let $(S_n)_n$ be a filtration of $\mathcal{B}$ by finite sets and let $N > 0$. For each $a \in \mathcal{A}_{S_N}^{\infty}$, there exists a finite primitive set $S_N(a)$ of positive integers such that*

$$\widetilde{\mathcal{H}}_N = \bigcup_{a \in \mathcal{A}_{S_N}^{\infty}} a\mathbb{Z} \setminus \mathcal{M}_{S_N(a)} \tag{51}$$

and all sets $a\mathbb{Z} \setminus \mathcal{M}_{S_N(a)}$ are non-empty. (An explicit construction of the sets $S_N(a)$ is given in the proof.) In particular, the assumption in equation (AS) of Theorem 2.31 is satisfied.

Proof. Consider any fixed $a \in \mathcal{A}_{S_N}^{\infty}$. In view of Proposition 3.16, the set $S_N(a)$ must be constructed in such a way that $\mathcal{M}_{S_N(a)} = \bigcap \mathcal{M}_{S_N((a_n)_{n \geq N})}$, where the intersection runs over all $(a, \mathcal{A})$-sequences with $a_N = a$. As all sets $S_N((a_n)_{n \geq N})$ consist of divisors of ℓ_{S_N}, this set is only a finite intersection, say of r sets of multiples $\mathcal{M}_{R_1}, \ldots, \mathcal{M}_{R_r}$. Hence, we may choose as $S_N(a)$ the primitivization of the set of all $c_1 \vee \cdots \vee c_r$, where $c_i \in R_i$ for $i = 1, \ldots, r$.

Suppose for a contradiction that $a\mathbb{Z} \subseteq \mathcal{M}_{S_N(a)}$. Then, $a\mathbb{Z} \subseteq \mathcal{M}_{S_N((a_n)_{n \geq N})}$ for each $(a, \mathcal{A})$-sequence $(a_n)_{n \geq N}$ with $a_N = a$. Hence, for each such sequence, there is $b \in \mathcal{B}$ such that $\gcd(b, \ell_{S_N}) \mid a = \gcd(a_n, \ell_{S_N})$ and $b \mid a_n \vee \ell_{S_N}$ for some $n \geq N$. It follows that $b^{\div \ell_{S_N}} \mid (a_n \vee \ell_{S_N})^{\div \ell_{S_N}} = a_n^{\div \ell_{S_N}}$, so that $b = b^{\div \ell_{S_N}} \cdot \gcd(b, \ell_{S_N}) \mid a_n^{\div \ell_{S_N}} \cdot \gcd(a_n, \ell_{S_N}) = a_n$. However, this is impossible, because $a_n \in \mathcal{A}_{S_n}^{\infty}$ and $\mathcal{B}$ is primitive. □

Remark 3.18.

(a) Each set $S_N((a_n)_{n \geq N})$ contains the set S_N. Hence, each of the sets $\mathcal{M}_{S_N(a)}$ contains $\mathcal{M}_{S_N}$.

(b) If $\sup_N |\mathcal{A}_{S_N}^{\infty}| < \infty$, then, for sufficiently large N, there is at most one $(a, \mathcal{A})$-sequence $(a_n)_{n \geq N}$ for each $a \in \mathcal{A}_{S_N}^{\infty}$. Hence, $\mathcal{M}_{S_N(a)} = \mathcal{M}_{S_N((a_n)_{n \geq N})}$ for each such a.

Under a special assumption (which is satisfied in all our examples except Example 3.29), we have a simplified description of the sets $\widetilde{\mathcal{H}}_n$.

LEMMA 3.19. *Assume that (S_n) is a filtration of $\mathcal{B}$ by finite saturated sets. Let $N \in \mathbb{N}$. The following conditions are equivalent:*

(a) *$\ell_{S_N} \vee a' \in \mathcal{F}_{\mathcal{B} \setminus S_N}$ for every $n > N$ and $a' \in \mathcal{A}_{S_n}^{\infty}$;*

(b) *$S_N((a_n)_{n \geq N}) = S_N$ for every $a \in \mathcal{A}_{S_N}^{\infty}$ and for all $(a, \mathcal{A})$-sequences $(a_n)_{n \geq N}$.*

If this the case, $\widetilde{\mathcal{H}}_N = \mathcal{M}_{\mathcal{A}_{S_N}^{\infty}} \setminus \mathcal{M}_{S_N}$.

Proof. Assume condition (a) and let $a \in \mathcal{A}_{S_N}^{\infty}$. Let $\gcd(b, \ell_{S_N}) \in S_N((a_n)_{n \geq N})$ for some $b \in \mathcal{B}$ and some $(a, \mathcal{A})$-sequence $(a_n)_{n \geq N}$. Then, $b \mid a_n \vee \ell_{S_N}$ for some $n \geq N$ and, as $a_n \in \mathcal{A}_{S_n}^{\infty}$ by Remark 3.15, condition (a) applied to $a' = a_n$ yields $b \in S_N$.

Conversely, assume that $b \mid \ell_{S_N} \vee a'$ for some $b \in \mathcal{B} \setminus S_N$ and $a' \in \mathcal{A}_{S_{n_0}}^{\infty}$, where $n_0 > N$. There exists an $(a', \mathcal{A})$-sequence $(a_n)_{n \geq n_0}$ (see Remark 3.15). Set $a_n = \gcd(a_{n_0}, \ell_{S_n})$ for $N \leq n \leq n_0$. Then, $(a_n)_{n \geq N}$ is an $(a_N, \mathcal{A})$-sequence such that $a_{n_0} = a'$. Moreover,

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$\gcd(\ell_{S_N}, b) \in S((a_n)_{n\geq N})$ and $\gcd(\ell_{S_N}, b) \notin S_N$ as S_N is saturated. It follows that $S_N((a_n)_{n\geq N}) \neq S_N$.

The remaining assertion follows by Proposition 3.16 and Remark 3.15. □

3.3. *Trivial centralizer.* We start with an example for which the simple Proposition 2.20 guarantees a trivial centralizer.

Example 3.20. Let $\mathcal{B} = \{2^n c_n : n > 0\}$, where $(c_n)_n$ is a pairwise coprime sequence of odd integers. We will show that the corresponding $\mathcal{B}$-free system has a trivial centralizer. Let $S_n = \{2^k c_k : 0 < k \leq n\}$. The sets S_n form a filtration by finite sets. Then, $\ell_{S_n} = 2^n \prod_{i=1}^n c_i$ and $\mathcal{A}_{S_n}^{\infty,p} = \mathcal{A}_{S_n} \setminus S_n = \{2^n\}$. Notice that the property in equation (TI) is satisfied for $A_n = \mathcal{A}_{S_n}^{\infty}$. Moreover, for each $N > 0$, there is only one $(a, \mathcal{A})$-sequence with $a \in \mathcal{A}_{S_N}^{\infty,p}$, namely the sequence $(2^n)_{n\geq N}$, and $S_N((2^n)_{n\geq N}) = S_N$ according to equation (49). Hence, $\widetilde{\mathcal{H}}_N = 2^N \setminus \mathcal{M}_{S_N} = \mathcal{H}_N$. So $\tilde{\tau}_n = \ell_{S_n}$. By Proposition 2.20, the centralizer is trivial, as shown previously in [10].

Next, we apply Theorem 2.31 to examples which violate properties (TI) and (Seh), and hence also condition (D).

Example 3.21. Let $\mathcal{B} = \{2^n c_n, 3^n d_n : n > 0\}$, where $(c_n)_n$ and $(d_n)_n$ are two sequences of integers coprime to 2 and 3, and such that the sequence $(c_n \vee d_n)_{n>0}$ is pairwise coprime. We will show that the corresponding $\mathcal{B}$-free system has a trivial centralizer. Let $S_n = \{2^k c_k, 3^k d_k : 0 < k \leq n\}$. The sets S_n form a filtration by finite sets. Notice that $\ell_{S_n} = 6^n \prod_{i=1}^n (c_i \vee d_i)$ and $\mathcal{A}_{S_n}^{\infty,p} = \mathcal{A}_{S_n} \setminus S_n = \{2^n, 3^n\}$. In particular, $\langle \mathcal{A}_{S_n}^{\infty} \rangle = \mathbb{Z}$ for all n, so that property (TI) is violated. Below we show that also condition (D) is violated, while Proposition 2.30 shows that conditions (D′) and (DD′) are satisfied.

We claim $\widetilde{\mathcal{H}}_n = \mathcal{H}_n = \mathcal{M}_{\mathcal{A}_{S_n}^{\infty,p}} \setminus \mathcal{M}_{S_n}$. Indeed, suppose that $b \mid s^n \vee \ell_{S_N}$ for some $b \in \mathcal{B}$ and some $s \in \{2, 3\}$. Since $c_j \nmid s^n \vee \ell_{S_N}$ and $d_j \nmid s^n \vee \ell_{S_N}$ for any $j > N$, we have $b \in S_N$. So condition (a) from Lemma 3.19 holds, and the claim follows observing also Remark 3.12.

