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FINITE-GAP INTEGRATION OF THE $SU(2)$ BOGOMOLNY EQUATIONS

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Abstract. The Atiyah–Drinfeld–Hitchin–Manin–Nahm (ADHMN) construction of magnetic monopoles is given in terms of the (normalizable) solutions of an associated Weyl equation. We focus here on solving this equation directly by algebro-geometric means. The (adjoint) Weyl equation is solved using an ansatz of Nahm in terms of Baker–Akhiezer functions. The solution of Nahm's equation is not directly used in our development.

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1. Introduction. Consider the dimensional reduction to three dimensions of the four-dimensional Yang–Mills Lagrangian with gauge group $SU(2)$ under the assumption that all fields are independent of time. Upon identifying the a_4-component of the gauge field with the Higgs field Φ we obtain the three-dimensional Yang–Mills–Higgs Lagrangian

$$L = -\frac{1}{2}\mathrm{Tr}\, F_{ij}F^{ij} + \mathrm{Tr}\, D_i\Phi\, D^i\Phi.$$

Here, $F_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j]$ is the curvature of the (spatial) connection of the gauge field $a_i(\boldsymbol{x})$ and D_i the covariant derivative $D_i\Phi = \partial_i\Phi + [a_i, \Phi]$, $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. We are interested in configurations minimizing the energy of the system. These are given by the *Bogomolny equation*

$$D_i\Phi = \pm \sum_{j,k=1}^{3} \epsilon_{ijk}F_{jk}, \quad i = 1, 2, 3. \tag{1.1}$$

A solution with the boundary conditions

$$\left.\sqrt{-\frac{1}{2}\mathrm{Tr}\,\Phi(r)^2}\right|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

is called a *monopole* of charge n.

In this paper we shall follow the Atiyah–Drinfeld–Hitchin–Manin–Nahm (ADHMN) construction (see the original papers [**4**], [**6**] and the recent review [**8**]).

This has its origin in the construction of instanton solutions to the (Euclidean) self-dual Yang–Mills equations by Atiyah–Drinfeld–Hitchin–Manin (ADHM): here the self-duality equations, partial differential equation in four variables, are transformed to an algebraic matrix equation. In the monopole setting the ADHMN construction reduces the Bogomolny equation, again a partial differential equation but now in three variables, to a system of ordinary differential equations. Our interest here will be to integrate these by algebro-geometric methods.

The standard approach to the integration of the Bogomolny equation within the ADHMN construction consists of two stages. First, an auxiliary equation known as Nahm's equation,

$$\frac{\mathrm{d}T_i(z)}{\mathrm{d}z} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(z), T_k(z)], \quad z \in [-1, 1] \tag{1.2}$$

is integrated for $n \times n$ matrices $T_i(z)$ subject to certain boundary conditions. The solution of this equation is then used to define a differential operator

$$\Delta^{\dagger} = \imath\, 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} - \sum_{j=1}^{3}(T_j(z) + \imath\, x_j 1_n) \otimes \sigma_j, \tag{1.3}$$

where σ_j are the Pauli matrices. The Higgs and gauge fields are then expressed as certain averages over the normalizable solutions $\boldsymbol{v}$ to the Weyl equation $\Delta^{\dagger}\boldsymbol{v}(\boldsymbol{x}, z) = 0$. Nahm's equation, introduced in the ADHMN construction, plays an important role in many problems of mathematical physics and the integration of this equation is of great significance. Its role in the integration of the Bogomolny equation is nevertheless an auxiliary one. In this paper, we shall concentrate on solving of the Weyl equation directly by algebro-geometric means using an ansatz (again of Nahm) that has not been considered previously in this light. Our new insight is that this ansatz may be solved in terms of a Baker–Akhiezer function. Although the associated spectral problem is equivalent to that appearing in the algebro-geometric integration of the Nahm equation, we do not use solutions of Nahm's equation directly in our development. To achieve our result we implement the θ-functional integration of the Nahm equation by Ercolani and Sinha [**3**] and our recent analysis [**1**]. The limitations of space in this volume prevent detailed examples being given and a fuller exposition will be given elsewhere.

2. The ADMHN construction. Set

$$\Delta = \imath\frac{\mathrm{d}}{\mathrm{d}z} + x - \imath\, T_4 + \boldsymbol{T}\cdot\boldsymbol{\sigma} = \imath\frac{\mathrm{d}}{\mathrm{d}z} - \imath\, R, \tag{2.1}$$

where

$$x = x_4 + \imath\,\boldsymbol{x}\cdot\boldsymbol{\sigma}, \quad T = T_4 + \imath\,\boldsymbol{T}\cdot\boldsymbol{\sigma}, \quad R = T + ix. \tag{2.2}$$

We will often assume we have chosen a gauge such that $T_4 = 0$ and that $x_4 = 0$. The ADHMN construction may be summarised in the following theorem:

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THEOREM 2.1 (ADMHN). *The charge n monopole solution of the Bogomolny equation is given by*

$$\Phi_{ab}(\boldsymbol{x}) = \imath \int_{-1}^{1} \mathrm{d}z\, z \boldsymbol{v}_a^{\dagger}(\boldsymbol{x}, z)\boldsymbol{v}_b(\boldsymbol{x}, z), \quad a, b = 1, 2, \tag{2.3}$$

$$A_{i\,ab}(\boldsymbol{x}) = \int_{-1}^{1} \mathrm{d}z\, \boldsymbol{v}_a^{\dagger}(\boldsymbol{x}, z)\frac{\partial}{\partial x_i}\boldsymbol{v}_b(\boldsymbol{x}, z), \quad i = 1, 2, 3, \quad a, b = 1, 2. \tag{2.4}$$

Here the two $(a = 1, 2)$ $2n$*-column vectors*[1] $\boldsymbol{v}_a(\boldsymbol{x}, z) = (v_1^{(a)}(\boldsymbol{x}, z), \ldots, v_{2n}^{(a)}(\boldsymbol{x}, z))^T$ *form an orthonormal basis on the interval* $z \in [-1, 1]$

$$\int_{-1}^{1} \mathrm{d}z\, \boldsymbol{v}_a^{\dagger}(\boldsymbol{x}, z)\boldsymbol{v}_b(\boldsymbol{x}, z) = \delta_{ab}, \tag{2.5}$$

for the normalizable solutions to the the Weyl equation

$$\Delta^{\dagger}\boldsymbol{v} = 0, \tag{2.6}$$

where $\Delta^{\dagger}$ *is given by* (1.3). *The normalizable solutions form a two-dimensional subspace of the solution space* $(\boldsymbol{v}^{(1)}(\boldsymbol{x}, z), \ldots, \boldsymbol{v}^{(2n)}(\boldsymbol{x}, z))$. *The* $n \times n$*-matrices* $T_j(z)$, *called Nahm data, satisfy Nahm's equation* (1.2) *and are required to satisfy the following boundary conditions: they are regular at* $z \in (-1, 1)$; *have simple poles at* $z = \pm 1$, *the residues of which form the irreducible n-dimensional representation of the* $su(2)$ *algebra; further*

$$T_i(z) = -T_i^{\dagger}(z), \quad T_i(z) = T_i^T(-z). \tag{2.7}$$

A proof consisting of direct verification may be found for example in the recent exposition by the Weinberg and Yi **[8]** and references therein.

