MODULAR REPRESENTATIONS OF S_n

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1. Introduction. The purpose of this paper is to clarify and sharpen the argument in the last two chapters of the author's *Representation theory of the symmetric group* **(3)**. When these chapters were written the peculiar properties of the case $p = 2$ were not fully appreciated. No difficulty arises in the definition of the block in terms of the p-core, or in the application of the general modular theory based on the formula

$$e(f^\lambda) = e(n!) - e((n-a)!) + e(f_p^\lambda).$$

Trouble comes for $p = 2$ in the application of the raising operator used to construct the decomposition matrices, and this leads to errors in Tables 2–7 to 2–10. Some corrected tables are given at the end of this paper and reasons for the anomaly are explained.

2. Background material. In the case of the *symmetric group* S_n the possibility of studying the representation theory from the actual matrices, without the use of characters, is based on the following theorem.

2.1. YOUNG'S FUNDAMENTAL THEOREM. *To construct the matrix representing the transposition* $(r, r+1)$ *in the irreducible representation* $[\lambda]$, *arrange the* f^λ *standard tableaux* $\ldots t_u^\lambda \ldots t_v^\lambda \ldots$ *in dictionary order and set:*

(i) 1 *in the leading diagonal where* t^λ *has* $r, r+1$ *in the same row.*

(ii) -1 *in the leading diagonal where* t^λ *has* $r, r+1$ *in the same column.*

(iii) *a quadratic matrix*

$$\begin{array}{c c} & \begin{array}{cc} t_u^\lambda & \quad t_v^\lambda \end{array} \\ \begin{array}{c} t_u^\lambda \\ t_v^\lambda \end{array} & \begin{bmatrix} -\rho & 1-\rho^2 \\ 1 & \rho \end{bmatrix} \end{array}$$

at the intersections of the rows and columns corresponding to t_u^λ, t_v^λ, *where* $u < v$ *and* t_v^λ *is obtainable from* t_u^λ *by interchanging* r *and* $r+1$. *If* r *appears in the* (i, j) *position and* $r+1$ *in the* (k, l) *position of* t_u^λ *with* $i < k$, $j > l$, *then*

$$\rho^{-1} = g_{ij} - g_{kl} = (j - i) - (l - k).$$

(iv) *zero elsewhere.*

That all these representations are *rational* is of great importance for the modular theory. In fact, the distribution of a prime p in the *denominators* determines the reduction of $[\lambda]$ modulo p. To make such rational coefficients p-integral we utilize the transformation[1]:

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[1] **(3**, p. 143**)**. Cf. also **(1**, p. 501**)** where an analogous transformation is used.

$$2.2 \qquad \begin{bmatrix} 1 & -\frac{m-1}{m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{m} & \frac{m^2-1}{m^2} \\ 1 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 1 & \frac{m-1}{m} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

for all positive integers $p^{-1} = m$, and relate 2.2 to the tableaux t_u^λ by setting

$$g_{ij} \equiv j - i \qquad (\text{mod } p).$$

By this means we have defined the *p-graph* $G_p[\lambda]$, which, for [5, 3, 1] with $p = 5$, is

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & & \\ 3 & & & & \end{matrix}$$

(This and all subsequent definitions are paraphrased here from **(3)** to which the reader is referred.)

We define a *p-hook* to consist of a portion of the rim of $[\lambda]$ which is (a) connected, (b) removable and so contains all p-residues in natural order. A diagram containing no p-hook is called a *p-core.*

The development of the modular theory of S_n begins with the separation of the representations $[\lambda]$ into blocks. We say that if b p-hooks can be removed from $[\lambda]$ leaving a p-core $[\tilde{\lambda}]$ then $[\lambda]$ is of *weight b*. All diagrams $[\lambda]$ of weight b which have the same p-core $[\tilde{\lambda}]$ are said to belong to the same *block* **B**. For example, [5, 3, 1] has the 5-core $[2, 1^2]$ as has also the diagram [4, 3, 2], so that these irreducible representations of S_9 belong to the same 5-block.

In general, this distribution of the irreducible representations of a given group G into blocks leads to the definition of the *decomposition matrix D* associated with **B**; the rows of D give the irreducible modular components of those $[\lambda]$'s belonging to **B** while the columns give the indecomposables of the regular representation of G which contain these $[\lambda]$'s of **B**.

3. Construction of *D*-matrices. Various methods have been used to construct the matrices D of S_n. In particular, for $b = 1$ the first result was due to Nakayama while for $b = 2$ the case $n = 2p$ was given by Chung. Extensions of their methods are described in **(3)**, but a more far-reaching analysis is obtained by superimposing a given tableau t^λ on the *p-graph* $G_p[\lambda]$. This associates with each standard tableau a permutation P of the residues $0, 1, \ldots, p-1$. For example, in the case of [5, 3, 1]

$$G_5[5,3,1]: \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & & \\ 3 & & & & \end{matrix} \qquad t: \begin{matrix} 1 & 2 & 4 & 5 & 6 \\ 3 & 7 & 8 & & \\ 9 & & & & \end{matrix}$$

lead to the permutation $P = 0\,1\,4\,2\,3\,4\,0\,1\,3$.