In view of Theorem 3.17, the condition in equation (AS) is satisfied, so that Theorem 2.31 implies $|K| = 1$ or $|K| = 2$. Below we will rule out the second possibility.

(i) If $K = \{k\}$, then $\tilde{\tau}_n \mid (y_F)_n - k$ for all $n > 0$, where $\tilde{\tau}_n$ is the minimal period of $\widetilde{\mathcal{H}}_n$, in this case, the minimal period of $\mathcal{M}_{\mathcal{A}_{S_n}^{\infty}} \setminus \mathcal{M}_{S_n}$. Since $S_n^{\div 2^n} = \{c_k, 3^k d_k : 0 < k \leq n\}$ and $S_n^{\div 3^n} = \{2^k c_k, d_k : 0 < k \leq n\}$ are primitive, Lemma 2.26(b) applies. So $\tilde{\tau}_n = \operatorname{lcm}(\mathcal{A}_{S_n}^{\infty} \cup S_n) = \operatorname{lcm}(S_n)$, and the triviality of the centralizer follows from Theorem 2.31.
(ii) Suppose for a contradiction that $|K| = 2$, say $K = \{k_2 = \kappa_n(2^n), k_3 = \kappa_n(3^n)\}$. Then by Theorem 2.31(b), $2^n \mid (y_F)_n - k_2$ and $3^n \mid (y_F)_n - k_3$ for all $n > 0$. For $s \in \{2, 3\}$, the sets $\widetilde{\mathcal{H}}_n^{k_s}$ (defined in Remark 2.17(a)) and $s^n\mathbb{Z} \cap \widetilde{\mathcal{H}}_n^{k_s}$ are invariant under translation by $\gcd((y_F)_n - k_s, p_n)$, see Remark 2.17(c), because $s^n \mid p_n$ and $s^n \mid (y_F)_n - k_s$. Hence, the same is true for the set $\widetilde{\mathcal{H}}_n^{k_s} \setminus s^n\mathbb{Z}$.

We claim that $\widetilde{\mathcal{H}}_n^{k_s} \subseteq s^n\mathbb{Z} \cap \widetilde{\mathcal{H}}_n$ for $s = 2, 3$. Indeed, if this is not the case, then for at least one $s \in \{2, 3\}$ and $\bar{s} = 5 - s$,

$$\varnothing \neq \widetilde{\mathcal{H}}_n^{k_s} \setminus s^n\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n \setminus s^n\mathbb{Z} \subseteq (2^n\mathbb{Z} \cup 3^n\mathbb{Z}) \setminus s^n\mathbb{Z} \subseteq \bar{s}^n\mathbb{Z},$$

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so that $\gcd((y_F)_n - k_s, p_n)$ must be a multiple of $\bar{s}^n$. It follows that $\bar{s}^n \mid (y_F)_n - k_s$. Since we observed above that $\bar{s}^n \mid (y_F)_n - k_{\bar{s}}$, we see that $\bar{s}^n \mid k_s - k_{\bar{s}} = \pm(k_2 - k_3)$ for all $n > 0$, which implies $k_2 = k_3$ in contradiction to $|K| = 2$.

Moreover, as $\widetilde{\mathcal{H}}_n = \widetilde{\mathcal{H}}_n^{k_s} \cup \widetilde{\mathcal{H}}_n^{k_{\bar{s}}}$,

$$\widetilde{\mathcal{H}}_n \setminus \bar{s}^n\mathbb{Z} \subseteq \widetilde{\mathcal{H}}_n^{k_s} \subseteq s^n\mathbb{Z} \cap \widetilde{\mathcal{H}}_n \quad \text{for } s = 2, 3.$$

Since we proved above that $\widetilde{\mathcal{H}}_n = \mathcal{M}_{\mathcal{A}_{S_n}^{\infty,p}} \setminus \mathcal{M}_{S_n} = \mathcal{M}_{\{2^n,3^n\}} \setminus \mathcal{M}_{S_n}$, this implies

$$s^n\mathbb{Z} \setminus \mathcal{M}_{S_n \cup \{\bar{s}^n\}} \subseteq \widetilde{\mathcal{H}}_n^{k_s} \subseteq s^n\mathbb{Z} \setminus \mathcal{M}_{S_n} \quad \text{for } s = 2, 3,$$

equivalently,

$$s^n\mathbb{Z} \cap \mathcal{M}_{S_n} \subseteq s^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_s} \subseteq s^n\mathbb{Z} \cap \mathcal{M}_{S_n \cup \{\bar{s}^n\}} \quad \text{for } s = 2, 3. \tag{52}$$

Denote the minimal period of $\widetilde{\mathcal{H}}_n^{k_s}$ by $\tilde{\tau}^{(s)}$. Our goal is to prove that $\gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)}) > 2m$, because the fact that $\gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)})$ divides $\gcd((y_F)_n - k_2, (y_F)_n - k_3)$ and hence also $k_3 - k_2$ then shows that $k_2 = k_3$, which is the desired contradiction. (Recall from Proposition 2.28 that $K \subseteq [-m, m]$.)

Observe first that $s^n \mid \tilde{\tau}^{(s)}$ and $t + \tilde{\tau}^{(s)}\mathbb{Z} \subseteq s^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_s} \subseteq s^n\mathbb{Z} \cap \mathcal{M}_{S_n \cup \{\bar{s}^n\}}$ for each $t \in s^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_s}$. Hence, there is $b \in S_n \cup \{\bar{s}^n\}$ such that $s^n \vee b \mid \gcd(t, \tilde{\tau}^{(s)})$.

(1) If the first inclusion in equation (52) is strict, there exists $t \in (s^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_s}) \setminus \mathcal{M}_{S_n}$. Hence, $s^n \vee \bar{s}^n \mid \gcd(t, \tilde{\tau}^{(s)})$.

(2) The same arguments apply when the roles of s and $\bar{s}$ are interchanged.

Therefore, if $2^n\mathbb{Z} \cap \mathcal{M}_{S_n} \subsetneq 2^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_2}$ or $3^n\mathbb{Z} \cap \mathcal{M}_{S_n} \subsetneq 3^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_3}$, then $3^n \mid \gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)})$ or $2^n \mid \gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)})$, respectively, and we are done.

It remains to treat the case where $2^n\mathbb{Z} \cap \mathcal{M}_{S_n} = 2^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_2}$ and $3^n\mathbb{Z} \cap \mathcal{M}_{S_n} = 3^n\mathbb{Z} \setminus \widetilde{\mathcal{H}}_n^{k_3}$. In this case, $\tilde{\tau}^{(s)}$ is the smallest period of $s^n\mathbb{Z} \setminus \mathcal{M}_{S_n}$, so $\tilde{\tau}^{(s)} = s^n \cdot \operatorname{lcm}((S_n^{\div s^n})^{\text{prim}})$ for $s = 2, 3$ by Lemma 2.26(a), where $S_n^{\div 2^n} = \{c_i, 3^i d_i : 1 \le i \le n\}$ and $S_n^{\div 3^n} = \{2^i c_i, d_i : 1 \le i \le n\}$.

(a) If there are infinitely many $i \in \mathbb{N}$ such that $c_i \nmid d_i$, then there are infinitely many $n \in \mathbb{N}$ such that $3^n \mid \operatorname{lcm}((S_n^{\div 2^n})^{\text{prim}})$, so that $3^n \mid \gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)})$.

(b) Analogously, if there are infinitely many $i \in \mathbb{N}$ such that $d_i \nmid c_i$, then there are infinitely many $n \in \mathbb{N}$ such that $2^n \mid \operatorname{lcm}((S_n^{\div 3^n})^{\text{prim}})$, so that $2^n \mid \gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)})$.

(c) It remains to treat the case where $c_i = d_i$ except for finitely many $i \in \mathbb{N}$, say for $i < N$. Then, $c_N \cdot \cdots \cdot c_n \mid \gcd(\tilde{\tau}^{(2)}, \tilde{\tau}^{(3)})$ for all $n \ge N$.

We complete this example by showing (for suitable choices of $c_n = d_n$) that conditions (Seh) and hence also (D) are violated. Suppose that $\prod_{n\in\mathbb{N}}(1 - 1/c_n) > \frac{1}{2}$. Then, for any $n \ge 1$, the equation $3^nT - 2^nM = 1$ has solutions T, M with

$$3^nT, 2^nM \in \mathcal{M}_{\mathcal{A}_{S_n}^{\infty}} \setminus \mathcal{M}_{S_n} = (2^n\mathbb{Z} \cup 3^n\mathbb{Z}) \setminus \mathcal{M}_{\{c_1,\ldots,c_n\}},$$

so that there are holes with distance 1 (in the sense of [2]), see Proposition 3.7. Indeed, there is a unique solution $X_0 \in \{1, \ldots, 6^n - 1\}$ of the equations $X \equiv 1 \bmod 2^n$ and $X \equiv 0 \bmod 3^n$. For $k = 0, \ldots, c_1 \cdots c_n - 1$, let $X_k = X_0 + k6^n$, $T_k = X_k/3^n$ and $M_k = X_k - 1/2^n$. As all X_k are further solutions of the same two equations, and as the c_i

are pairwise coprime and also coprime to 2 and to 3, exactly $\gamma_n := c_1 \cdots c_n \prod_{i=1}^{n}(1 - 1/c_i)$ of these solutions are $\{c_1, \ldots, c_n\}$-free, and also exactly γ_n of the $c_1 \cdots c_n$ numbers $X_k - 1$ are $\{c_1, \ldots, c_n\}$-free. Hence, exactly γ_n of the numbers T_k and γ_n of the numbers M_k are $\{c_1, \ldots, c_n\}$-free. Since $2\gamma_n > c_1 \cdots c_n$, there exists at least one $k \in \{0, \ldots, c_1 \cdots c_n - 1\}$ such that T_k and M_k are $\{c_1, \ldots, c_n\}$-free. This proves the claim.