The integrals in (2.3), (2.4) and (2.5) may be computed in the closed form **[7]** in the following way. Denote by

$$\mathcal{H}(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{\sigma} \otimes 1_n, \quad \mathcal{T}(z) = \imath \boldsymbol{\sigma} \otimes \boldsymbol{T}(z), \tag{2.8}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$, $\boldsymbol{T} = (T_1, T_2, T_3)^T$ and $\boldsymbol{x} \cdot \boldsymbol{\sigma} \otimes 1_n = \sum_{i=1}^{3} x_i\sigma_i \otimes 1_n$, $\boldsymbol{\sigma} \otimes \boldsymbol{T}(z) = \sum_{i=1}^{3} \sigma_i \otimes T_i(z)$. Introduce the $2n \times 2n$ matrix $\mathcal{Q}(\boldsymbol{x}, z)$,

$$\mathcal{Q}(\boldsymbol{x}, z) = \frac{1}{r^2}\mathcal{H}(\boldsymbol{x})\mathcal{T}(z)\mathcal{H}(\boldsymbol{x}) - \mathcal{T}(z). \tag{2.9}$$

Then the following formulae of Panagopoulos (see Appendix A) are valid for the normalizable Weyl spinors, $\boldsymbol{v}_{1,2}(\boldsymbol{x}, z)$,

[1] Throughout the paper vectors are column-vectors and printed in bold, e.g. $\boldsymbol{a}$; the superscript † means conjugated and transposed, e.g. for vector $\boldsymbol{a}^{\dagger} = \overline{\boldsymbol{a}}^T$, and this holds similarly for matrices.

$$\int \boldsymbol{v}_a^\dagger(\boldsymbol{x}, z)\boldsymbol{v}_b(\boldsymbol{x}, z)\mathrm{d}z = \boldsymbol{v}_a^\dagger(\boldsymbol{x}, z)\mathcal{Q}^{-1}(\boldsymbol{x}, z)\boldsymbol{v}_b(\boldsymbol{x}, z); \tag{2.10}$$

$$\int z\boldsymbol{v}_a^\dagger(\boldsymbol{x}, z)\boldsymbol{v}_b(\boldsymbol{x}, z)\mathrm{d}z = \boldsymbol{v}_a^\dagger(\boldsymbol{x}, z)\mathcal{Q}^{-1}(\boldsymbol{x}, z)\left[z + 2\mathcal{H}(\boldsymbol{x})\frac{\mathrm{d}}{\mathrm{d}(r^2)}\right]\boldsymbol{v}_b(\boldsymbol{x}, z); \tag{2.11}$$

$$\begin{aligned}&\int \boldsymbol{v}_a^\dagger(\boldsymbol{x}, z)\frac{\partial}{\partial x_i}\boldsymbol{v}_b(\boldsymbol{x}, z)\mathrm{d}z\\ &\quad = \boldsymbol{v}_a^\dagger(\boldsymbol{x}, z)\mathcal{Q}^{-1}(\boldsymbol{x}, z)\left[\frac{\partial}{\partial x_i} + \mathcal{H}(\boldsymbol{x})\frac{zx_i + \imath(\boldsymbol{x}\times\nabla)_i}{r^2}\right]\boldsymbol{v}_b(\boldsymbol{x}, z).\end{aligned} \tag{2.12}$$

Therefore, only the boundary values of the normalized Weyl spinors $\boldsymbol{v}_{1,2}(\boldsymbol{x}, \pm 1)$ together with their derivatives need be computed to find solutions to the Bogomolny equation. The Nahm data at these boundary values is also needed, and this involves the residues noted earlier.

3. The Nahm Ansatz. Although, we wish to solve

$$\Delta^\dagger \boldsymbol{v} = 0,$$

Nahm introduced an ansatz that provides solutions to

$$\Delta \boldsymbol{w} = 0$$

that we now recall. Consider solutions of the form

$$\boldsymbol{w} = (1_2 + \hat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})\, e^{\imath x_4 z}|s> \otimes \hat{\boldsymbol{w}}(z), \tag{3.1}$$

where $|s>$ is an arbitrary normalized spinor not in $\ker(1_2 + \hat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})$ and $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is (as we shall see) a unit vector independent of z. Substituting in $\Delta\boldsymbol{w} = 0$ we find

$$0 = |s> \otimes\left(\imath\frac{\mathrm{d}}{\mathrm{d}z} + \hat{\boldsymbol{u}}\cdot\boldsymbol{R}\right)\hat{\boldsymbol{w}}(z) + \sigma_k|s> \otimes\left(\imath\hat{u}^k\frac{\mathrm{d}}{\mathrm{d}z} + R^k + \imath(\boldsymbol{R}\times\hat{\boldsymbol{u}})^k\right)\hat{\boldsymbol{w}}(z)$$

and so we require

$$0 = \left(\imath\frac{\mathrm{d}}{\mathrm{d}z} + \hat{\boldsymbol{u}}\cdot\boldsymbol{R}\right)\hat{\boldsymbol{w}}(z), \tag{3.2}$$

$$0 = \mathcal{L}_k\hat{\boldsymbol{w}}(z) := \left(\imath\hat{u}^k\frac{\mathrm{d}}{\mathrm{d}z} + R^k + \imath(\boldsymbol{R}\times\hat{\boldsymbol{u}})^k\right)\hat{\boldsymbol{w}}(z). \tag{3.3}$$

The consistency of these equations imposes various constraints. First consider

$$\begin{aligned}[\mathcal{L}_1, \mathcal{L}_2] = {}&(\imath\hat{u}^1 + \hat{u}^2\hat{u}^3)(\dot{T}_2 - [T_3, T_1]) - (\imath\hat{u}^2 - \hat{u}^1\hat{u}^3)(\dot{T}_1 - [T_2, T_3])\\ &- (1 - (\hat{u}^3)^2)(\dot{T}_3 - [T_1, T_2]) + (1 - \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}})\,\dot{T}_3.\end{aligned}$$

Thus, provided $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is a unit vector and the T_i's satisfy the Nahm equations we have consistency of the equations $\mathcal{L}_k\hat{\boldsymbol{w}}(z) = 0$.

At this stage we introduce a convenient parameterization (reflected in Hitchin's minitwistor construction). Let $\boldsymbol{y} \in \mathbb{C}^3$ be a null vector. We may consider $\boldsymbol{y} \in \mathbb{P}^2$ and

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parameterize $\boldsymbol{y}$ as

$$\boldsymbol{y} = \left(\frac{1+\zeta^2}{2\imath}, \frac{1-\zeta^2}{2}, -\zeta \right). \tag{3.4}$$

Then

$$\boldsymbol{y} \cdot \overline{\boldsymbol{y}} = \frac{(1+|\zeta|^2)^2}{2}, \qquad \boldsymbol{y} \cdot \boldsymbol{y} = 0.$$

The signs here have been chosen so that

$$L(\zeta) := 2\imath\, \boldsymbol{y} \cdot \boldsymbol{T} = (T_1 + \imath T_2) - 2\imath T_3 \zeta + (T_1 - \imath T_2)\zeta^2.$$

In due course we will see this to be our Lax matrix. Set

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}(\zeta) := \imath \frac{\boldsymbol{y} \times \overline{\boldsymbol{y}}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}} = \frac{1}{1+|\zeta|^2}(\imath(\zeta - \overline{\zeta}),\ (\zeta + \overline{\zeta}), 1 - |\zeta|^2). \tag{3.5}$$

Then

$$\hat{\boldsymbol{u}} \times \boldsymbol{y} = -\imath\, \boldsymbol{y}, \qquad \hat{\boldsymbol{u}} \times \overline{\boldsymbol{y}} = \imath\, \overline{\boldsymbol{y}}.$$