In the ordinary representation theory of S_n, Young's *raising operator* R_{ij} plays a significant role in determining the irreducible components of permutation and other induced representations. If we restrict the operator R_{ij}

so that it *preserves the residue class* modulo p we remain within the same p-block and permutations P remain unchanged. By seeking those $[\lambda]$'s which "admit" a given permutation P we have a method of constructing the indecomposables of S_n, i.e. the columns of the decomposition matrices D.

Many examples of this construction were given in (**3**, VII) while the analogous problem of finding the modularly irreducible components of $[\lambda]$ was studied in (**3**, VIII). The approach adopted there was to utilize the raising operator to construct a transforming matrix L which would divide the tableaux of $[\lambda]$ into sets corresponding to the "indecomposables" previously obtained, though no explicit argument was given to show *modular irreducibility*.

It is significant that the permutations P play the role of the variables in the modular theory in the sense that they remain invariant under the raising operator.

We begin by assuming that these "variables" are divided into sets $\bar{\Sigma}_{ij}$ according to the construction applied to S_{n-1}. Moreover, we assume that they are associated with modular irreducible components so that there is no "*cross-over*" above the diagonal between the $\bar{\Sigma}_{ij}$. If $[\bar{\lambda}_i]$ is an ordinary irreducible representation of S_{n-1} and

$$[\lambda] \downarrow [\bar{\lambda}_1] + [\bar{\lambda}_2] + \ldots + [\bar{\lambda}_s],$$

it is assumed that the $\bar{\Sigma}_{ij}$ correspond to the modular components of $[\bar{\lambda}_i]$, and finally that the modular representations on the $\bar{\Sigma}_{ij}$ and $\bar{\Sigma}_{kj}$ which belong to $[\bar{\lambda}_i]$ and $[\bar{\lambda}_k]$ are identical. The theorems (**3**, 8.21 and 8.22) are crucial and we quote them here.

3.1. *If $n-1$ and n both occupy r-positions in t_u^λ and t_v^λ, then $\rho = 1/kp$ in that part of the matrix representing $(n-1, n)$ constructed according to* 2.1. *Both t_u^λ and t_v^λ belong to the same set Σ of* $[\lambda]$.

3.2. *If $n-1$ occupies an r-position and n an s-position in t_u^λ where r and s are neither equal nor successive residues modulo p, then that part of the matrix $(n-1, n)$ constructed according to* 2.1 *is p-integral and t_u^λ and t_v^λ both belong to the same set Σ of* $[\lambda]$.

The proofs given were in terms of the raising operator and so far as this is concerned nothing further needs to be said.

Case $p = 2$. In this case we need only consider 3.1 since every two residues are either equal or successive, and only one significant possibility arises, represented by the following skew tableaux:

$$\begin{array}{lll}
2k - - (r-1) - r & & \\
\mid \ \ldots (n-2)\ (n-1) & \ldots (n-2)n & \ldots (n-1)n \\
r & & \\
n & (n-1) & (n-2)
\end{array}$$

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The possibility that the $(r-1)$-position might be differently located is covered in **(3**, 8.32**)** by setting $p = 2$ so that $q \equiv 1 \pmod 2$. The corresponding matrices are:

$$(n-2, n-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\dfrac{1}{2k-1} & \dfrac{(2k-1)^2-1}{(2k-1)^2} \\ 0 & 1 & \dfrac{1}{2k-1} \end{bmatrix},$$

$$(n-1, n) = \begin{bmatrix} -\dfrac{1}{2k} & \dfrac{(2k)^2-1}{(2k)^2} & 0 \\ 1 & \dfrac{1}{2k} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the appropriate part of L_n must be

$$L = \begin{bmatrix} 1 & -\dfrac{2k-1}{2k} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad L^{-1} = \begin{bmatrix} 1 & \dfrac{2k-1}{2k} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $L_R = I$, so that

$$L(n-2, n-1)L^{-1} \equiv \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad L(n-1, n)L^{-1} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \pmod 2.$$

Actually, it is only the first two rows and columns of these matrices which interest us and we could suppose that $k = 1$ and draw our conclusion from the irreducibility modulo 2 of the 2-core [2, 1], for which

$$L(12)L^{-1} \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad L(23)L^{-1} \equiv \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \pmod 2.$$

Since the first tableau belongs to some $\bar{\Sigma}_{1j}$ and the second to another $\bar{\Sigma}_{2k}$ of the same block of S_{n-1}, we conclude that both $\bar{\Sigma}_{1j}$ and $\bar{\Sigma}_{2k}$ belong to the same Σ of S_n. We make no statement regarding the $\bar{\Sigma}$ containing the third tableau since $n-1$ and $n-2$ are associated with *successive* residues and our matrices yield no conclusion in this case.