3.4. *Non-trivial centralizer.* The purpose of this section is to treat simple examples of the type

$$\mathcal{B}_1^N = \{2^k c_k : k \in \mathbb{N}\} \cup \{2^{k-1} c_k^2 : k \in \mathbb{N}, k < N\}, \tag{53}$$

where $N \in \mathbb{N} \cup \{\infty\}$, and the numbers c_n are odd and pairwise coprime. Observe that $\mathcal{B}_1^1$ and $\mathcal{B}_1^2$ are the sets $\mathcal{B}_1$ and $\mathcal{B}_1'$ from the introduction, respectively. Let $S_n = \{2^k c_k : k \leqslant n\} \cup \{2^{k-1} c_k^2 : k \leqslant \min\{n, N-1\}\}$. Then, $\mathrm{lcm}(S_n) = 2^n c_1^2 \ldots c_n^2$ for $n < N$, $\mathrm{lcm}(S_n) = 2^n c_1^2 \ldots c_{N-1}^2 c_N \ldots c_n$ for $n \geq N$ and $\mathcal{A}_{S_n} = S_n \cup \{2^n\}$, so that $\mathcal{A}_{S_n} \setminus S_n = \mathcal{A}_{S_n}^{\infty} = \mathcal{A}_{S_n}^{\infty,p} = \{2^n\}$. As $\min(\mathcal{A}_{S_n} \setminus S_n)$ obviously tends to infinity, these are examples of Toeplitz type by Proposition 3.5(d). Along the same lines as in Example 3.21, one can show that all periods are essential and $\widetilde{\mathcal{H}}_n = \mathcal{H}_n = 2^n\mathbb{Z} \setminus \mathcal{M}_{S_n}$.

Let $\tilde{\tau}_n$ be the smallest period of $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$. We will show that $\tilde{\tau}_n = \mathrm{lcm}(S_n)/c_1 \ldots c_{N-1}$. Notice that $2^n k \in \mathcal{M}_{S_n}$ if and only if $2^n k \in \mathcal{M}_{\{c_1,\ldots,c_n\}}$. So, $\widetilde{\mathcal{H}}_n = 2^n\mathbb{Z} \setminus \mathcal{M}_{\{c_1,\ldots,c_n\}}$. Since $c_1, \ldots, c_n$ are odd and pairwise coprime, Lemma 2.26(a) implies $\tilde{\tau}_n = 2^n c_1 \ldots c_n = \mathrm{lcm}(S_n)/c_1 \ldots c_{N-1}$.

Notice that property (TI) holds for $A_n = \mathcal{A}_{S_n}^{\infty,p}$. By Proposition 2.20, there exists a unique $k \in \mathbb{Z}$ such that $\partial W + y_F \subseteq \partial W + \Delta(k)$, and $\mathrm{lcm}(S_n)/c_1 \ldots c_{N-1} = \tilde{\tau}_n \mid (y_F)_{S_n} - k$. It follows at once that $y_F = \Delta(k)$ for $\mathcal{B}_1^1$ and that $y_F - \Delta(k)$ has order at most $\prod_{i=1}^{N-1} c_i$ in G in the case of $\mathcal{B}_1^N$ and finite N, while $y_F - \Delta(k)$ may have any order in the case of $\mathcal{B}_1^{\infty}$.

In the remainder of this subsection, we show that such non-trivial centralizers as allowed above really exist. To this end, fix $N \in \mathbb{N} \cup \{\infty\}$ and $1 \leq \ell < N$, and consider $\mathcal{B}_1^N$ as before. Denote $p_m = \mathrm{lcm}(S_m)$ for any $m \in \mathbb{N}$. Let

$$q = \frac{p_\ell}{c_\ell} = 2^\ell c_1^2 \ldots c_{\ell-1}^2 c_\ell. \tag{54}$$

We define $F_\ell \colon X_\eta \to \{0, 1\}^{\mathbb{Z}}$ by

$$(F_\ell x)_s = \begin{cases} x_s & \text{if } s \notin c_\ell\mathbb{Z} - \pi_{\mathrm{lcm}(S_\ell)}(x), \\ x_{s+q} & \text{if } s \in c_\ell\mathbb{Z} - \pi_{\mathrm{lcm}(S_\ell)}(x). \end{cases}$$

This map is continuous, because $\pi_{\mathrm{lcm}(S_\ell)}$, the ℓth coordinate of the MEF map, is continuous. Recall that the MEF map $\pi : X_\eta \to G$ is chosen such that $\pi(\eta) = \Delta(0)$. As $\Delta(0) \in C_\phi$ [18, Lemma 3.5] for $\mathcal{B}$-free Toeplitz subshifts, $|\pi^{-1}\{\pi(F(\eta))\}| = |\pi^{-1}\{\pi(\eta)\}| = 1$, that is, $\pi(F(\eta)) \in C_\phi$ and, observing also Remark 3.1,

$$F(\eta) = \phi(\pi(F(\eta))) = \phi(y_F) = \mathbb{1}_{\mathbb{Z} \setminus \bigcup_{b \in \mathcal{B}}(b\mathbb{Z} - (y_F)_b)}. \tag{55}$$

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LEMMA 3.22. *Define $y \in G$ by*

$$y_b = \begin{cases} 0 & \text{if } b \neq 2^{\ell-1}c_\ell^2, \\ q & \text{if } b = 2^{\ell-1}c_\ell^2. \end{cases} \tag{56}$$

Then y coincides with the rotation y_{F_ℓ} associated to F_ℓ.

Proof. As $\pi_{\mathrm{lcm}(S_\ell)}(\eta) = 0$,

$$(F_\ell \eta)_s = \begin{cases} \eta_s & \text{if } s \notin c_\ell\mathbb{Z}, \\ \eta_{s+q} & \text{if } s \in c_\ell\mathbb{Z}. \end{cases}$$

In view of equation (55) and of the injectivity of ϕ (see Remark 3.1), we must show that $(F_\ell\eta)_s = 0$ if and only if $s \in \bigcup_{b\in\mathcal{B}}(b\mathbb{Z} - y_b)$ for all $s \in \mathbb{Z}$.

(1) Assume that $s \notin c_\ell\mathbb{Z}$. Then,

$$(F_\ell\eta)_s = 0 \Leftrightarrow \eta_s = 0 \Leftrightarrow s \in \mathcal{M}_\mathcal{B} \Leftrightarrow s \in 2^{\ell-1}c_\ell^2\mathbb{Z} \cup \bigcup_{b\in\mathcal{B}\setminus\{2^{\ell-1}c_\ell^2\}} (b\mathbb{Z} - y_b).$$

As $s \notin c_\ell\mathbb{Z}$ by assumption and $s + q \notin c_\ell\mathbb{Z}$ because $c_\ell \mid q$, this is equivalent to $s \in \bigcup_{b\in\mathcal{B}}(b\mathbb{Z} - y_b)$.

(2) Now assume that $s \in c_\ell\mathbb{Z}$. Let $T = \{2c_1, \ldots, 2^\ell c_\ell, c_1^2, \ldots, 2^{\ell-2}c_{\ell-1}^2\}$. Observe that $q = \mathrm{lcm}(T)$ and T is saturated. Observe also that $(F_\ell\eta)_s = 0$ if and only if $s + q \in \mathcal{M}_\mathcal{B}$.

So let $b \in \mathcal{B}$ and assume that $b \mid s + q$.

(i) If $b \in T$, then $b \mid q$, and hence $s \in b\mathbb{Z} = b\mathbb{Z} - y_b$.

(ii) If $b = 2^{\ell-1}c_\ell^2$, then $s \in b\mathbb{Z} - q = b\mathbb{Z} - y_b$.

(iii) If $b \in \mathcal{B} \setminus (T \cup \{2^{\ell-1}c_\ell^2\})$, then $2^\ell \mid b$ and, as $c_\ell \mid s$ and $2^\ell c_\ell \mid q$, it follows that $s \in 2^\ell c_\ell\mathbb{Z} = 2^\ell c_\ell\mathbb{Z} - y_{2^\ell c_\ell}$.

We have proved that $s \in \bigcup_{b\in\mathcal{B}}(b\mathbb{Z} - y_b)$.

