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BERNSTEIN AND MARKOV TYPE INEQUALITIES FOR GENERALIZED NON-NEGATIVE POLYNOMIALS

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ABSTRACT. Generalized polynomials are defined as products of polynomials raised to positive real powers. The generalized degree can be introduced in a natural way. Several inequalities holding for ordinary polynomials are expected to be true for generalized polynomials, by utilizing the generalized degree in place of the ordinary one. Based on Remez-type inequalities on the size of generalized polynomials, we establish Bernstein and Markov type inequalities for generalized non-negative polynomials, obtaining the best possible result up to a multiplicative absolute constant.

1. **Introduction.** Bernstein's inequality asserts that

$$\max_{-\pi \le t \le \pi} |p'(t)| \le n \max_{-\pi \le t \le \pi} |p(t)| \tag{1.1}$$

for all real trigonometric polynomials of degree at most n. The corresponding algebraic result is known as Markov's inequality and states that

$$\max_{-1 \le x \le 1} |p'(x)| \le n^2 \max_{-1 \le x \le 1} |p(x)| \tag{1.2}$$

for all real algebraic polynomial of degree at most n. These results show how fast a polynomial of degree at most n can change, and play a very significant role in approximation theory. In the next section we introduce generalized polynomials of generalized degree and we extend the validity of the above inequalities for them up to a multiplicative constant. To prove our theorems the classical methods fail to work, and a completely different approach is required.

2. **Generalized non-negative polynomials: notations and definitions.** Denote by Π_n the set of all real algebraic polynomials of degree at most n. The family of all real trigonometric polynomials of degree at most n will be denoted by T_n. The functions

$$f = \prod_{j=1}^{k} P_{n_j}^{r_j} \quad \left(P_{n_j} \in \Pi_{n_j} \setminus \Pi_{n_j-1},\ r_j > 0,\ j = 1, 2, \ldots, k\right) \tag{2.1}$$

and

$$f = \prod_{j=1}^{k} P_{n_j}^{r_j} \left(P_{n_j} \in T_{n_j} \setminus T_{n_j-1},\ r_j > 0,\ j = 1, 2, \ldots, k\right) \tag{2.2}$$

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will be called generalized real algebraic polynomials and generalized real trigonometric polynomials, respectively, of (generalized) degree

$$N = \sum_{j=1}^{k} r_j n_j. \tag{2.3}$$

To be precise, in this paper we will use the definition

$$z^r = \exp(r \log |z| + ir \arg z) \qquad (z \in \mathbb{C},\ r \in \mathbb{R},\ -\pi \leq \arg z < \pi). \tag{2.4}$$

Obviously

$$|f| = \prod_{j=1}^{k} |P_{n_j}|^{r_j}. \tag{2.5}$$

We will denote by $GRAP_N$ the set of all generalized real algebraic polynomials of degree at most N. The family of all generalized real trigonometric polynomials of degree at most N will be denoted by $GRTP_N$. We introduce the classes $|GRAP|_N = \{\,|f| : f \in GRAP_N\}$ and $|GRTP|_N = \{\,|f| : f \in GRTP_N\}$. The function

$$f(z) = c \prod_{j=1}^{k} (z - z_j)^{r_j} \qquad (0 \neq c \in \mathbb{C},\ z_j \in \mathbb{C},\ r_j > 0,\ j = 1, 2, \ldots, k) \tag{2.6}$$

will be called a generalized complex algebraic polynomial of (generalized) degree

$$N = \sum_{j=1}^{k} r_j. \tag{2.7}$$

We have

$$|f(z)| = |c| \prod_{j=1}^{k} |z - z_j|^{r_j}. \tag{2.8}$$

Denote by $GCAP_N$ the set of all generalized complex algebraic polynomials of degree at most N. The set $\{\,|f| : f \in GCAP_N\}$ will be denoted by $|GCAP|_N$.