The three vectors $\mathrm{Re}(\boldsymbol{y})$, $\mathrm{Im}(\boldsymbol{y})$ and $\hat{\boldsymbol{u}}$ form an orthogonal basis in $\mathbb{R}^3$ with $|\hat{\boldsymbol{u}}| = 1$, whence any $\boldsymbol{v} \in \mathbb{R}^3$ may be written as

$$\boldsymbol{v} = \hat{\boldsymbol{u}}\,(\hat{\boldsymbol{u}} \cdot \boldsymbol{v}) + \overline{\boldsymbol{y}} \left(\frac{\boldsymbol{y} \cdot \boldsymbol{v}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}} \right) + \boldsymbol{y} \left(\frac{\overline{\boldsymbol{y}} \cdot \boldsymbol{v}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}} \right).$$

In particular,

$$\boldsymbol{v} + \imath\, \boldsymbol{v} \times \hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}\,(\hat{\boldsymbol{u}} \cdot \boldsymbol{v}) + 2\overline{\boldsymbol{y}} \left(\frac{\boldsymbol{y} \cdot \boldsymbol{v}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}} \right). \tag{3.6}$$

We record that

$$\overline{\boldsymbol{y}(\zeta)} = -\overline{\zeta}^2 \boldsymbol{y}(-1/\overline{\zeta}), \qquad \hat{\boldsymbol{u}}(-1/\overline{\zeta}) = -\hat{\boldsymbol{u}}(\zeta),$$
$$\hat{\boldsymbol{u}} = (-\imath\zeta^{-1}, \zeta^{-1}, -1) - \frac{2\boldsymbol{y}}{\zeta(1+|\zeta|^2)} = (\imath\zeta, \zeta, 1) + \frac{2\overline{\zeta}\,\boldsymbol{y}}{1+|\zeta|^2},$$
$$\hat{\boldsymbol{u}} \cdot \boldsymbol{T} = -\imath\left[(T_1 + \imath T_2)\zeta^{-1} - \imath T_3\right] - \frac{2\boldsymbol{y} \cdot \boldsymbol{T}}{\zeta(1+|\zeta|^2)} = \imath[(T_1 - \imath T_2)\zeta - \imath T_3] - \frac{2\overline{\zeta}\,\boldsymbol{y} \cdot \boldsymbol{T}}{1+|\zeta|^2}.$$

Parameterizing $\hat{\boldsymbol{u}}$ as above and using (3.6) we may write

$$\begin{aligned} \imath\,\hat{\boldsymbol{u}} \frac{\mathrm{d}}{\mathrm{d}z} + \boldsymbol{R} + \imath\, \boldsymbol{R} \times \hat{\boldsymbol{u}} &= \imath\,\hat{\boldsymbol{u}} \frac{\mathrm{d}}{\mathrm{d}z} + \hat{\boldsymbol{u}}\,(\hat{\boldsymbol{u}} \cdot \boldsymbol{R}) + 2\overline{\boldsymbol{y}} \left(\frac{\boldsymbol{y} \cdot \boldsymbol{R}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}} \right) \\ &= \hat{\boldsymbol{u}} \left(\imath \frac{\mathrm{d}}{\mathrm{d}z} + \hat{\boldsymbol{u}} \cdot \boldsymbol{R} \right) + 2\overline{\boldsymbol{y}} \left(\frac{\boldsymbol{y} \cdot \boldsymbol{R}}{\boldsymbol{y} \cdot \overline{\boldsymbol{y}}} \right) \end{aligned}$$

and as a consequence (3.2) and (3.3) are equivalent to

$$0 = \left(\imath \frac{\mathrm{d}}{\mathrm{d}z} + \hat{\boldsymbol{u}} \cdot \boldsymbol{R} \right) \hat{\boldsymbol{w}}(z), \tag{3.7}$$

$$0 = (\boldsymbol{y} \cdot \boldsymbol{R})\, \hat{\boldsymbol{w}}(z). \tag{3.8}$$

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The remaining consistency to be checked is then

$$\left[\imath\,\frac{\mathrm{d}}{\mathrm{d}z}+\hat{\boldsymbol{u}}\cdot\boldsymbol{R},\ \boldsymbol{y}\cdot\boldsymbol{R}\right]=\imath\,\boldsymbol{y}\cdot\dot{\boldsymbol{T}}+[\hat{\boldsymbol{u}}\cdot\boldsymbol{T},\ \boldsymbol{y}\cdot\boldsymbol{T}]=0,$$

which upon use of $\hat{\boldsymbol{u}}\times\boldsymbol{y}=-\imath\,\boldsymbol{y}$ is equivalent to Nahm's equation.

Equally from

$$\hat{\boldsymbol{u}}\cdot\boldsymbol{R}=-\imath[(R_1+\imath R_2)\zeta^{-1}-\imath R_3]-\frac{2\boldsymbol{y}\cdot\boldsymbol{R}}{\zeta(1+|\zeta|^2)}=\imath[(R_1-\imath R_2)\zeta-\imath R_3]-\frac{2\bar{\zeta}\,\boldsymbol{y}\cdot\boldsymbol{R}}{1+|\zeta|^2},$$

we may write the equations as

$$0=\left(\frac{\mathrm{d}}{\mathrm{d}z}+[(R_1-\imath R_2)\zeta-\imath R_3]\right)\hat{\boldsymbol{w}}(z)=\left(\frac{\mathrm{d}}{\mathrm{d}z}+M+\imath[(x_1-\imath x_2)\zeta-\imath x_3]\right)\hat{\boldsymbol{w}}(z),$$
$$0=(\boldsymbol{y}\cdot\boldsymbol{R})\,\hat{\boldsymbol{w}}(z),$$

where

$$M=(T_1-\imath T_2)\zeta-\imath T_3. \tag{3.9}$$

The equations we have obtained are just the Lax equations

$$0=2\imath(\boldsymbol{y}\cdot\boldsymbol{R})\,\hat{\boldsymbol{w}}(z)=(L(\zeta)-\eta)\,\hat{\boldsymbol{w}}(z),\qquad \eta=2\boldsymbol{y}\cdot\boldsymbol{x},$$
$$0=\left(\imath\,\frac{\mathrm{d}}{\mathrm{d}z}+\hat{\boldsymbol{u}}\cdot\boldsymbol{R}\right)\hat{\boldsymbol{w}}(z),$$

and

$$\dot{L}=[L,M].$$

From the first of these we see that

$$0=\det\left(L(\zeta)-\eta\right),$$

which gives the equation of the spectral curve $\mathcal{C}$. Upon using $\overline{\boldsymbol{y}(\zeta)}=-\bar{\zeta}^2\,\boldsymbol{y}(-1/\bar{\zeta})$ we see from

$$0=\det\left(L(\zeta)-\eta\right)^\dagger=\det(L(\zeta)^\dagger-\bar{\eta})=\det(2\imath\,\overline{\boldsymbol{y}(\zeta)}\cdot\boldsymbol{T}-\bar{\eta})$$
$$=\det(-2\imath\bar{\zeta}^2\boldsymbol{y}(-1/\bar{\zeta})\cdot\boldsymbol{T}-\bar{\eta})$$

that the spectral curve is invariant under

$$(\zeta,\eta)\to\left(-\frac{1}{\bar{\zeta}},-\frac{\bar{\eta}}{\bar{\zeta}^2}\right).$$

The spectral curve then has the form

$$\eta^n+a_1(\zeta)\eta^{n-1}+\cdots+a_n(\zeta)=0,\quad \deg a_k(\zeta)\leq 2k, \tag{3.10}$$

and the genus of $\mathcal{C}$ is $g=(n-1)^2$.