It remains to prove that if $s \in c_\ell\mathbb{Z}$ and $s \in \bigcup_{b\in\mathcal{B}}(b\mathbb{Z} - y_b)$, then $s + q \in \mathcal{M}_\mathcal{B}$. Indeed,

$$\begin{aligned} s + q &\in \bigg(c_\ell\mathbb{Z} \cap \bigcup_{b\in\mathcal{B}}(b\mathbb{Z} - y_b)\bigg) + q \\ &= \bigg(\bigcup_{b\in T}(b \vee c_\ell)\mathbb{Z} + q\bigg) \cup 2^{\ell-1}c_\ell^2\mathbb{Z} \cup \bigg(\bigcup_{b\in\mathcal{B}\setminus(T\cup\{2^{\ell-1}c_\ell^2\})} c_\ell b\mathbb{Z} + q\bigg) \\ &\subseteq \bigcup_{b\in T} b\mathbb{Z} \cup 2^{\ell-1}c_\ell^2\mathbb{Z} \cup 2^\ell c_\ell\mathbb{Z} \subseteq \mathcal{M}_\mathcal{B}, \end{aligned}$$

because $\mathrm{lcm}(T \cup \{c_\ell\}) = q$ and $2^\ell \mid b$ for each $b \in \mathcal{B} \setminus (T \cup \{2^{\ell-1}c_\ell^2\})$. □

PROPOSITION 3.23. *For each $1 \leqslant \ell < N \leqslant \infty$, the map F_ℓ belongs to the centralizer of the $\mathcal{B}_1^N$-free subshift. It satisfies $F_\ell^{c_\ell} = \mathrm{id}_{X_\eta}$, but $F_\ell^i \neq \mathrm{id}_{X_\eta}$ for all $1 \leqslant i < c_\ell$.*

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Proof. We show first that $F_\ell(\eta) \in X_\eta$. Recall that $\pi_{\mathrm{lcm}(S_\ell)}(\eta) = 0$. Fix $n \in \mathbb{N}$ and choose $t \geqslant \ell$ such that $2^t > n$. As $\gcd(p_\ell, p_t/c_\ell) = q$, there exists $z \in \mathbb{Z}$ such that

$$z \equiv 0 \bmod \frac{p_t}{c_\ell} \quad \text{and} \quad z \equiv q \bmod p_\ell. \tag{57}$$

We claim that $(F_\ell\eta)[-n, n] = \eta[-n+z, n+z]$. Let $s \in [-n, n]$. There are two cases:

(1) $s \notin c_\ell\mathbb{Z}$. Then, $2^{\ell-1}c_\ell^2$ and $2^\ell c_\ell$ do not divide neither s nor $s+z$ by the second of the congruences in equation (57).

- (i) Assume that $\eta_s = 0$. Then, $2^{j-\varepsilon}c_j^{1+\varepsilon} \mid s$ for some $j \neq \ell$ and $\varepsilon \in \{0, 1\}$. Observe that $j \leqslant t$, since $|s| \leq n < 2^t$. Then, $2^{j-\varepsilon}c_j^{1+\varepsilon} \mid s+z$ by the first of the congruences in equation (57), and hence $\eta_{s+z} = 0$.
- (ii) Conversely, assume that $\eta_{s+z} = 0$. Then, $2^{j-\varepsilon}c_j^{1+\varepsilon} \mid s+z$ for some $j \neq \ell$ and $\varepsilon \in \{0, 1\}$, and if $j \leq t$, then $2^{j-\varepsilon}c_j^{1+\varepsilon} \mid s$ by the first of the congruences in equation (57), and hence $\eta_s = 0$. However, $j > t$ is impossible, because then $2^t \mid s+z$, so that $2^t \mid s$ by the first of the congruences in equation (57) again, in contradiction to $0 < |s| < 2^t$.

We have shown that $\eta_{s+z} = \eta_s = (F_\ell\eta)_s$.

(2) $s \in c_\ell\mathbb{Z}$. We use the fact that $z \equiv q \bmod p_\ell$ in view of equation (57) repeatedly.

- (i) Assume that $\eta_{s+q} = 0$, so that $2^{j-\varepsilon}c_j^{1+\varepsilon} \mid s+q$ for some $j \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. If $j \leq \ell$, then $2^{j-\varepsilon}c_j^{1+\varepsilon} \mid s+z$ because $z \equiv q \bmod p_\ell$. If $j > \ell$, then $2^\ell \mid s+q$, and hence $2^\ell \mid s$ and $2^\ell c_\ell \mid s$. As $z \equiv q \bmod p_\ell$, also $2^\ell c_\ell \mid z$, so that $2^\ell c_\ell \mid s+z$. In both cases, $\eta_{s+z} = 0$.
- (ii) Conversely, assume that $\eta_{s+z} = 0$. The same arguments as before, with roles of q and z interchanged, show that $\eta_{s+q} = 0$.

We have shown that $\eta_{s+z} = \eta_{s+q} = (F_\ell\eta)_s$, and thus the claim follows.

As $\pi(\eta) = \Delta(0)$, we have $\pi(\sigma^k\eta) = \Delta(k)$ for any $k \in \mathbb{Z}$. So, $\pi_{\mathrm{lcm}(S_\ell)}(\sigma^k\eta) = k \bmod p_\ell$ for any $k \in \mathbb{Z}$. Hence, for any $k \in \mathbb{Z}$,

$$\begin{aligned}(F_\ell(\sigma^k\eta))_s &= \begin{cases}(\sigma^k\eta)_s & \text{if } s \notin c_\ell\mathbb{Z} - k,\\ (\sigma^k\eta)_{s+q} & \text{if } s \in c_\ell\mathbb{Z} - k,\end{cases}\\ &= \begin{cases}\eta_{s+k} & \text{if } s+k \notin c_\ell\mathbb{Z},\\ \eta_{s+q+k} & \text{if } s+k \in c_\ell\mathbb{Z},\end{cases} = (F_\ell\eta)_{s+k} = (\sigma^k(F_\ell\eta))_s.\end{aligned}$$

So, $F_\ell(\sigma^k\eta) \in X_\eta$ for each $k \in \mathbb{Z}$. The denseness of the orbit of η and the continuity of F_ℓ imply that $F_\ell(X_\eta) \subseteq X_\eta$ and F_ℓ commutes with σ.

Since F_ℓ corresponds to y_{F_ℓ} given by equation (56), and q has order c_ℓ in the group $\mathbb{Z}/2^{\ell-1}c_\ell^2\mathbb{Z}$, it follows that F_ℓ has order c_ℓ.

This proves in particular that F_ℓ is a homeomorphism. □

Remark 3.24. Consider the case $N < \infty$. As the numbers $c_1, \ldots, c_{N-1}$ are pairwise coprime, the group generated by the automophisms $F_1, \ldots, F_{N-1}$ is cyclic of order $c_1 \ldots c_{N-1}$. Let F be a generator of this group. Then, in view of Corollary 2.21, $\mathrm{Aut}_\sigma(X_\eta) = \langle\sigma\rangle \oplus \langle F\rangle$ in the case of $\mathcal{B}_1^N$.

https://doi.org/10.1017/etds.2023.43 Published online by Cambridge University Press

COROLLARY 3.25. *Consider $\mathcal{B}_1^\infty$. Proposition 3.23 shows that the group $Aut_\sigma(X_\eta)/\langle\sigma\rangle$ contains the infinite direct sum of finite cyclic groups*

$$\mathbb{Z}/c_1\mathbb{Z} \oplus \mathbb{Z}/c_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/c_\ell\mathbb{Z} \oplus \ldots .$$

Remark 3.26. One can show that the element $y \in G$ given by equation (56) satisfies the sufficient conditions from [**2**, Theorem 1] for representing a (non-trivial) element of the centralizer of the $\mathcal{B}_1^N$-free subshift. However, the methods from [**2**] do not limit the order of elements from this centralizer as in Theorem 2.8. It is shown in [**2**] that Toeplitz subshifts with skeletons with equidistant holes have only elements of finite order in their centralizer. However, this does not apply to minimal $\mathcal{B}$-free subshifts, see Proposition 3.7.