In the trigonometric case we say that the function

$$f(z) = c \prod_{j=1}^{k} \Big(\sin\big((z - z_j)/2\big)\Big)^{r_j} \qquad (0 \neq c \in \mathbb{C},\ z_j \in \mathbb{C},\ r_j > 0,\ j = 1, 2, \ldots, k) \tag{2.9}$$

is a generalized complex trigonometric polynomial of (generalized) degree

$$N = \frac{1}{2} \sum_{j=1}^{k} r_j. \tag{2.10}$$

We have

$$|f(z)| = |c| \prod_{j=1}^{k} \big|\sin\big((z - z_j)/2\big)\big|^{r_j}. \tag{2.11}$$

Denote the set of all generalized complex trigonometric polynomials of degree at most N by $GCTP_N$. The set $\{|f| : f \in GCTP_N\}$ wil be denoted by $|GCTP|_N$. We remark that if $f \in |GCAP|_N$, then restricted to the real line we have $f \in |GRAP|_N$. Similarly, if $f \in |GCTP|_N$, then restricted to the real line we have $f \in |GRTP|_N$. These follow from the observations

$$|z - z_j| = \big((z - z_j)(z - \bar{z}_j)\big)^{1/2} \qquad (z \in \mathbb{R}) \tag{2.12}$$

and

$$\begin{aligned} |\sin\big((z - z_j)/2\big)| &= |\sin\big((z - z_j)/2\big)\sin\big((z - \bar{z}_j)/2\big)|^{1/2} \\ &= \frac{1}{\sqrt{2}}\big(\cosh(\operatorname{Im} z_j) - \cos(z - \operatorname{Re} z_j)\big) \end{aligned} \tag{2.13}$$

Using (2.12) and (2.13) one can easily check that restricted to the real line

$$|GCAP|_N = \left\{f = \prod_{j=1}^{k} P_j^{r_j/2} : 0 \le P_j \in \Pi_2,\ r_j > 0, j = 1, 2, \ldots, k;\ \sum_{j=1}^{k} r_j \le N\right\}$$

and

$$|GCTP|_N = \left\{f = \prod_{j=1}^{k} P_j^{r_j/2} : 0 \le P_j \in T_1,\ r_j > 0, j = 1, 2, \ldots, k;\ \sum_{j=1}^{k} r_j \le 2N\right\}$$

The subject of this paper is the classes $|GCAP|_N$ and $|GCTP|_N$ restricted to the real line, and the elements of these classes can be considered as generalized non-negative polynomials in the above sense. This explains the title. To express our information on the Lebesgue measure of the subset of $[-1, 1]$, or $[-\pi, \pi)$, respectively, where the modulus of a generalized polynomial is not greater than 1, we introduce the notation

$$GCAP_N(\delta) = \Big\{f \in GCAP_N : m\big(\{x \in [-1, 1] : |f(x)| \le 1\}\big) \ge 2 - \delta\Big\}\ (0 < \delta < 2)$$

and

$$GCTP_N(\delta) = \Big\{f \in GCTP_N : m\big(\{t \in [-\pi, \pi) : |f(t)| \le 1\}\big) \ge 2\pi - \delta\Big\}\ (0 < \delta < 2\pi).$$

In what follows we study every function restricted to the real line. Throughout this paper c_i will denote positive absolute constants.

3. **New Results.** In this paper f' will mean either the left or the right hand side derivative with respect to the real variable. Our main result is the following extension of Bernstein's inequality.

THEOREM 3.1. *Let $f \in |GCTP|_N$ be of the form (2.11) with each $r_j \ge 1$ $(1 \le j \le k)$. Then*

$$\max_{-\pi \le t \le \pi} |f'(t)| \le c_1 N \max_{-\pi \le t \le \pi} |f(t)|,$$

where $c_1 > 0$ is an absolute constant.

The problem arises how to define f' for an $f \in |GCTP|_N$. Observe that though f' may not exist at the zeros of f, the one-sided derivatives exist and their absolute values are equal to each other. This means $|f'|$ is well-defined on the real line. Using the substitution $x = \cos t$, from Theorem 3.1 we immediately obtain

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THEOREM 3.2. *Let $f \in |GCAP|_N$ be of the form (2.8) with each $r_j \geq 1$ $(1 \leq j \leq k)$. Then*

$$|f'(y)| \leq \frac{c_1 N}{\sqrt{1-y^2}} \max_{-1\leq x\leq 1} |f(x)| \qquad (-1 < y < 1),$$

where $c_1 > 0$ is the same absolute constant as in Theorem 3.1.

This pointwise inequality does not give any information at the endpoints. The extension of Markov's inequality is established by

THEOREM 3.3. *Let $f \in |GCAP|_N$ be of the form (2.8) with each $r_j \geq 1$ $(1 \leq j \leq k)$. Then*

$$\max_{-1\leq x\leq 1} |f'(x)| \leq c_2 N^2 \max_{-1\leq x\leq 1} |f(x)|,$$

where $c_2 > 0$ is an absolute constant.