It is worth remarking that Nahm's ansatz only yields solutions of $\Delta\boldsymbol{w}=0$ and does not yield solutions of $\Delta^\dagger\boldsymbol{v}=0$.

3.1. Strategy of solution. The strategy for constructing solutions involves three steps. We have seen that finding solutions to $\Delta \boldsymbol{w} = 0$ reduces to solving

$$0 = (L(\zeta) - \eta)\, \hat{\boldsymbol{w}}(z), \tag{3.11}$$

$$0 = \left(\frac{d}{dz} + M\right) \hat{\boldsymbol{w}}(z), \tag{3.12}$$

upon using the (slightly modified) ansatz

$$\boldsymbol{w} = (1_2 + \hat{\boldsymbol{u}}(\boldsymbol{x}) \cdot \boldsymbol{\sigma})\, e^{-\imath z[(x_1 - \imath x_2)\zeta - \imath x_3 - x_4]} |s> \otimes \hat{\boldsymbol{w}}(z).$$

Here $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is a unit vector and $\eta = 2\boldsymbol{y} \cdot \boldsymbol{x}$. We might construct a solution as follows.

(1) Given a spectral curve $0 = \det(L(\zeta) - \eta)$ and a position $\boldsymbol{x}$ we substitute $\eta = 2\boldsymbol{y} \cdot \boldsymbol{x}$ using the expression for $\boldsymbol{y}$ in terms of ζ. This is an equation of degree $2n$ in ζ which we shall refer to as the *Atiyah–Ward* constraint, this equation having appeared in their work. The $2n$ solutions give us $2n$ associated values $\hat{\boldsymbol{u}}^a$, $a = 1, \ldots, 2n$. For each of these we solve for $\hat{\boldsymbol{w}}(z)$ yielding a $2n \times 1$ matrix $\boldsymbol{w}^a$. Taking each of the $2n$ solutions we obtain a $2n \times 2n$ matrix of solutions W.

(2) As $0 = \Delta W = \imath(\frac{\mathrm{d}}{\mathrm{d}z} - R)W$, then

$$\frac{\mathrm{d}}{\mathrm{d}z} W = RW, \qquad \frac{\mathrm{d}}{\mathrm{d}z} W^\dagger = W^\dagger R, \qquad \frac{\mathrm{d}}{\mathrm{d}z} (W^\dagger)^{-1} = -R(W^\dagger)^{-1},$$

whence

$$0 = \Delta^\dagger (W^\dagger)^{-1} = \imath \left(\frac{\mathrm{d}}{\mathrm{d}z} + R\right)(W^\dagger)^{-1}. \tag{3.13}$$

So given W we may construct $V = (W^\dagger)^{-1}$.

(3) To reconstruct the gauge and Higgs fields using the formulae of the previous section we must extract from V the two normalizable solutions.

The new insight that the study of integrable systems brings to this problem is that $\hat{\boldsymbol{w}}(z)$ may be understood as a Baker–Akhiezer function constructed explicitly. Before considering this, we conclude the section by noting Nahm's construction for $\hat{\boldsymbol{w}}(z)$.

3.2. Constructing $\hat{\boldsymbol{w}}(z)$ using the adjoint equation. We begin with several simple observations. First, assuming L is invertible, the Lax equation $\dot{L} = [L, M]$ means also that

$$\frac{\mathrm{d}}{\mathrm{d}z} L^{-1} = -L^{-1} \dot{L} L^{-1} = [L^{-1}, M],$$

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Adj} L = \frac{\mathrm{d}}{\mathrm{d}z} (\det(L)\, L^{-1}) = [\operatorname{Adj} L, M].$$

Second, suppose λ_i is an eigenvalue of L with associated eigenvector f_i, $Lf_i = \lambda_i f_i$. Then f_i is only determined up to a scale $f_i \to f_i h_i(z)$ which may differ from eigenvector to eigenvector. Set

$$F = (f_1, \ldots, f_n), \qquad \Lambda = \operatorname{Diag}(\lambda_1, \ldots, \lambda_n).$$

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Then

$$LF = F\Lambda$$

is compatible with the Lax equation if and only if $F = F(z)$ is governed by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + M\right) F = F\,\mathrm{Diag}(\alpha_1, \ldots, \alpha_n), \tag{3.14}$$

for some $\alpha_i(z)$. Conversely, given a solution of this equation we may reconstruct L satisfying $\dot{L} = [L, M]$ via $L = F\Lambda F^{-1}$. Third, if λ_i is an eigenvalue of L we may construct a corresponding eigenvector f_i via

$$f_i = \mathrm{Adj}(L - \lambda_i)\nu h_i(z), \tag{3.15}$$

where ν is any constant vector. This follows as

$$(L - \lambda_i)f_i = (L - \lambda_i)\,\mathrm{Adj}(L - \lambda_i)\nu h_i(z) = \det(L - \lambda_i)\nu h_i(z) = 0.$$

With such eigenvectors f_i we see that

$$\begin{aligned}\dot{F} &= [\mathrm{Adj}(L - \lambda_i), M]\nu h_i(z) + \mathrm{Adj}(L - \lambda_i)\nu \dot{h}_i(z)\\ &= -MF + Fh_i^{-1}\dot{h}_i + \mathrm{Adj}(L - \lambda_i)M\nu h_i(z)\end{aligned}$$

and for this to be of the form (3.14) we require

$$0 = \mathrm{Adj}(L - \lambda_i)\nu(\dot{h}_i - h_i\alpha_i) + \mathrm{Adj}(L - \lambda_i)M\nu h_i(t).$$

Taking the inner product with an arbitrary vector μ then yields the differential equation

$$h_i^{-1}\frac{\mathrm{d}h_i}{\mathrm{d}z} = \alpha_i(z) - \frac{\mu^T\,\mathrm{Adj}(L - \lambda_i)M\nu}{\mu^T\,\mathrm{Adj}(L - \lambda_i)\nu}.$$

Therefore requiring the differential equation for F leads to a differential equation for h_i. Suppose we write

$$h_i(z) = \frac{\exp[-\theta_i(z) + \int^z \alpha_i(z)\,\mathrm{d}z]}{\sqrt{\mu^T\,\mathrm{Adj}(L - \lambda_i)\nu}}$$

then

$$h_i^{-1}\frac{\mathrm{d}h_i}{\mathrm{d}z} = \alpha_i(z) - \frac{\mathrm{d}\theta_i}{\mathrm{d}z} - \frac{1}{2}\frac{\mu^T\,[\mathrm{Adj}(L - \lambda_i), M]\,\nu}{\mu^T\,\mathrm{Adj}(L - \lambda_i)\nu}$$

which provides a solution if

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}z} = \frac{1}{2}\frac{\mu^T\{M, \mathrm{Adj}(L - \lambda_i)\}\,\nu}{\mu^T\,\mathrm{Adj}(L - \lambda_i)\nu}.$$

Nahm's approach to construct $\hat{\boldsymbol{w}}(\boldsymbol{z})$ was to express this in the form (3.15) together with the one-dimensional differential equations for θ_i. This method has only been implemented in the charge two case and we now propose an alternative approach.