3.5. *Holes versus essential holes: examples.* We start with an example for which there is *no period structure such that all holes are essential and the centralizer is trivial.*

Example 3.27. Assume that $c_1, \ldots, c_n, \ldots, q_1, \ldots, q_n, \ldots$ are pairwise coprime natural numbers. Let

$$b_1 = q_1c_1,\ b_2 = q_2c_2,\ b_3 = q_1q_3c_3,\ b_4 = q_1q_2q_4c_4, \ldots, b_m = q_1 \ldots q_{m-2}q_mc_m, \ldots$$

and set $\mathcal{B} = \{b_m : m \in \mathbb{N}\}$. Let (p_n) be any period structure for $\eta = \eta_{\mathcal{B}}$. Observe that for every $m \in \mathbb{N}$, there exists $n, n' \in \mathbb{N}$ such that

$$p_m \mid q_1 \ldots q_nc_1 \ldots c_n \text{ and } q_1 \ldots q_mc_1 \ldots c_m | p_{n'}.$$

Fix N such that $q_1 | p_N$ and let

$$m = \max\{i \in \mathbb{N} : q_1 \ldots q_i | p_N\}.$$

Then, $q_{m+1} \nmid p_N$ and hence

$$k := \gcd(b_{m+1}, p_N) = q_1 \ldots q_{m-1}c_{m+1}^{\varepsilon},$$

where $\varepsilon \in \{0, 1\}$. Note that

$$(k + p_N\mathbb{Z}) \cap q_m\mathbb{Z} = \varnothing. \tag{58}$$

We claim that $k \in \mathcal{H}_N \setminus \widetilde{\mathcal{H}}_N$. Clearly, $k \in \mathcal{F}_{\mathcal{B}}$, so $\eta_k = 1$. By the definition of k, it follows that $(k + p_N\mathbb{Z}) \cap b_{m+1}\mathbb{Z} \neq \varnothing$, thus η is not constant along $k + p_N\mathbb{Z}$ and hence $k \in \mathcal{H}_N$. Now take $n > N$ such that

$$q_1 \ldots q_{m+1}c_1 \ldots c_{m+1} | p_n. \tag{59}$$

We claim that $(k + p_N\mathbb{Z}) \cap \mathcal{H}_n = \varnothing$. Let $l \in \mathbb{Z}$ and assume first that $k + lp_N \in \mathcal{F}_{\mathcal{B}}$. Suppose that $k + lp_N + l'p_n \in \mathcal{M}_{\mathcal{B}}$ for some $l' \in \mathbb{Z}$. So $b \mid k + lp_N + l'p_n$ for some $b \in \mathcal{B}$. By equations (58) and (59), $q_m \nmid b$, so $b \in \{b_1, \ldots, b_{m+1}\}$. However then, by equation (59), $b \mid k + lp_N$, which is a contradiction. It follows that $k + lp_N \notin \mathcal{H}_n$. Now assume that $k + lp_N \in \mathcal{M}_{\mathcal{B}}$. So, $b \mid k + lp_N$ for some $b \in \mathcal{B}$. Then by equation (58), $b \in \{b_1, \ldots, b_{m+1}\}$, and hence $b \mid p_n$ and $k + lp_N + p_n\mathbb{Z} \subseteq \mathcal{M}_{\mathcal{B}}$ by equation (59). Again we see that $k + lp_N \notin \mathcal{H}_n$. The claim follows.

Let $S_n = \{b_1, \ldots, b_n\}$. Then $p_n := \operatorname{lcm}(S_n) = q_1 \ldots q_nc_1 \ldots c_n$ defines a period structure, $\mathcal{A}_{S_n} \setminus S_n = \{q_1 \ldots q_{n-1}, q_1 \ldots q_n\}$ and $\mathcal{A}_{S_n}^\infty = \{q_1 \ldots q_n\}$. By Proposition 3.7, with respect to this period structure, $\mathcal{H}_n = q_1 \ldots q_{n-1}\mathbb{Z} \setminus \mathcal{M}_{S_n}$. Since $c_m \nmid \ell_{S_N} \vee q_1 \ldots q_n$ for any $m, n > N$, $\ell_{S_N} \vee q_1 \ldots q_n \in \mathcal{F}_{\mathcal{B}\setminus S_N}$. So item (a) from Lemma 3.19 holds. Hence, $\widetilde{\mathcal{H}}_n = q_1 \ldots q_n\mathbb{Z} \setminus \mathcal{M}_{S_n}$. Since $S_n^{\div q_1 \ldots q_n} = \{c_1, \ldots, c_n\}$ and $S_n^{\div q_1 \ldots q_{n-1}} = \{c_1, \ldots, c_{n-1}, q_n c_n\}$ are both primitive, Lemma 2.26 shows that the minimal periods of $\widetilde{\mathcal{H}}_n$ and $\mathcal{H}_n$ are both equal $p_n = q_1 \ldots q_n c_1 \ldots c_n$, although $\widetilde{\mathcal{H}}_n \neq \mathcal{H}_n$. Proposition 2.20 shows that the centralizer of X_η is trivial.

We continue with an example for which *the validity of the identities $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$ depends on the choice of the period structure.*

Example 3.28. Assume that we have a collection $\{q_i, c_i, d_i : i \geq 1\}$ of pairwise coprime odd natural numbers greater than 1. Let

$$b_i = 2^i q_i c_i,\ b_i' = 2^i q_i d_i,\ b_i'' = 2^{i+1} q_i$$

for $i \geq 1$ and

$$S_n = \{b_i, b_i', b_i'' : 1 \leq i \leq n\},\ S_n' = S_n \cup \{b_{n+1}\}$$

for $n \geq 1$. Finally, we set

$$\mathcal{B} = \bigcup_{n\geq 1} S_n = \bigcup_{n\geq 1} S_n'.$$

Clearly, $\mathcal{B}$ contains no scaled copy of an infinite coprime set.

The sets S_n and S_n' are saturated and

$$\operatorname{lcm}(S_n) = 2^{n+1} \cdot q_1 \cdot \ldots \cdot q_n \cdot c_1 \cdot \ldots \cdot c_n \cdot d_1 \cdot \ldots \cdot d_n,\ \operatorname{lcm}(S_n') = q_{n+1}c_{n+1} \operatorname{lcm}(S_n), \tag{60}$$

and hence

$$\mathcal{A}_{S_n} \setminus S_n = \mathcal{A}_{S_n}^\infty = \{2^{n+1}\} \tag{61}$$

and

$$\mathcal{A}_{S_n'} \setminus S_n' = \{2^{n+1}, 2^{n+1}q_{n+1}\},\ \mathcal{A}_{S_n'}^\infty = \{2^{n+1}\}. \tag{62}$$

It follows that $\widetilde{\mathcal{H}}_N = \mathcal{H}_N = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N\}})$, where the sets of holes and essential holes are calculated with respect to the period structure $p_n = \operatorname{lcm}(S_n)$. Indeed,

$$\mathcal{H}_N = 2^{N+1}\mathbb{Z} \setminus \mathcal{M}_{S_N} = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{(S_N^{\div 2^{N+1}})^{\text{prim}}}) = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N\}})$$

by equation (61) and Proposition 3.7. Suppose that $b \mid \ell_{S_N} \vee 2^{n+1}$ for some $b \in \mathcal{B}$ and some $n > N$. Since $q_i \nmid \ell_{S_N} \vee 2^{n+1}$ for any $i > N$, $b \in S_N$. So item (a) from Lemma 3.19 holds and the assertion follows. By Lemma 2.26(a), we obtain $\tilde{\tau}_N = \tau_N = 2^{N+1} q_1 \cdots q_N$.

Let $\mathcal{H}_n'$, $\widetilde{\mathcal{H}}_n'$ be the sets of holes and essential holes, respectively, calculated with respect to the period structure $p_n' = \operatorname{lcm}(S_n')$. We will show that

$$\mathcal{H}_N' = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N,q_{N+1}c_{N+1}\}}) \quad \text{and} \quad \widetilde{\mathcal{H}}_N' = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N,q_{N+1}\}}),$$

so that $\tilde{\tau}_N' = \tau_N'/c_{N+1} = 2^{N+1} q_1 \cdots q_{N+1}$ by Lemma 2.26(a).

Indeed,

$$\mathcal{H}'_N = 2^{N+1}\mathbb{Z} \setminus \mathcal{M}_{S'_N} = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{(S'_N \div 2^{N+1})^{\rm prim}}) = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N,q_{N+1}c_{N+1}\}})$$

by equation (62) and Proposition 3.7. Let $n > N$. (Notice that item (a) from Lemma 3.19 does not hold because $2^{N+2}q_{N+1} \mid 2^{n+1} \vee \mathrm{lcm}(S'_N)$ and $2^{N+2}q_{N+1} \notin S'_N$.) Suppose that for some $k \in \mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N,q_{N+1}c_{N+1}\}}$,

$$(2^{N+1}k + \mathrm{lcm}(S'_N)\mathbb{Z}) \cap \mathcal{H}'_n = \varnothing.$$

Then, in particular,

$$(2^{N+1}k + \mathrm{lcm}(S'_N)\mathbb{Z}) \cap 2^{n+1}\mathbb{Z} \subseteq \mathcal{M}_{S'_n}.$$

By Lemma 2.24, there exists $b \in S'_n$ such that

$$b \mid \gcd(2^{N+1}k, \mathrm{lcm}(S'_N)) \vee 2^{n+1} = \gcd(2^{N+1}k \vee 2^{n+1}, \mathrm{lcm}(S'_N) \vee 2^{n+1}).$$