We remark that in Theorems 3.1, 3.2 and 3.3 the condition $r_j \geq 1$ $(1 \leq j \leq k)$ is needed to guarantee the differentiability of f.

4. **Lemmas for Theorems 3.1, 3.2 and 3.3.** To prove Theorems 3.1, 3.2 and 3.3 we need a series of lemmas. To obtain Theorem 3.3 from Theorem 3.2, we will need a Chebyshev-type inequality, namely we will need to estimate the maximum modulus of an $f \in GCAP_N$ on $[-1, 1]$ if its maximum modulus on $[-\alpha, \alpha]$, is 1 for some $0 < \alpha < 1$. In [2] a much more general Remez-type inequality is established for generalized complex algebraic polynomials. How large can the maximum modulus of an $f \in GCAP_N$ on $[-1, 1]$ be if the measure of the subset of $[-1, 1]$, where $|f|$ is not greater than 1, is prescribed, that is $f \in GCAP_N(\delta)$ for some $0 < \delta < 2$? This was answered by Remez [5] for ordinary polynomials, and proofs are available in [3] and [4] as well. The following extension for generalized complex algebraic polynomials was proved in [2] which preserves the best possible order of magnitude.

LEMMA 4.1. *We have*

$$\max_{-1\leq x\leq 1} |f(x)| \leq \exp(5N\sqrt{\delta}) \qquad (0 < \delta \leq 1)$$

and

$$\max_{-1\leq x\leq 1} |f(x)| \leq \exp\left(\frac{8N}{2-\delta}\right) \qquad (1 < \delta \leq 2)$$

for every $f \in GCAP_N(\delta)$.

When $\delta = 1/(25N^2 + 1)$, Theorem 4.1 yields

COROLLARY 4.2. *Let $\alpha = 1 - 1/(50N^2 + 2)$. Then*

$$\max_{-1\leq x\leq 1} |f(x)| \leq e \max_{-\alpha\leq x\leq \alpha} |f(x)|$$

for every $f \in GCAP_N$.

In the proof of Theorem 3.1 a trigonometric Remez-type inequality, established in [2], will play an important role. How large can the maximum modulus of an $f \in GCTP_N$ on $[-\pi, \pi]$ be if the measure of the subset of $[-\pi, \pi]$, where $|f|$ is not greater than 1, is prescribed, that is $f \in GCTP_N(\delta)$ for some $0 < \delta < 2\pi$? A satisfactory answer is given by

LEMMA 4.3. *There is an absolute constant* c_3 *such that*

$$\max_{-\pi \le t \le \pi} |f(t)| \le \exp(c_3 N\delta) \qquad (0 < \delta \le \pi/2)$$

for every $f \in GCTP_N(\delta)$.

Our next lemma gives an upper bound for the number of the zeros of a trigonometric polynomial on an interval with prescribed length.

LEMMA 4.4. *Let* $p \in T_n$ *and* $t_0 \in \mathbb{R}$. *Then* p *has at most*

$$\frac{3nh \max_{-\pi \le t \le \pi} |p(t)|}{|p(t_0)|}$$

zeros (counting multiplicities) in the interval $[t_0 - h, t_0 + h]$.

Though this lemma can be found in [1] (see Lemma 1), we will present its short proof here as well. Using Lemmas 4.3 and 4.4 we will prove

LEMMA 4.5. *Assume that* $p \in T_n$ *has only real zeros and at least one of any two adjacent zeros of* p *has multiplicity at least* s. *Then there is an interval* I *such that* $m(I) \ge c_4 s/n$ *and*

$$|p(\tau)| \ge \exp(-s) \max_{-\pi \le t \le \pi} |p(t)| \qquad (\tau \in I),$$

where $c_4 > 0$ *is an absolute constant.*

In the proof of Lemma 4.5 it will be crucial to find a $Q_{n,\omega} \in T_n$ $(0 < \omega < \pi)$ for which $|Q_{n,\omega}(x)| \le 1$ on $[-\omega, \omega]$, $|Q_{n,\omega}(x)| \le |Q_{n,\omega}(\pi)|$ on $[-\pi, \pi]$, and $|Q_{n,\omega}(\pi)|$ is as large as possible. In [2] it is shown that the trigonometric polynomial

$$Q_{n,\omega}(t) = Q_{2n}\left(\frac{\sin(t/2)}{\sin(\omega/2)}\right) \in T_n,$$

where $Q_{2n}(x) = \cos(2n \arccos x)$ $(-1 \le x \le 1)$ is the Chebyshev polynomial of degree $2n$, solves this extremal problem. We remark that the above polynomial was used by V. S. Videnskii [6] to establish Markov and Bernstein type inequalities for the derivative of trigonometric polynomials on an interval shorter than the period. In the proof of Lemma 4.5 we will need the order of magnitude of $|Q_{n,\omega}(\pi)|$ which is given by