4. A spectral problem. We have identified $\hat{\boldsymbol{w}}(z)$ with the Baker–Akhiezer function and now must ask whether this can be constructed. Set

$$A_{-1} = T_1 + iT_2,\ A_0 = -2iT_3,\ A_1 = T_1 - iT_2,$$

and so

$$L(\zeta) = A_{-1} + A_0\zeta + A_1\zeta^2, \qquad M = \frac{1}{2}A_0 + A_1\zeta.$$

Viewing equation (3.12) as a spectral problem

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}A_0(z)\right)\hat{\boldsymbol{w}}(z) = -\zeta A_1(z)\hat{\boldsymbol{w}}(z),$$

we seek to solve this. The z-dependence of the right-hand side means however this is not a standard eigenvalue problem, but it may be reduced to such using the trick by Ercolani and Sinha [**3**]. With the notation introduced, Nahm's equation yield

$$\frac{\mathrm{d}}{\mathrm{d}z}A_1(z) = \frac{1}{2}[A_0(z), A_1(z)] \tag{4.1}$$

and so by introducing the matrix $C(z)$ with

$$\frac{\mathrm{d}}{\mathrm{d}z}C(z) = \frac{1}{2}A_0(z)C(z), \qquad C(0) = 1_n$$

we may write

$$A_1(z) = C(z)A_1(0)C(z)^{-1}.$$

Then upon performing a gauge transformation

$$Q_0(z) = C(z)A_0(z)C(z)^{-1}, \qquad \Phi(z) = C(z)^{-1}\hat{w}(z)$$

we obtain the spectral problem

$$\left(\frac{d}{dz} + Q_0(z)\right)\boldsymbol{\Phi}(z) = -\zeta A_1(0)\boldsymbol{\Phi}(z). \tag{4.2}$$

Here, $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}(\zeta, \eta, z) = \boldsymbol{\Phi}(P, z)$ is given by the Baker–Akhiezer function on the curve, $P = (\zeta, \eta) \in \mathcal{C}$. Then

$$\hat{\boldsymbol{w}}(z) = C(z)\boldsymbol{\Phi}(\zeta, \eta, z)$$

and

$$\begin{aligned}\boldsymbol{w} &= (1_2 + \hat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})\,e^{-\imath z[(x_1-\imath x_2)\zeta-\imath x_3-x_4]}|s> \otimes C(z)\boldsymbol{\Phi}(\zeta, \eta, z),\\ &= 1_2 \otimes C(z)\left((1_2 + \hat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})\,e^{-\imath z[(x_1-\imath x_2)\zeta-\imath x_3-x_4]}|s> \otimes \boldsymbol{\Phi}\left(\zeta, \eta = \frac{2\boldsymbol{y}\cdot\boldsymbol{x}}{\zeta}, z\right)\right).\end{aligned}$$

Again, if we group all $2n$ solutions $\boldsymbol{\Phi}$ together into a $n \times 2n$ matrix $\hat{\boldsymbol{\Phi}}$ we then obtain

$$W = (1_2 \otimes C(z))\,\varphi, \qquad \varphi = (1_2 + \hat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})\,e^{-\imath z[(x_1-\imath x_2)\zeta-\imath x_3-x_4]}|s> \otimes\hat{\boldsymbol{\Phi}},$$

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where φ is a $2n \times 2n$ matrix. Then

$$V^{\dagger} = W^{-1} = \varphi^{-1}(1_2 \otimes C(z)^{-1})$$

will be in terms of the Baker–Akhiezer function. It remains then to construct $\boldsymbol{\Phi}$.

4.1. The Baker–Akhiezer function. By a constant gauge transformation we may assume that $A_1(0)$ is diagonal. Its behaviour may be read from the spectral curve (3.10),

$$A_1(0) = \mathrm{Diag}\,(\rho_1, \ldots, \rho_m)\,, \qquad \rho_m = \mathrm{Res}_{P \to \infty_m} \frac{\eta}{\zeta},$$

where ∞_m $(m = 1, \ldots, n)$ are the n points above $\zeta = \infty$. Thus the integration of the Adjoint Weyl equation reduces to the matrix spectral problem (4.2). The same problem appeared in [**1**] and [**3**] when focussing on the algebro-geometric integration of the Nahm equation and we shall use the results of our recent paper [**1**] for the integration of the Weyl equation.

Let θ be the canonical θ-function of the curve $\mathcal{C}$ and let τ be its period matrix. The period lattice is then generated by $\Lambda = (1_g, \tau)$ and

$$\theta(\boldsymbol{w}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^g} \exp\{\imath\pi\, \boldsymbol{k}^T \tau \boldsymbol{k} + 2\,\imath\pi\, \boldsymbol{w}^T \boldsymbol{k}\}.$$

Denote by $\Theta = \{\boldsymbol{w} | \theta(\boldsymbol{w}) = 0\}$ the θ-divisor in the Jacobi variety of the curve $\mathcal{C}$, $\mathbb{C}^g/\Lambda$.

THEOREM 4.1. *Let* $\boldsymbol{\Phi}(P, z) = (\Phi_1(P, z), \ldots, \Phi_n(P, z))^T$ *be the eigenfunction (or Baker–Akhiezer function) of the standard spectral problem* (4.2). *The components* $\Phi_j(P, z)$ *are given by*

$$\Phi_j\,(P, z) = g_j(P)\, \frac{\theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\infty_j) + (z+1)\,\boldsymbol{U} - \widetilde{\boldsymbol{K}})\theta(\boldsymbol{U} - \widetilde{\boldsymbol{K}})}{\theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\infty_j) + \boldsymbol{U} - \widetilde{\boldsymbol{K}})\theta((z+1)\,\boldsymbol{U} - \widetilde{\boldsymbol{K}})}\, e^{z \int_{P_0}^{P} \gamma_\infty - z\,\nu_j}. \tag{4.3}$$

Here $\boldsymbol{\phi}(P)$ *is the Abel map,* $z \in (-1, 1)$*, and* $P \in \mathcal{C}$*. The vector* $\widetilde{\boldsymbol{K}}$ *is defined by*

$$\widetilde{\boldsymbol{K}} = \boldsymbol{K} + \boldsymbol{\phi}\left((n-2)\sum_{k=1}^{n} \infty_k\right),$$

where $\boldsymbol{K}$ *is the vector of Riemann constants. We have that*

(1) $\widetilde{\boldsymbol{K}}$ *is independent of the choice of base point of the Abel map;*
(2) $\theta(\widetilde{\boldsymbol{K}}) = 0$*;*
(3) $2\widetilde{\boldsymbol{K}} \in \Lambda$*;*
(4) *for* $n \geq 3$ *we have* $\widetilde{\boldsymbol{K}} \in \Theta_{\mathrm{singular}}$.

For each j *the function* $g_j(P)$ *is meromorphic on* $\mathcal{C}$*,* $g_j(\infty_j) = 1$*, and has a zero-divisor of degree* $g + n - 1$ *that includes the* $n - 1$ *points* $(\infty_1, \ldots, \widehat{\infty_j}, \ldots, \infty_n)$.