Since $k \in \mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N,q_{N+1}c_{N+1}\}}$ and $b \mid 2^{N+1}k \vee 2^{n+1}$, we have $b \in S'_n \setminus S'_N$. However, $b \mid \mathrm{lcm}(S'_N) \vee 2^{n+1}$ but $q_i \nmid \mathrm{lcm}(S'_N) \vee 2^{n+1}$ for any $i > N+1$, $d_{N+1} \nmid \mathrm{lcm}(S'_N) \vee 2^{n+1}$ and $2^{N+1}q_{N+1}c_{N+1} \in S'_N$. So, $b = 2^{N+2}q_{N+1}$ and $q_{N+1} \mid k$. Conversely,

$$(2^{N+1}q_{N+1}m + \mathrm{lcm}(S'_N)\mathbb{Z}) \cap 2^{n+1} \subseteq 2^{n+1}q_{N+1}\mathbb{Z} \subseteq 2^{N+2}q_{N+1}\mathbb{Z} \subseteq \mathcal{M}_{S'_n}$$

for any $m \in \mathbb{Z}$. Hence,

$$\begin{aligned}\widetilde{\mathcal{H}}'_N &= 2^{N+1}\mathbb{Z} \setminus \mathcal{M}_{S'_N \cup \{2^{N+1}q_{N+1}\}} = 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{((S'_N \cup \{2^{N+1}q_{N+1}\}) \div 2^{N+1})^{\rm prim}}) \\ &= 2^{N+1}(\mathbb{Z} \setminus \mathcal{M}_{\{q_1,\ldots,q_N,q_{N+1}\}}).\end{aligned}$$

For both filtrations, property (TI) from §2.5 is obviously satisfied so that Proposition 2.20 applies. However, $p_N/\tilde{\tau}_N = c_1 \cdots c_N \cdot d_1 \cdots d_N$ and $p'_N/\tilde{\tau}'_N = c_1 \cdots c_{N+1} \cdot d_1 \cdots d_N$ are both unbounded in N, so that only very weak conclusions can be drawn from this proposition. In particular, no bound on the size of the centralizer can be deduced from it.

Finally, we provide an example for which $\widetilde{\mathcal{H}}_n \subsetneq \mathcal{M}_{\mathcal{A}^\infty_{S_n}} \setminus \mathcal{M}_{S_n}$ *for any saturated filtration* (S_n) *of* $\mathcal{B}$*, so that* $S_n \subsetneq S_n(a)$ *for some* $a \in \mathcal{A}^\infty_{S_n}$ *(and hence also for some* $a \in \mathcal{A}^{\infty,p}_{S_n}$*)*, see Theorem 3.17 and Remark 3.18(a).

Example 3.29. Let $s_i, s'_i, q_i, r_i, d_{i-1} (i \in \mathbb{N})$ be pairwise different primes. Let

$$b_1 = s_1 \cdot s'_1 \cdot q_1 \cdot d_0 \tag{63}$$

and, for $m \in \mathbb{N} \cup \{0\}$ and $i \geq 2$, let

$$b_{i,m} = \frac{1}{s'_{i-1}} s_1 \cdot s'_1 \cdot \ldots \cdot s_i \cdot s'_i \cdot q_i \cdot r_i^m \cdot d_m. \tag{64}$$

We set

$$\mathcal{B} = \{b_1\} \cup \{b_{i,m} : i \geq 2, m \geq 0\}. \tag{65}$$

Then, $\mathcal{B}$ is primitive. It is easy to show that $\mathcal{B}$ contains no scaled copy of an infinite coprime set, so the $\mathcal{B}$-free shift is Toeplitz by Proposition 3.5.

Let (S_n) be any saturated filtration of $\mathcal{B}$ by finite sets. With no loss of generality, we can assume that $b_1, b_{2,0} \in S_1$. Let k be the minimal number such that $b_{k+1,0} \notin S_1$. It follows that $b_{2,0}, \ldots, b_{k,0} \in S_1$. Let n be maximal such that $b_{k+1,0} \notin S_n$. Then,

$$b_1, b_{2,0}, \ldots, b_{k,0} \in S_n \quad \text{and} \quad b_{k+1,0} \in S_{n+1} \setminus S_n \tag{66}$$

and

$$s_1 \cdot s_1' \cdot \ldots \cdot s_k \cdot s_k' \cdot q_1 \cdot \ldots \cdot q_k \cdot d_0 \mid \ell_{S_n}. \tag{67}$$

Observe that $b_{k+1,m} \notin S_n$ for every $m \in \mathbb{N}$. Otherwise, as $d_0|\ell_{S_n}$ and S_n is saturated, $b_{k+1,0} \in S_n$ in contrast to our assumption. Thus, $r_{k+1} \nmid \ell_{S_n}$. For m large enough, say for $m \geq m_0$, the number $\gcd(b_{k+1,m}, \ell_{S_{n+1}})$ does not depend on m.

A case by case analysis of prime divisors of $b_{k+1,0}$ and $b_{k+1,m}$ shows that

$$d_0 \gcd(b_{k+1,m}, \ell_{S_n}) = \gcd(b_{k+1,0}, \ell_{S_n}) \quad \text{for } m \geq m_0. \tag{68}$$

Indeed, $s_i | b_{k+1,0}$ (respectively $s_i' | b_{k+1,0}$) if and only if $s_i | b_{k+1,m}$ (respectively $s_i' | b_{k+1,m}$) for every $i \in \mathbb{N}$. Moreover, $r_{k+1} \nmid \ell_{S_n}$, $d_0 \nmid b_{k+1,m}$ and q_{k+1} divides both $b_{k+1,0}$ and $b_{k+1,m}$.

We prove that

$$\begin{aligned} &\text{for all } a \in \mathcal{A}^{\infty}_{S_{n+1}} : \gcd(a, \ell_{S_n}) | \gcd(b_{k+1,0}, \ell_{S_n}) \\ &\quad \Leftrightarrow a = \gcd(b_{k+1,m}, \ell_{S_{n+1}}) \quad \text{for } m \geq m_0. \end{aligned} \tag{69}$$

In view of equation (68), it is enough to prove '$\Rightarrow$'. We can assume that $a = \gcd(b_{i,m}, \ell_{S_{n+1}})$ for some $i > 1$ and $m \in \mathbb{N}$, and $\gcd(b_{i,m}, \ell_{S_n}) = \gcd(a, \ell_{S_n}) \mid \gcd(b_{k+1,0}, \ell_{S_n})$. As $s_k' \nmid b_{k+1,0}$ and $s_k' | \ell_{S_n}$, we have $i \leq k-1$ or $i = k+1$. However, $i \leq k-1$ implies that $q_i \mid \gcd(b_{i,m}, \ell_{S_n}) \mid b_{k+1,0}$, which is a contradiction. Thus, $i = k+1$ and since $a \in \mathcal{A}^{\infty}_{S_{n+1}}$, $a = \gcd(b_{k+1,m}, \ell_{S_{n+1}})$ for $m \geq m_0$.

Note that as $b_{k+1,0} \in S_{n+1}$,

$$\text{for all } m \in \mathbb{N} : \ s_{k+1} \cdot s_{k+1}' \cdot q_{k+1} \mid \gcd(b_{k+1,m}, \ell_{S_{n+1}}). \tag{70}$$

Let $a \in \mathcal{A}^{\infty}_{S_{n+1}}$ be such that $(\gcd(b_{k+1,0}, \ell_{S_n}) + \ell_{S_n}\mathbb{Z}) \cap a\mathbb{Z} \neq \varnothing$. Then, $\gcd(a, \ell_{S_n}) | \gcd(b_{k+1,0}, \ell_{S_n})$ and by equation (69), $a = \gcd(b_{k+1,m}, \ell_{S_{n+1}})$ for $m \geq m_0$. By equations (67) and (70),

$$b_{k+1,0} \mid a \vee \ell_{S_n}. \tag{71}$$

It follows from Lemma 2.25 that $(\gcd(b_{k+1,0}, \ell_{S_n}) + \ell_{S_n}\mathbb{Z}) \cap a\mathbb{Z} \subseteq b_{k+1,0}\mathbb{Z}$. Hence,

$$(\gcd(b_{k+1,0}, \ell_{S_n}) + \ell_{S_n}\mathbb{Z}) \cap (\mathcal{M}_{\mathcal{A}^{\infty}_{S_{n+1}}} \setminus \mathcal{M}_{S_{n+1}}) = \varnothing,$$

which, because of Remark 3.12, implies $\gcd(b_{k+1,0}, \ell_{S_n}) \notin \widetilde{\mathcal{H}}_n$. Moreover, by equation (68) and the primitivity of $\mathcal{B}$, $\gcd(b_{k+1,0}, \ell_{S_n}) \in \mathcal{M}_{\mathcal{A}^{\infty}_{S_n}} \setminus \mathcal{M}_{S_n}$. Thus, $\gcd(b_{k+1,0}, \ell_{S_n}) \in \mathcal{M}_{\mathcal{A}^{\infty}_{S_n}} \setminus (\mathcal{M}_{S_n} \cup \widetilde{\mathcal{H}}_n)$.