LEMMA 4.6. *We have*

$$\exp\big(c_5 n(\pi-\omega)\big) \le Q_{n,\omega}(\pi) \le \exp\big(c_6 n(\pi-\omega)\big) \qquad (\pi/2 \le \omega < \pi),$$

where $0 < c_5 < c_6$ *are absolute constants.*

Approximating the exponents by rational numbers not less than 1, from Lemma 4.5 we will conclude

LEMMA 4.7. *Assume that* $g \in |GCTP|_N$ *has only real zeros and at least one of any two adjacent zeros of g has multiplicity at least 1 (in case of* $g \in |GCTP|_N$ *the multiplicity may be any positive real number). Then there is an interval I such that* $m(I) \ge c_4/N$ *and*

$$g(\tau) \ge e^{-1} \max_{-\pi \le t \le \pi} g(t) \qquad (\tau \in I),$$

where c_4 *is the same as in Theorem 4.5.*

As we have remarked in Section 2, if $f \in |GCTP|_N$ is of the form (2.11), then, restricted to the real line, $f \in |GRTP|_N$ is of the form

$$f = \prod_{j=1}^{k} P_j^{r_j/2} \quad (0 \le P_j \in T_1,\ r_j > 0,\ j = 1,2,\ldots,k).$$

By the following two lemmas, in the proof of Theorem 3.1 we may assume that $f \in |GCTP|_N$ has only real zeros.

LEMMA 4.8. *Let* $r_j \ge 1$ *be fixed real numbers for* $j = 1,2,\ldots,k$. *Then there exists a generalized non-negative real trigonometric polynomial F of the form*

$$F = \prod_{j=1}^{k} \tilde{P}_j^{r_j/2} \quad (\tilde{P}_j \in T_1,\ \tilde{P}_j(z) \ge 0,\ j = 1,2,\ldots,k,\ z \in \mathbb{R})$$

such that

$$\frac{|F'(\pi)|}{\max_{-\pi \le t \le \pi} F(t)} = \sup_{|f|} \frac{|f'(\pi)|}{\max_{-\pi \le t \le \pi} f(t)} = L,$$

where the supremum is taken for all not identically zero $f \in |GCTP|_N$ *of the form*

$$f = \prod_{j=1}^{k} P_j^{r_j/2} \quad (P_j \in T_1,\ P_j(z) \ge 0,\ j = 1,2,\ldots,k,\ z \in \mathbb{R}).$$

LEMMA 4.9. *The function F, defined by Lemma 4.8, has only real zeros.*

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5. **Proof of the Lemmas.**

PROOF OF LEMMA 4.4. For the sake of brevity let $m = [3nhw] + 1$ with

$$w = \frac{\max_{-\pi\le t\le\pi} |p(t)|}{|p(t_0)|},$$

where $[x]$ denotes the greatest integer not exceeding x. Assume that the statement of the lemma is false, thus there are $t_1 < t_2 < \cdots < t_\nu$ in $[t_0 - h, t_0 + h]$ such that p has at least μ_i repeated roots at t_i $(1 \le i \le \nu)$ and $\sum_{i=1}^{\nu} \mu_i = m$. We introduce the polynomial

$$\Omega(x) = \prod_{i=1}^{\nu} (x - t_i)^{\mu_i}.$$

By a well-known relation for the remainder of the Hermite interpolating polynomial, there exists a $\xi \in [t_1, t_\nu] \subset [t_0 - h, t_0 + h]$ such that

$$p(t_0) - H(t_0) = \frac{1}{m!} p^{(m)}(\xi)\Omega(t_0),$$

where $H \in \Pi_{m-1}$ and $H^{(j)}(t_i) = p^{(j)}(t_i) = 0$ $(1 \le i \le \nu,\ 0 \le j \le \mu_i - 1)$, thus $H \equiv 0$. Hence, by $m! > (m/e)^m$ and $s \ge 1$ we have