The matrix $Q_0(z)$ *(which has poles of first order at* $z = \pm 1$*) is given by*

$$Q_0(z)_{jl} = \epsilon_{jl}\, \frac{\rho_j - \rho_l}{\mathcal{E}(\infty_j, \infty_l)}\, e^{i\pi\, \tilde{\boldsymbol{q}} \cdot (\boldsymbol{\phi}(\infty_l) - \boldsymbol{\phi}(\infty_j))}\, \frac{\theta(\boldsymbol{\phi}(\infty_l) - \boldsymbol{\phi}(\infty_j) + (z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})}{\theta((z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})}\, e^{z(\nu_l - \nu_j)}. \tag{4.4}$$

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Here $E(P, Q) = \mathcal{E}(P, Q)/\sqrt{\mathrm{d}x(P)\mathrm{d}x(Q)}$ *is the Schottky–Klein prime form,* $\boldsymbol{U} - \widetilde{\boldsymbol{K}} = \frac{1}{2}\tilde{\boldsymbol{p}} + \frac{1}{2}\tau\tilde{\boldsymbol{q}}$ *(* $\tilde{\boldsymbol{p}},\ \tilde{\boldsymbol{q}} \in \mathbb{Z}^g$ *) is a non-singular even* θ*-characteristic, and* $\epsilon_{jl} = \epsilon_{lj} = \pm 1$ *is determined (for* $j < l$*) by* $\epsilon_{jl} = \epsilon_{jj+1}\epsilon_{j+1j+2}\cdots\epsilon_{l-1l}$. *The* $n-1$ *signs* $\epsilon_{jj+1} = \pm 1$ *are arbitrary.*

In passing we note that a formula with similar features was obtained by Dubrovin [**2**] when giving a θ-functional solution to the Euler equation describing motion of the n-dimensional rigid body. The essential difference is that the curve $\mathcal{C}$ here should be subjected to the the following three constraints **H1**, **H2**, **H3** of Hitchin who showed a bijection between such curves and magnetic monopoles [**5**]:

H1 $\mathcal{C}$ admits the involution: $(\zeta, \eta) \longrightarrow (-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2)$.

H2 Let $\gamma_\infty(P)$ be the unique differential of the second kind on $\mathcal{C}$ defined by the conditions

$$\gamma_\infty(P)|_{P\to\infty_i} = \left(\frac{\rho_i}{\xi^2} + O(1)\right)\mathrm{d}\xi, \qquad \oint_{\mathfrak{a}_k} \gamma_\infty(P) = 0, \quad i, k = 1, \ldots, g,$$

where ξ is a local coordinate and $\rho_i = \mathrm{Res}_{P\to\infty_i}\eta/\zeta$. Then $\mathfrak{b}$-periods defining the winding vector $\boldsymbol{U}$ are to be half-periods,

$$\boldsymbol{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau\boldsymbol{m}. \tag{4.5}$$

The vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$ are called *Ercolani–Sinha vectors*. They should be *primitive*, i.e. $s\boldsymbol{U}$ belongs to the period lattice Λ if and only if $s = 0$ or $s = 2$ (equivalently, $z = s - 1 = \pm 1$). (Hitchin's original constraint was reformulated to this form in [**1**].)

H3 All components of the Baker–Akhiezer function $\Phi_j(P, z)$ are real and smooth for $z \in (-1, 1)$.

Bringing the previous results together then yields

PROPOSITION 4.2. *Let* $\boldsymbol{w}^{(k)}(\boldsymbol{x}, z)$*,* $k = 1, \ldots, 2n$ *be the column vectors*

$$\boldsymbol{w}^{(k)}(\boldsymbol{x}, z) = (1_2 + \hat{\boldsymbol{u}}(P_k)\cdot\boldsymbol{\sigma})\, e^{-\imath z[(x_1 - \imath x_2)\zeta - \imath x_3 - x_4]}|s> \otimes C(z)\boldsymbol{\Phi}(P_k, z) \tag{4.6}$$

where $P_k = (\zeta_k, \eta_k) \in \mathcal{C}$ *are solutions to the Atiyah–Ward constraint,* $C(z)^{-1}$ *is the fundamental solution to the ODE*

$$\frac{\mathrm{d}}{\mathrm{d}z}C(z)^{-1} + \frac{1}{2}Q_0(z)C(z)^{-1} = 0$$

normalized by the condition $C(0) = 1_n$*, and the* $n \times n$*-matrix* $Q_0(z)$ *and* n*-vector* $\boldsymbol{\Phi}(P, z)$ *are given by the* θ*-functional formulae* (4.3) *and* (4.4) *respectively. Then the* $2n \times 2n$ *matrix*

$$V(\boldsymbol{x}, z) = \left\{\left(\boldsymbol{w}^{(1)}(\boldsymbol{x}, z), \boldsymbol{w}^{(2)}(\boldsymbol{x}, z), \ldots, \boldsymbol{w}^{(2n)}(\boldsymbol{x}, z)\right)^{-1}\right\}^\dagger \tag{4.7}$$

defines the fundamental solution to the Weyl equation, $\Delta^\dagger V = 0$.

5. Conclusions. Although non-abelian magnetic monopoles have been objects of fascination for some decades now, very few explicit solutions are known. This note fits into our longer programme of seeing how far the techniques from integrable

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systems will allow us to construct such solutions. Here, we have considered the explicit construction of magnetic monopoles using algebro-geometric constructions coming from integrable systems. Previous studies along these lines have focussed on the construction of solutions to Nahm's equation which is an auxiliary problem to that of the explicit integration of the Bogomolny equations. Although the ADHM construction is based upon normalizable solutions of the equation $\Delta^{\dagger}\boldsymbol{v}=0$ an ansatz of Nahm naturally gives solutions of the adjoint equation $\Delta\boldsymbol{w}=0$: the matrices of fundamental solutions of these equations are related by $V=(W^{\dagger})^{-1}$. Here we have expressed $\boldsymbol{w}$ in terms of a Baker–Akhiezer function and given explicit expressions for this. Assuming one has a spectral curve these expressions may be evaluated algorithmically. Unfortunately the curves characterizing magnetic monopoles are often restricted by transcendental constraints (**H2** and **H3**), but this is a separate and interesting story to the one presented here. Finally we have not addressed here the remaining problem of extracting the normalizable solutions from this data. Details and examples of this approach will be given elsewhere.

Appendix A. The Panagopolous formulae. We have

$$\Delta = \imath\frac{\mathrm{d}}{\mathrm{d}z} + x_4 + \imath\,\boldsymbol{x}\cdot\boldsymbol{\sigma} - \imath\,T_4 + \boldsymbol{T}\cdot\boldsymbol{\sigma}.$$

Thus

$$\Delta^{\dagger} = \imath\frac{\mathrm{d}}{\mathrm{d}z} + x_4 - \imath\,\boldsymbol{x}\cdot\boldsymbol{\sigma} - \imath\,T_4 - \boldsymbol{T}\cdot\boldsymbol{\sigma}.$$

Set

$$\Delta^{\dagger} = \imath\left[1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} - \imath\,x_4 - T_4 + \mathcal{H} + \mathcal{F}\right]$$

with Hermitian

$$\mathcal{H} = -\sum_{j=1}^{3} x_j\sigma_j\otimes 1_n, \qquad \mathcal{F} = \imath\sum_{j=1}^{3}\sigma_j\otimes T_j.$$

Then if $\boldsymbol{v}$ is any solution of $\Delta^{\dagger}\boldsymbol{v}=0$ we have that

$$1_{2n}\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{v} = [\imath\,x_4 + T_4 - (\mathcal{H}+\mathcal{F})]\,\boldsymbol{v}.$$

For completeness we prove here the Panagopolous formulae [7] using the method described in this reference (extending very slightly to the case x_4, T_4 possibly non-zero). These integral formulae reduce to the problem of finding for any given operator $\mathcal{A}$ and any two solutions $\boldsymbol{v}_{a,b}$ of $\Delta^{\dagger}\boldsymbol{v}=0$ an operator $\mathcal{B}$ such that

$$\boldsymbol{v}_a^{\dagger}\mathcal{A}\boldsymbol{v}_b = \frac{\mathrm{d}}{\mathrm{d}z}(\boldsymbol{v}_a^{\dagger}\mathcal{B}\boldsymbol{v}_b). \tag{A.1}$$