We claim that η is a regular Toeplitz sequence. Indeed, let $S_n = \{b_1\} \cup \{b_{i,m} : 2 \leq i \leq n, 0 \leq m \leq n\}$. Then, $\ell_{S_n} = s_1 s_1' \ldots s_n s_n' q_1 \ldots q_n r_1^n \ldots r_n^n d_0 \ldots d_n$ and for $b_{i,m} \notin S_n$,

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$$\gcd(b_{i,m}, \ell_{S_n}) = \begin{cases} \dfrac{1}{s'_{i-1}} s_1 s'_1 \ldots s_i s'_i q_i r_i^n & \text{if } i \le n, m > n, \\ s_1 s'_1 \ldots s_{n-1} s'_{n-1} s_n d_m & \text{if } i = n+1, m \le n, \\ s_1 s'_1 \ldots s_{n-1} s'_{n-1} s_n & \text{if } i = n+1, m > n, \\ s_1 s'_1 \ldots s_n s'_n d_m & \text{if } i > n+1, m \le n, \\ s_1 s'_1 \ldots s_n s'_n & \text{if } i > n+1, m > n. \end{cases}$$

Hence, $\mathcal{A}_{S_n}^{\infty,p} = \{1/s'_{i-1} s_1 s'_1 \ldots s_i s'_i q_i r_i^n : 2 \le i \le n\} \cup \{s_1 s'_1 \ldots s_{n-1} s'_{n-1} s_n\}$. Since $d(\mathcal{M}_{\mathcal{A}_{S_n}^{\infty,p}}) \le 1 - \prod_{a \in \mathcal{A}_{S_n}^{\infty,p}} (1 - 1/a) \le 1 - (1 - 1/\min_{2 \le i \le n} {r_i}^n)^{n-1} (1 - 1/s_1 s'_1 \ldots s_{n-1} s'_{n-1} s_n) \to 0$ as $n \to \infty$, by Corollary 3.11, $m_G(\partial W) = 0$. So the $\mathcal{B}$-free Toeplitz shift is regular, see e.g. [8, Theorem 13.1]. Similarly, one can show that the condition in equation (42) of Proposition 2.28 and Theorem 2.31 is satisfied for $A_n = \mathcal{A}_{S_n}^{\infty,p}$. We do not attempt to determine the sets $\widetilde{\mathcal{H}}_n$ (and their periods) according to the prescription in Theorem 3.17 explicitly.

Note that $a_{n+1} = s_1 s'_1 \cdots s_n s'_n s_{n+1}$ is an example of a number in $\mathcal{A}_{S_{n+1}}^{\infty,p}$ for which $\gcd(a_{n+1}, \ell_{S_n})$ belongs to $\mathcal{A}_{S_n}^{\infty}$ but not to $\mathcal{A}_{S_n}^{\infty,p}$, compare Lemma 3.9 items (c) and (e).

We complete this example by showing (for suitable choices of d_i) that conditions (Seh) and hence also (D) (the equivalence of conditions (Seh) and (D) was claimed without proof in §2.2, see also Lemma 2.9) and property (TI), see Proposition 2.20, are violated. Let $a_n = s_1 s'_1 s_2 s'_2 \ldots s_{n-1} s'_{n-1} s_n$ and $a'_n = s_1 s_2 s'_2 q_2 r_2^n$. Suppose that $\prod_{n \ge 0} (1 - 1/d_n) > \frac{1}{2}$. Then the equation $a_n / s_1 s_2 s'_2 T - a'_n / s_1 s_2 s'_2 M = 1$ has solutions T, M with

$$a_n T, a'_n M \in \mathcal{M}_{\mathcal{A}_{S_n}^{\infty}} \setminus \mathcal{M}_{S_n}$$

so that, for all n, there are holes with distance $s_1 s_2 s'_2$ (in the sense of [2]) in $\mathcal{H}_n$, see Proposition 3.7. Indeed, there is a unique solution $X_0 \in \{1, \ldots, \operatorname{lcm}(a_n, a'_n)/ s_1 s_2 s'_2 - 1\}$ of the equations $X \equiv 1 \bmod a_n / s_1 s_2 s'_2$ and $X \equiv 0 \bmod a'_n / s_1 s_2 s'_2$. For $k = 0, \ldots, d_0 d_1 \cdots d_n - 1$, let $X_k = X_0 + k \operatorname{lcm}(a_n, a'_n)/s_1 s_2 s'_2$, $T_k = X_k s_1 s_2 s'_2 / a_n$ and $M_k = (X_k - 1) s_1 s_2 s'_2 / a'_n$. As all X_k are further solutions of the same two equations, and as the d_i are pairwise different primes, exactly $\gamma_n := d_0 d_1 \cdots d_n \prod_{i=0}^n (1 - 1/d_i)$ of these solutions are $\{d_0, d_1, \ldots, d_n\}$-free, and also exactly γ_n of the $d_0 d_1 \cdots d_n$ numbers $X_k - 1$ are $\{d_0, d_1, \ldots, d_n\}$-free. Hence, exactly γ_n of the numbers T_k and γ_n of the numbers M_k are $\{d_0, d_1, \ldots, d_n\}$-free. Since $2\gamma_n > d_0 d_1 \cdots d_n$, there exists at least one $k \in \{0, \ldots, d_0 d_1 \cdots d_n - 1\}$ such that T_k and M_k are $\{d_0, d_1, \ldots, d_n\}$-free, so then $a_n T_k$ and $a'_n M_k$ are S_n-free. This proves the claim.

3.6. *Superpolynomial complexity.* We consider the example $\mathcal{B} = \mathcal{B}_2 = \{2^i c_i, 3^i c_i : i \in \mathbb{N}\}$, where c_i are odd pairwise coprime numbers not divisible by 3. Recall from Example 3.21 that our Theorem 2.31 applies to this $\mathcal{B}$ and ensures that the $\mathcal{B}$-free subshift has a trivial centralizer. Here we show that it has superpolynomial complexity.

We denote by ρ the complexity function of X_η for $\eta = \mathbb{1}_{\mathcal{F}_\mathcal{B}}$, that is,

$$\rho(n) = |\{\eta[k+1, k+n] : k \in \mathbb{Z}\}|$$

for $n \in \mathbb{N}$.

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PROPOSITION 3.30. *Assume that $2c_1 < 2^2c_2 < 2^3c_3 < \cdots$ are such that*

$$\sum_{i=1}^{\infty} \frac{1}{c_i} < \frac{1}{2}, \tag{72}$$

and there exists a real number $\alpha > 1$ such that

$$c_j \leq \alpha^j \tag{73}$$

for $j \gg 0$. Then, for each $\varepsilon \in (0, 1)$,

$$\liminf_{n\to+\infty} \frac{\rho(n)}{n^{\varepsilon \lg_{2\alpha} \lg_{2\alpha} n}} = +\infty.$$

Remark 3.31. If c_i is the square of the $(i+2)$th odd prime number, for $i \in \mathbb{N}$ (that is, $c_1 = 25$ etc.), the assumptions of the proposition are satisfied (equation (73) holds for $\alpha \geq 25$).

Remark 3.32. For any $j \in \mathbb{N}$, under the assumption in equation (72),

$$c_1 \ldots c_j \geq (2j)^j > j^j.$$

This inequality is a consequence of the fact that the arithmetic mean of positive numbers is greater than or equal to their geometric mean.

The following lemma is elementary.

LEMMA 3.33. *For every $n, b \in \mathbb{N}$ and $k, r \in \mathbb{Z}$:*

$$\frac{n}{b} - 1 < |(b\mathbb{Z} + r) \cap [k+1, k+n]| < \frac{n}{b} + 1.$$

Let

$$\delta = \frac{1}{2} - \sum_{i=1}^{\infty} \frac{1}{c_i}.$$

Given $n \in \mathbb{N}$ let $m_n = [\lg_2 n]$. Note that $\delta > 0$ by equation (72) and $2^{m_n+1} > n$.

LEMMA 3.34. *Assume that $j_0 \in \{1, \ldots, m_n\}$ satisfies*

$$2^{j_0} c_{j_0} < \frac{\delta n}{2 \lg_2 n}.$$

If

$$[1, n] \cap (2^{j_0} c_{j_0}\mathbb{Z} + r) \subseteq \bigcup_{i=1}^{m_n} (2^i c_i \mathbb{Z} + s_i) \cup \bigcup_{i=1}^{m_n} (3^i c_i \mathbb{Z} + t_i)$$

for some $r, s_1, \ldots, s_{m_n}, t_1, \ldots, t_{m_n} \in \mathbb{Z}$, then $r \equiv s_{j_0} \bmod 2^{j_0} c_{j_0}$.