$$\begin{aligned}|p^{(m)}(\xi)| &\ge (m/e)^m h^{-m} |p(t_0)| \ge (m/e)^m (3wn/m)^m |p(t_0)| \\ &= (3w/e)^m \frac{1}{w} n^m \max_{-\pi\le t\le\pi} |p(t)| > n^m \max_{-\pi\le t\le\pi} |p(t)|\end{aligned}$$

which contradicts Bernstein's inequality. Thus the lemma is proved. ■

PROOF OF LEMMA 4.6. Observe that

$$\frac{\sin(\pi/2)}{\sin(\omega/2)} = 1 + \frac{1 - \sin(\omega/2)}{\sin(\omega/2)} = 1 + \frac{2\sin^2\frac{\pi-\omega}{4}}{\sin(\omega/2)},$$

therefore

$$1 + \frac{c_7(\pi - \omega)^2}{\omega} \le \frac{\sin(\pi/2)}{\sin(\omega/2)} \le 1 + \frac{c_8(\pi - \omega)^2}{\omega}.$$

Hence, using the well-known formula

$$Q_n(x) = \frac{1}{2}\Big(\big(x + (x^2 - 1)^{1/2}\big)^n + \big(x - (x^2 - 1)^{1/2}\big)^n\Big) \quad \big(x \in \mathbb{R} \setminus (-1, 1)\big),$$

we get the desired inequalities by a straightforward calculation. ■

PROOF OF LEMMA 4.5. Because of the periodicity we may asume that

$$p(\pi) = \max_{-\pi\le t\le\pi} |p(t)|. \tag{5.1}$$

We define

$$Q_{n,\omega}(t) = Q_{2n}\Big(\frac{\sin(t/2)}{\sin(\omega/2)}\Big) \text{ with } \omega = \pi - s/(3n), \tag{5.2}$$

where $Q_{2n}(x) = \cos(2n \arccos x)$ $(-1 \leq x \leq 1)$ is the Chebyshev polynomial of degree $2n$. We introduce the set

$$A = \{t \in [\pi - s/(3n), \pi + s/(3n)] : |p(t)| \geq \exp(-s)p(\pi)\}. \tag{5.3}$$

We study the trigonometric polynomial $q = pQ_{n,\omega} \in T_{2n}$. Assumption (5.1) and the inequality $Q_{n,\omega}(\pi) \geq \exp(c_5 s/3)$ $\big(\omega = \pi - s/(3n)\big)$ imply

$$\begin{aligned} |q(\tau)| &\leq |p(\tau)| \leq \exp(-c_5 s/3) Q_{n,\omega}(\pi) p(\pi) \\ &= \exp(-c_9 s) \max_{-\pi \leq t \leq \pi} |q(t)| \quad (-\omega \leq \tau \leq \omega). \end{aligned} \tag{5.4}$$

Further, by the definition of the set A, we obtain

$$\begin{aligned} |q(\tau)| &\leq \exp(-s) p(\pi) Q_{n,\omega}(\pi) \\ &= \exp(-s) \max_{-\pi \leq t \leq \pi} |q(t)| \quad \big(\tau \in (\omega, 2\pi - \omega) \setminus A\big). \end{aligned} \tag{5.5}$$

Summarizing (5.4) and (5.5), we conclude

$$|q(\tau)| \leq \exp(-c_{10} s) \max_{-\pi \leq t \leq \pi} |q(t)| \quad (\tau \in [-\pi, \pi] \setminus A). \tag{5.6}$$

Now by Lemma 4.3, from (5.6) we easily deduce that $m(A) \geq c_{11} s/n$. To finish the proof of the lemma we show that A is the union of at most two intervals (hence the larger one satisfies the requirements). Indeed, Lemma 4.4, (5.1) and the prescribed information on the multiplicity of the zeros of p imply that p has at most one zero (possibly with higher multiplicity) in $[\pi - s/(3n), \pi + s/(3n)]$. Since p has only real zeros, A is the union of at most two intervals, indeed. Thus the lemma is proved. ∎

PROOF OF LEMMA 4.7. Let $g \in |GCTP|_N$ be of the form

$$g = \prod_{j=1}^{k} P_j^{r_j/2} \quad (P_j \in T_1, P_j(z) \geq 0,\ j = 1, 2, \ldots, k,\ z \in \mathbb{R}). \tag{5.7}$$