In this case

$$\boldsymbol{v}_a^{\dagger}\mathcal{A}\boldsymbol{v}_b = \frac{\mathrm{d}\boldsymbol{v}_a^{\dagger}}{\mathrm{d}z}\mathcal{B}\boldsymbol{v}_b + \boldsymbol{v}_a^{\dagger}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z}\boldsymbol{v}_b + \boldsymbol{v}_a^{\dagger}\mathcal{B}\frac{\mathrm{d}\boldsymbol{v}_b}{\mathrm{d}z} = \boldsymbol{v}_a^{\dagger}\left(\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z} - (\mathcal{H}+\mathcal{F})\mathcal{B} - \mathcal{B}(\mathcal{H}+\mathcal{F})\right)\boldsymbol{v}_b,$$

and thus we seek to relate the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A} = \frac{d\mathcal{B}}{dz} - (\mathcal{H} + \mathcal{F})\mathcal{B} - \mathcal{B}(\mathcal{H} + \mathcal{F}).$$

Introduce the operator $\mathcal{D}$ by

$$\mathcal{D}(\mathcal{R}) = \frac{d\mathcal{R}}{dz} - (\mathcal{H} + \mathcal{F})\mathcal{R} - \mathcal{R}(\mathcal{H} + \mathcal{F}).$$

We shall use the following relations:

$$\mathcal{F}^2 = -1_2 \otimes \sum_{i=1}^{3} T_i T_i - \imath \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \sigma_k \otimes T_i T_j \tag{A.2}$$

and

$$\frac{d\mathcal{F}}{dz} = \imath \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \sigma_k \otimes T_i T_j \tag{A.3}$$

Therefore

$$\mathcal{F}^2 + \frac{d\mathcal{F}}{dz} = -1_2 \otimes \sum_{i,j=1}^{3} T_i T_j$$

and

$$\left[\mathcal{F}^2 + \frac{d\mathcal{F}}{dz}, \mathcal{H}\right] = 0.$$

PROPOSITION A. 1. *Let*

$$\mathcal{Q} = \frac{1}{r^2}\mathcal{H}\mathcal{F}\mathcal{H} - \mathcal{F}.$$

Then we have the anti-derivative

$$\int dz\, \boldsymbol{v}_a^{\dagger}\boldsymbol{v}_b = \boldsymbol{v}_a^{\dagger}\mathcal{Q}^{-1}\boldsymbol{v}_b. \tag{A.4}$$

Proof. In this case $\mathcal{A} = 1_{2n}$ and we must show that

$$\frac{d\mathcal{Q}^{-1}}{dz} - (\mathcal{H} + \mathcal{F})\mathcal{Q}^{-1} - \mathcal{Q}^{-1}(\mathcal{H} + \mathcal{F}) = 1_{2n}. \tag{A.5}$$

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The left-hand side of (A.5) may be rewritten as follows:

$$\begin{aligned}
&\mathcal{Q}^{-1}\left[-\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{r^2}\mathcal{HFH}-\mathcal{F}\right)-(\mathcal{H}+\mathcal{F})\left(\frac{1}{r^2}\mathcal{HFH}-\mathcal{F}\right)-\left(\frac{1}{r^2}\mathcal{HFH}-\mathcal{F}\right)(\mathcal{H}+\mathcal{F})\right]\mathcal{Q}^{-1}\\
&\quad=\mathcal{Q}^{-1}\left[-\frac{1}{r^2}\mathcal{H}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z}\mathcal{H}+\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z}-\frac{1}{r^2}(\mathcal{H}+\mathcal{F})\mathcal{HFH}-\frac{1}{r^2}\mathcal{HFH}(\mathcal{H}+\mathcal{F})\right.\\
&\qquad\left.+(\mathcal{H}+\mathcal{F})\mathcal{F}+\mathcal{F}(\mathcal{H}+\mathcal{F})\right]\mathcal{Q}^{-1}\\
&\quad=\mathcal{Q}^{-1}\left[\frac{1}{r^2}\mathcal{H}\left(\mathcal{F}^2+1_2\otimes\sum_{i,j=1}^{3}T_iT_j\right)\mathcal{H}-\left(\mathcal{F}^2+1_2\otimes\sum_{i,j=1}^{3}T_iT_j\right)\right.\\
&\qquad\left.-\frac{1}{r^2}\mathcal{H}^2\mathcal{FH}-\frac{1}{r^2}\mathcal{FHFH}+\mathcal{HF}+\mathcal{F}^2-\frac{1}{r^2}\mathcal{HFH}^2-\frac{1}{r^2}\mathcal{HFHF}+\mathcal{FH}+\mathcal{F}^2\right]\mathcal{Q}^{-1}.
\end{aligned}$$

Now $\mathcal{H}^2=r^2 1_{2n}$ and

$$\begin{aligned}
\mathcal{Q}^2&=\left(\frac{1}{r^2}\mathcal{HFH}-\mathcal{F}\right)^2=\frac{1}{r^4}\mathcal{HFH}^2\mathcal{FH}+\mathcal{F}^2-\frac{1}{r^2}\mathcal{HFHF}-\frac{1}{r^2}\mathcal{FHFH}\\
&=\frac{1}{r^2}\mathcal{HF}^2\mathcal{H}-\frac{1}{r^2}\mathcal{HFHF}-\frac{1}{r^2}\mathcal{FHFH}+\mathcal{F}^2.
\end{aligned}$$

Performing the appropriate cancellations we obtain the necessary result. □

PROPOSITION A. 2. *Let $\mathcal{Q}$ be as in the Proposition A. 1 and*

$$\mathcal{S}=\mathcal{Q}^{-1}\left(z+2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}\right).$$

Then we have the anti-derivative

$$\int \mathrm{d}z\, z\boldsymbol{v}_a^{\dagger}\boldsymbol{v}_b=\boldsymbol{v}_a^{\dagger}\mathcal{S}\boldsymbol{v}_b. \tag{A.6}$$

Proof. Denote

$$\mathcal{S}_1=\mathcal{Q}^{-1}z,\qquad \mathcal{S}_2=\mathcal{Q}^{-1}2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}.$$

Then

$$\mathcal{D}(\mathcal{S}_1)=\frac{\mathrm{d}}{\mathrm{d}z}(z\mathcal{Q}^{-1})-z(\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1}-z\mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})=z\mathcal{D}(\mathcal{Q}^{-1})+\mathcal{Q}^{-1}=z1_{2n}+\mathcal{Q}^{-1}.$$

Further, using (A.5),

$$\begin{aligned}
\mathcal{D}(\mathcal{S}_2)&=\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{Q}^{-1}2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}\right)-(\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1}2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}-\mathcal{Q}^{-1}\,2\mathcal{H}\,\frac{\mathrm{d}}{\mathrm{d}r^2}(\mathcal{H}+\mathcal{F})\\
&=\mathcal{D}(\mathcal{Q}^{-1})\,2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}+\mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}-\mathcal{Q}^{-1}\,2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}(\mathcal{H}+\mathcal{F})\\
&=2\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}r^2}+\mathcal{Q}^{-1}(2r^2+2\mathcal{FH})\frac{\mathrm{d}}{\mathrm{d}r^2}-\mathcal{Q}^{-1}(2r^2+2\mathcal{HF})\frac{\mathrm{d}}{\mathrm{d}r^2}-\mathcal{Q}^{-1}2\mathcal{H}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}r^2}.
\end{aligned}$$