Proof. Suppose otherwise, then $(2^{j_0} c_{j_0}\mathbb{Z} + r)$ is disjoint to $(2^{j_0} c_{j_0}\mathbb{Z} + s_{j_0})$ and hence

$$[1, n] \cap (2^{j_0} c_{j_0}\mathbb{Z} + r) \subseteq \bigcup_{i \in \{1,\ldots,m_n\}\setminus\{j_0\}} (2^i c_i \mathbb{Z} + s_i) \cup \bigcup_{i=1}^{m_n} (3^i c_i \mathbb{Z} + t_i). \tag{74}$$

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If follows that

$$
\begin{aligned}
&|[1, n] \cap (2^{j_0} c_{j_0}\mathbb{Z} + r)| \\
&\quad \leq \sum_{i \in \{1,\ldots,m_n\}\setminus\{j_0\}} |[1, n] \cap (2^i c_i \mathbb{Z} + s_i) \cap (2^{j_0} c_{j_0}\mathbb{Z} + r)| \\
&\qquad + \sum_{i=1}^{m_n} |[1, n] \cap (3^i c_i \mathbb{Z} + t_i) \cap (2^{j_0} c_{j_0}\mathbb{Z} + r)|. \qquad (75)
\end{aligned}
$$

For $i \neq j_0$, $(2^i c_i \mathbb{Z} + s_i) \cap (2^{j_0} c_{j_0}\mathbb{Z} + r)$ is either empty or equal to $\operatorname{lcm}(2^i c_i, 2^{j_0} c_{j_0})\mathbb{Z} + r'$ for some $r' \in \mathbb{Z}$. Similarly, for $i = 1, \ldots, m_n$, $(3^i c_i \mathbb{Z} + r_i) \cap (2^{j_0} c_{j_0}\mathbb{Z} + r)$ is either empty or equal to $\operatorname{lcm}(3^i c_i, 2^{j_0} c_{j_0})\mathbb{Z} + r''$ for some $r'' \in \mathbb{Z}$. Then, equation (75) and Lemma 3.33 yield

$$
\frac{n}{2^{j_0} c_{j_0}} - 1 \leq \sum_{i \in \{1,\ldots,m_n\}\setminus\{j_0\}} \left(\frac{n}{\operatorname{lcm}(2^i c_i, 2^{j_0} c_{j_0})} + 1 \right) + \sum_{i=1}^{m_n} \left(\frac{n}{3^i 2^{j_0} \operatorname{lcm}(c_i, c_{j_0})} + 1 \right). \qquad (76)
$$

For $i \neq j_0$, $\operatorname{lcm}(2^i c_i, 2^{j_0} c_{j_0}) = 2^{\max\{i,j_0\}} c_i c_{j_0} \geq 2^{j_0} c_i c_{j_0}$ and $\operatorname{lcm}(c_i, c_{j_0}) \geq c_{j_0}$ for every i, thus

$$
\frac{n}{2^{j_0} c_{j_0}} - 1 \leq \sum_{i \in \{1,\ldots,m_n\}\setminus\{j_0\}} \left(\frac{n}{2^{j_0} c_i c_{j_0}} + 1 \right) + \sum_{i=1}^{m_n} \left(\frac{n}{3^i 2^{j_0} c_{j_0}} + 1 \right), \qquad (77)
$$

which, as $\sum_{i=1}^{m_n} 1/3^i < \frac{1}{2}$, implies

$$
\frac{n}{2^{j_0} c_{j_0}} - 1 \leq \frac{n}{2^{j_0} c_{j_0}} \left(\frac{1}{2} + \sum_{i=1}^{m_n} \frac{1}{c_i} \right) + 2m_n - 1, \qquad (78)
$$

and hence

$$
\frac{\delta n}{2^{j_0} c_{j_0}} \leq 2m_n. \qquad (79)
$$

It follows that

$$
2^{j_0} c_{j_0} \geq \frac{\delta n}{2m_n} \geq \frac{\delta n}{2 \lg_2 n}, \qquad (80)
$$

which is in contrast to the assumption. □

Let $n \in \mathbb{N}$ be big enough to satisfy $2c_1 < \delta n / 2 \lg_2 n$, and let j_n be the greatest natural number such that

$$
2^{j_n} c_{j_n} < \frac{\delta n}{2 \lg_2 n}. \qquad (81)
$$

It follows by equation (81) that $j_n \leq m_n$. Moreover, (j_n) is a non-decreasing (starting from n big enough) sequence such that $\lim_{n \to +\infty} j_n = +\infty$.

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Let N be a natural number such that $c_j \leq \alpha^j$ for every $j \geq j_N$ and $2c_1 < \delta N/2 \lg_2 N$. If $n \geq N$, then

$$j_n \geq \lg_{2\alpha}\left(\frac{\delta n}{2\lg_2 n}\right) - 1. \tag{82}$$

Indeed, otherwise

$$2^{j_n+1}c_{j_n+1} \leq (2\alpha)^{j_n+1} < \frac{\delta n}{2\lg_2 n}, \tag{83}$$

which is a contradiction with the choice of j_n.

LEMMA 3.35. *For any sequence* $\mathbf{r} = (r_1, \ldots, r_{m_n})$, *there exists* $x_\mathbf{r} \in \mathbb{Z}$ *such that*

$$\begin{cases} x_\mathbf{r} \equiv 2^j r_j \bmod 2^j c_j & \text{for } j = 1, \ldots, m_n, \\ x_\mathbf{r} \equiv 0 \bmod 2^{m_n+1}3^{m_n+1}. \end{cases}$$

Moreover, if $\mathbf{r}' = (r'_1, \ldots, r'_{m_n})$ *is another sequence of integers and*

$$\eta[x_\mathbf{r}+1, x_\mathbf{r}+n] = \eta[x_{\mathbf{r}'}+1, x_{\mathbf{r}'}+n], \tag{84}$$

where $x_{\mathbf{r}'}$ *is defined analogously, then*

$$r_j \equiv r'_j \bmod c_j$$

for $j \leq j_n$, *provided* $n \geq N$.

Proof. The existence of $x_\mathbf{r}$ follows by the Chinese Remainder Theorem. Assume equation (84). Since $2^{m_n+1}3^{m_n+1} \mid x_\mathbf{r}$, $2^{m_n+1}3^{m_n+1} \mid x_{\mathbf{r}'}$ and $2^{m_n+1} > n$, the sets $[x_\mathbf{r}+1, x_\mathbf{r}+n]$ and $[x_{\mathbf{r}'}+1, x_{\mathbf{r}'}+n]$ are disjoint to $\bigcup_{i>m_n}(2^i c_i\mathbb{Z} \cup 3^i c_i\mathbb{Z})$. Therefore, equation (84) implies

$$[1,n] \cap \bigcup_{i=1}^{m_n}((2^i c_i\mathbb{Z} \cup 3^i c_i\mathbb{Z}) - x_\mathbf{r}) = [1,n] \cap \bigcup_{i=1}^{m_n}((2^i c_i\mathbb{Z} \cup 3^i c_i\mathbb{Z}) - x_{\mathbf{r}'}).$$

In particular,

$$[1,n] \cap (2^j c_j\mathbb{Z} - x_\mathbf{r}) \subseteq \bigcup_{i=1}^{m_n}((2^i c_i\mathbb{Z} \cup 3^i c_i\mathbb{Z}) - x_{\mathbf{r}'})$$

for every $j = 1, \ldots, m_n$. If $j \leq j_n$, then $2^j c_j < \delta n/2\lg_2 n$ and by Lemma 3.34, we conclude that $x_\mathbf{r} \equiv x_{\mathbf{r}'} \bmod 2^j c_j$. As $x_\mathbf{r} \equiv 2^j r_j \bmod 2^j c_j$ and $x_{\mathbf{r}'} \equiv 2^j r'_j \bmod 2^j c_j$, it follows that $r_j \equiv r'_j \bmod c_j$. □

Proof of Proposition 3.30. Take $n \geq N$ big enough. By Lemma 3.35, to every sequence $\mathbf{r} = (r_1, \ldots, r_{m_n})$ of integers, we can associate a block of length n on η, and the remainders of r_j modulo c_j for $j \leq j_n$ are determined uniquely by the block. (The choice is not unique. The conditions on $x_\mathbf{r}$ given in Lemma 3.35 do not determine $\eta[x_\mathbf{r}+1, x_\mathbf{r}+n]$ uniquely.) It follows that

$$\rho(n) \geq c_1 \ldots c_{j_n}. \tag{85}$$

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Remark 3.32 yields that

$$c_1 \ldots c_{j_n} \geq j_n^{j_n}. \tag{86}$$

We observed in equation (82) that $j_n \geq \lg_{2\alpha}(\delta n/(2\lg_2 n)) - 1$. Let $0 < \varepsilon < 1$. The right-hand side of this inequality is greater than $\lg_{2\alpha}(n^\varepsilon)$ for $n \gg 0$. Thus, for n big enough, we have

$$j_n^{j_n} \geq (\varepsilon \lg_{2\alpha} n)^{(\varepsilon \lg_{2\alpha} n)} = \varepsilon^{(\varepsilon \lg_{2\alpha} n)} n^{\varepsilon \lg_{2\alpha} \lg_{2\alpha} n}. \tag{87}$$

Putting this together with equations (85) and (86), we finish the proof of the proposition. □

Remark 3.36. Analogous (even simpler) arguments can be applied to the example $\mathcal{B}_1^1$, with the same conclusion about the complexity.

Acknowledgements. The research of the first and second authors is supported by Narodowe Centrum Nauki grant UMO-2019/33/B/ST1/00364. We are indebted to Daniel Sell for several very helpful remarks and corrections to a previous version of this manuscript.

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https://doi.org/10.1017/etds.2023.43 Published online by Cambridge University Press