Because of the periodicity of g we may assume that

$$|g(\pi)| = \max_{-\pi \leq t \leq \pi} |g(t)|. \tag{5.8}$$

If each $r_j = s_j/s$ is rational in the representation (5.7) of $g \in |GCTP|_N$, then the trigonometric polynomial

$$p = g^{2s} = \prod_{j=1}^{k} P_j^{s_j} \in T_n \text{ with } n = \sum_{j=1}^{k} s_j$$

satisfies the conditions of Lemma 4.5 with $2s$ instead of s. Therefore there is an interval I such that $m(I) \geq 2c_4 s/n = c_4/N$ and

$$|p(\tau)| \geq \exp(-2s) \max_{-\pi \leq t \leq \pi} |p(t)| \qquad (\tau \in I).$$

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Hence

$$g(\tau) = |p(\tau)|^{1/(2s)} \geq e^{-1} \max_{-\pi \leq t \leq \pi} |p(t)|^{1/(2s)} = \max_{-\pi \leq t \leq \pi} g(t) \qquad (\tau \in I)$$

with $m(I) \geq c_4/N$. Since $c_4 > 0$ is an absolute constant, we can eliminate the assumption that each r_j is rational in the representation (5.7) of $g \in |GCTP|_N$ by a density argument. Thus the lemma is proved. ∎

PROOF OF LEMMA 4.8. Let f_i $(i = 1, 2, \ldots)$ be such that

$$\frac{|f_i'(\pi)|}{\max_{-\pi \leq t \leq \pi} |f_i(t)|} \geq \min\left\{i, L - i^{-1}\right\},$$

where

$$f_i = \prod_{j=1}^{k} P_{j,i}^{r_j/2} \quad \left(P_{j,i} \in T_1,\ P_{j,i}(z) \geq 0,\ j = 1, 2, \ldots, k,\ z \in \mathbb{R}\right).$$

Without loss of generality we may assume that

$$\max_{-\pi \leq t \leq \pi} |P_{j,i}(t)| = 1 \qquad (j = 1, 2, \ldots, k,\ i = 1, 2, \ldots).$$

Then there is a subsequence $\{i_m\}$ and polynomials $\tilde{P}_j \in T_1$ such that

$$\lim_{m \to \infty} \max_{-\pi \leq t \leq \pi} |(P_{j,i_m} - \tilde{P}_j)(t)| = 0 \qquad (j = 1, 2, \ldots, k).$$

Now it is easy to check that

$$F = \prod_{j=1}^{k} \tilde{P}_j^{r_j/2}$$

satisfies the requirements of the lemma. ∎

PROOF OF LEMMA 4.9. Assume to the contrary that there is a nonreal a such that $\tilde{P}_m(a) = 0$ for some $1 \leq m \leq k$. Then

$$F_\varepsilon(z) = \left(\tilde{P}_m(z)\left(1 - \frac{\varepsilon \sin^2\left((z - \pi)/2\right)}{\sin\left((z - a)/2\right)\sin\left((z - \bar{a})/2\right)}\right)\right)^{r_m/2} \prod_{\substack{j=1 \\ j \neq m}}^{k} \tilde{P}_j^{r_j/2}(z)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of F. (Observe that F does not take its maximum on $[-\pi, \pi]$ at π, since $F'(\pi)$ does not vanish because of its maximality.)

∎

6. **Proof of the Theorems.**

PROOF OF THEOREM 3.1. If $f \in |GCTP|_N$ is of the form (2.11) with each $r_j \geq 1$, then it can be written as

$$f = \prod_{j=1}^{k} P_j^{r_j/2} \quad (P_j \in T_1,\ P_j(z) \geq 0,\ j = 1, 2, \ldots, k,\ z \in \mathbb{R}) \tag{6.1}$$

with each $r_j \geq 1$. Because of the periodicity of $|f'|$ it is sufficient to prove that

$$|f'(\pi)| \leq c_1 N \max_{-\pi \leq t \leq \pi} |f(t)|. \tag{6.2}$$

To show this, by Lemmas 4.8 and 4.9 we may assume that f has only real zeros, hence $f \in |GCTP|_N$ is of the form (2.11) with each $r_j \geq 1$. Because of the periodicity of $|f'|$ we may assume that