The last term may be expressed as

$$-\mathcal{Q}^{-1}2\mathcal{H}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}r^2} = -\mathcal{Q}^{-1}\frac{\mathrm{d}\mathcal{H}^2}{\mathrm{d}r^2} = -\mathcal{Q}^{-1},$$

whence

$$\mathcal{D}(\mathcal{S}) = \mathcal{D}(\mathcal{S}_1 + \mathcal{S}_2) = z1_{2n} + (2\mathcal{H} + 2\mathcal{Q}^{-1}\mathcal{F}\mathcal{H} - 2\mathcal{Q}^{-1}\mathcal{H}\mathcal{F})\frac{\mathrm{d}}{\mathrm{d}r^2}.$$

Now the expression in brackets vanishes as a consequence of

$$\begin{aligned}2\mathcal{H} + 2\mathcal{Q}^{-1}\mathcal{F}\mathcal{H} - 2\mathcal{Q}^{-1}\mathcal{H}\mathcal{F} &= 2\mathcal{Q}^{-1}\left[\left(\frac{1}{r^2}\mathcal{H}\mathcal{F}\mathcal{H} - \mathcal{F}\right)\mathcal{H} + \mathcal{F}\mathcal{H} - \mathcal{H}\mathcal{F}\right]\\ &= 2\mathcal{Q}^{-1}\left[\mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H} + \mathcal{F}\mathcal{H} - \mathcal{H}\mathcal{F}\right] = 0\end{aligned}$$

and the result follows. □

PROPOSITION A. 3. *Let $\mathcal{Q}$ be as in the Proposition A. 1. Then the anti-derivative*

$$\int \boldsymbol{v}_a^\dagger \frac{\partial}{\partial x_i}\boldsymbol{v}_b \mathrm{d}z = \boldsymbol{v}_a^\dagger \mathcal{Q}^{-1}\left[\frac{\partial}{\partial x_i} + \mathcal{H}\frac{z}{r^2}x_i + \mathcal{H}\frac{\imath}{r^2}(\boldsymbol{x}\times\nabla)_i\right]\boldsymbol{v}_b. \qquad \text{(A.7)}$$

Proof. Let $L = L_1 + L_2 + L_3$ with

$$L_1 = \mathcal{Q}^{-1}\frac{\partial}{\partial x_i}, \quad L_2 = \mathcal{Q}^{-1}\mathcal{H}\frac{z}{r^2}x_i, \quad L_3 = \mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}(\boldsymbol{x}\times\nabla)_i\,.$$

We compute $\mathcal{D}(L_i)$, $i = 1, 2, 3$. First

$$\begin{aligned}\mathcal{D}(L_1) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{Q}^{-1}\frac{\partial}{\partial x_i}\right) - (\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1}\frac{\partial}{\partial x_i} - \mathcal{Q}^{-1}\frac{\partial}{\partial x_i}(\mathcal{H}+\mathcal{F})\\ &= \left[\frac{\mathrm{d}\mathcal{Q}^{-1}}{\mathrm{d}z} - (\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1} - \mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})\right]\frac{\partial}{\partial x_i} - \mathcal{Q}^{-1}\frac{\partial\mathcal{H}}{\partial x_i}\\ &= 1_{2n}\frac{\partial}{\partial x_i} - \mathcal{Q}^{-1}\frac{\partial\mathcal{H}}{\partial x_i},\end{aligned}$$

where we use Proposition A. 1. Next

$$\begin{aligned}\mathcal{D}(L_2) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{Q}^{-1}\mathcal{H}\frac{z}{r^2}x_i\right) - (\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1}\mathcal{H}\frac{z}{r^2}x_i - \mathcal{Q}^{-1}\mathcal{H}\frac{z}{r^2}x_i(\mathcal{H}+\mathcal{F})\\ &= \left[\frac{\mathrm{d}\mathcal{Q}^{-1}}{\mathrm{d}z} - (\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1} - \mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})\right]\mathcal{H}\frac{z}{r^2}x_i\\ &\quad + \mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})\mathcal{H}\frac{z}{r^2}x_i - \mathcal{Q}^{-1}\mathcal{H}\frac{z}{r^2}x_i(\mathcal{H}+\mathcal{F}) + \mathcal{Q}^{-1}\mathcal{H}\frac{x_i}{r^2}\\ &= \mathcal{H}\frac{z}{r^2}x_i + \mathcal{Q}^{-1}(r^2 + \mathcal{F}\mathcal{H})\frac{z}{r^2}x_i - \mathcal{Q}^{-1}(r^2 + \mathcal{H}\mathcal{F})\frac{z}{r^2}x_i + \mathcal{Q}^{-1}\mathcal{H}\frac{x_i}{r^2}\\ &= \mathcal{Q}^{-1}\mathcal{H}\frac{x_i}{r^2} + \frac{z}{r^2}x_i(\mathcal{H} + \mathcal{Q}^{-1}\mathcal{F}\mathcal{H} - \mathcal{Q}^{-1}\mathcal{H}\mathcal{F}).\end{aligned}$$

Now the expression in brackets vanishes (see the proof of Proposition A. 2). Therefore,

$$\mathcal{D}(L_2) = \mathcal{Q}^{-1}\mathcal{H}\frac{x_i}{r^2}.$$

Next we calculate $\mathcal{D}(T_3)$ for $i = 1$. We have

$$\begin{aligned}
\mathcal{D}(L_3) &= \frac{\mathrm{d}}{\mathrm{d}z}\left[\mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\right] - (\mathcal{H}+\mathcal{F})\left[\mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\right] \\
&\quad - \left[\mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\right](\mathcal{H}+\mathcal{F}) \\
&= \left[\frac{\mathrm{d}\mathcal{Q}^{-1}}{\mathrm{d}z} - (\mathcal{H}+\mathcal{F})\mathcal{Q}^{-1} - \mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})\right]\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) \\
&\quad + \mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) - \mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)(\mathcal{H}+\mathcal{F}) \\
&= \mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) + \mathcal{Q}^{-1}(\mathcal{H}+\mathcal{F})\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) \\
&\quad - \mathcal{Q}^{-1}\mathcal{H}(\mathcal{H}+\mathcal{F})\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) - \mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\mathcal{H} \\
&= [\mathcal{H} + \mathcal{Q}^{-1}(\mathcal{H}^2 + \mathcal{F}\mathcal{H}) - \mathcal{Q}^{-1}(\mathcal{H}^2 + \mathcal{H}\mathcal{F})]\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) \\
&\quad - \mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\mathcal{H}.
\end{aligned}$$

Finally we obtain

$$\mathcal{D}(L_3) = -\mathcal{Q}^{-1}\mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\mathcal{H}.$$

Altogether we have

$$\mathcal{D}(T) = 1_{2n}\frac{\partial}{\partial x_1} + \mathcal{Q}^{-1}\left\{-\frac{\partial\mathcal{H}}{\partial x_1} + \mathcal{H}\frac{x_1}{r^2} - \mathcal{H}\frac{\imath}{r^2}\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right)\mathcal{H}\right\}.$$

Multiplying the expression in the parentheses by $-r^2$ gives

$$\begin{aligned}
-\{\cdot\}\, r^2 &= -\sigma_1 \otimes 1_2\big(x_1^2 + x_2^2 + x_3^2\big) + \sigma_1 \otimes 1_2 x_1^2 + \sigma_2 \otimes 1_2 x_1 x_2 + \sigma_3 \otimes 1_2 x_1 x_3 \\
&\quad - \imath(\sigma_1 \otimes 1_2 x_1 + \sigma_2 \otimes 1_2 x_2 + \sigma_3 \otimes 1_2 x_3) \times (x_2\sigma_3 \otimes 1_2 - x_3\sigma_2 \otimes 1_2)
\end{aligned}$$

which vanishes by standard relations, proving the result. □

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