$$|f'(\pi)| = \max_{-\pi \leq t \leq \pi} |f'(t)|. \tag{6.3}$$

Since $f \in |GCTP|_N$ has only real zeros and each $r_j \geq 1$ in its representation (6.1), a routine application of Rolle's Theorem and a simple calculation show that $|f'| \in |GCTP|_N$ has only real zeros, and at least one of any two adjacent zeros of $|f'|$ has multiplicity at least 1, thus $g = |f'|$ satisifies the conditions of Lemma 4.7. Denote the endpoints of the interval I coming from Lemma 4.7 by a and b. From Lemma 4.7 we deduce

$$\begin{aligned} \max_{-\pi \leq t \leq \pi} |f'(t)| = |f'(\pi)| &\leq \frac{e}{b-a} \int_a^b |f'(t)|\, dt \leq \frac{N}{c_4} \int_a^b |f'(t)|\, dt \\ &\leq c_{12} N |f(b) - f(a)| \leq c_1 N \max_{-\pi \leq t \leq \pi} |f(t)| \end{aligned}$$

which proves the theorem. ∎

PROOF OF THEOREM 3.2. If $f \in |GCAP|_N$ is of the form (2.8) with each $r_j \geq 1$, then it can be written as

$$f = \prod_{j=1}^{k} P^{r_j/2} \quad (P_j \in \Pi_2,\ P_j(z) \geq 0,\ j = 1, 2, \ldots, k,\ z \in \mathbb{R}) \tag{6.4}$$

with each $r_j \geq 1$. Hence it is easy to see that $g(z) = f(\cos z) \in |GCTP|_N$ is of the form (2.11) with each $r_j \geq 1$ if z is real. Applying Theorem 3.1 to g, we obtain Theorem 3.2 immediately. ∎

PROOF OF THEOREM 3.3. Let α be the same as in Corollary 4.2. From Theorem 3.1 and Corollary 4.2 we easily obtain

$$\max_{-\alpha \leq x \leq \alpha} |f'(x)| \leq c_{13} N^2 \max_{-1 \leq x \leq 1} |f(x)| \leq c_{13} e N^2 \max_{-\alpha \leq x \leq \alpha} |f(x)| \tag{6.5}$$

for every $f \in |GCAP|_N$ of the form (2.8) with each $r_j \geq 1$. The theorem can be obtained from (6.5) by a linear transformation. ∎

7. **Immediate consequences.** In this section $c(m)$ denotes a suitable constant depending only on m. From Theorems 3.1, 3.2 and 3.3, by induction on m, we easily obtain the following.

COROLLARIES. *We have*

$$\max_{-\pi\le t\le\pi} |f^{(m)}(t)| \le c(m)N^m \max_{-\pi\le t\le\pi} |f(t)|$$

for every $f \in |GCTP|_N$ such that each z_j in (2.11) is real and each r_j in (2.11) is either positive integer or real greater than m. Further, the inequalities

$$|f^{(m)}(y)| \le c(m)\Big(\frac{N}{\sqrt{1-y^2}}\Big)^m \max_{-1\le x\le 1} |f(x)| \quad (-1 < y < 1)$$

and

$$\max_{-1\le x\le 1} |f^{(m)}(x)| \le c(m)N^{2m} \max_{-1\le x\le 1} |f(x)|$$

hold for every $f \in |GCAP|_N$ such that each z_j in (2.8) is real and each r_j in (2.8) is either positive integer or real greater than m.

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REFERENCES

1. T. Erdélyi, *Markov-type estimate for certain classes of constrained polynomials*, Constructive Approximation **5**(1989), 347–356.

2. ———, *Remez-type inequalities on the size of generalized polynomials*, J. London Math. Soc., to appear.

3. ———, *The Remez inequality on the size of polynomials*. Approximation Theory VI, (C K. Chui, L. L. Schumaker and J. D. Ward eds.), Academic Press, Boston, 1989, **1**, 243–246.

4. G. Freud, *Orthogonal polynomials*. Pergamon Press, Oxford, 1971.

5. E. J. Remez, *Sur une propriété des polynômes de Tchebycheff*, Communications de l'Inst. des Sci. Kharkow **13**(1936), 93–95.

6. V. S. Videnskii, *Markov and Bernstein type inequalities for derivatives of trigonometric polynomials on an interval shorter than the period*, (Russian) Dokl. Acad. Nauk, USSR (1)**130**(1960), 13–16.

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