Before passing on to the case $p \neq 2$ it is worth relating this approach to the skew diagram [2] . [1] to that given in **(3**, 3.12**)**, according to which the corresponding matrices would be

$$(n-2, n-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad (n-1, n) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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It can be verified that transformation by

$$L_1 = \begin{bmatrix} 1 & -1/2k & -1/2k \\ 0 & 1 & -1/(2k-1) \\ 0 & 0 & 1 \end{bmatrix}, \qquad L_1^{-1} = \begin{bmatrix} 1 & 1/2k & 1/(2k-1) \\ 0 & 1 & 1/(2k-1) \\ 0 & 0 & 1 \end{bmatrix},$$

yields the matrices given above. Appropriate transformations are available in each such case considered below.

Case $p \neq 2$. So far as 3.1 is concerned, the agrument we have just given remains valid, leading to matrices

$$L(n-2, n-1)L^{-1} \equiv \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad L(n-1, n)L^{-1} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{mod } p).$$

These are the same partial matrices as those arising in the case of the p-core $[p, 1]$ proving irreducibility, as before. In this case, however, the third skew tableau also belongs to Σ. Again, the more general case is covered by **(3**, 8.32**)** in which we may take $s = 0$ and obtain the given matrices from $[r + 1, 2, 1^{p-r-1}]$ which is a p-core for $r \neq 1$ or $p - 1$. This rules out the case when s and r are successive but this is covered in the text. If we assume that the residues attached to $n - 2$, $n - 1$, n are all equal as in the case of **(3**, 8.33**)**, the significant conclusion was not drawn that *all* the corresponding tableaux of $[\lambda]$ belong to the same set since $[p + 1, 2, 1^{p-1}]$ is a p-core for *all* p.

Turning to 3.2, since r and s are neither equal nor successive we must have $p > 3$ so that $(n - 1, n)$ is p-integral as it stands and is not modified by L_n. Actually, the irreducibility of the p-core $[q, 1]$ for $q < p - 1$ covers all the possibilities which can arise, proving that t_u^λ and t_v^λ do belong to the same set Σ of S_n.

4. Irreducibility. In order to answer the converse question: "Is the binding of the sets $\bar{\Sigma}$ considered in § 3 necessary as well as sufficient to yield Σ?" we can at least make the following statement.

4.1. *If* t_u^λ *and* t_v^λ *belong to distinct sets* Σ *of* $[\lambda]$, *then the residues of* $n - 1$ *and* n *must be consecutive, so that* $1 - \rho^2 \equiv 0$ $(\text{mod } p)$ *in the partial matrix concerned.*

If $n - 1$ and n are contiguous:

$$\ldots (n-1)n, \qquad \begin{matrix} \ldots (n-1), \\ n \end{matrix}$$

then clearly they cannot be interchanged because of the standardness condition so that the raising operator R has reached an impasse. This much was remarked in **(3**, p. 146**)**, but it was not made clear how the situation is affected by the introduction of a permutation associated with R. We assume for the time being that $p \neq 2$.

As in (**3**, p. 150), consider the skew tableaux

$$
\begin{array}{lll}
\begin{array}{l} kp - - (r-1) - r \\ \mid \qquad (n-3)(n-1) \\ (r-1)r \\ (n-2)n \end{array} &
\begin{array}{l} \ldots\ (n-2)(n-1) \\ \\ (n-3)n \end{array} &
\begin{array}{l} \ldots\ (n-3)(n-2) \\ \\ (n-1)n \end{array} \\
\\
\begin{array}{l} \qquad \ldots\ (n-3)n \\ (n-2)(n-1) \end{array} &
\begin{array}{l} \qquad \ldots\ (n-2)n \\ (n-3)(n-1) \end{array} &
\begin{array}{l} \qquad \ldots\ (n-1)n \\ (n-3)(n-2) \end{array}
\end{array}
$$

such that the transposition $(n - 1, n)$ is equivalent to a raising operator R applied to the fourth and fifth tableaux to yield the first and second, whereas a permutation (ρ) must be included to "raise" the sixth to the third. We have $L = L_R L_n \bar{L}$, where

$$
L_R = \left[\begin{array}{ccc|ccc}
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & -\dfrac{kp-1}{kp} \\
\hline
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{array}\right],
$$

$$
L_n = \left[\begin{array}{ccc|ccc}
1 & & & -\dfrac{kp-1}{kp} & & \\
 & 1 & & & -\dfrac{kp-1}{kp} & \\
 & & 1 & & & \\
\hline
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1
\end{array}\right],
$$

$$
\bar{L} = \left[\begin{array}{ccc|ccc}
1 & -\dfrac{kp-1}{kp} & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
\hline
 & & & 1 & -\dfrac{kp-1}{kp} & \\
 & & & & 1 & \\
 & & & & & \nu
\end{array}\right]
$$

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with $\nu = -(kp-2)/(kp-1)$. It may easily be verified that

$$L(n-3, n-2)L^{-1} \equiv \begin{bmatrix} -1 & 0 & 0 & & & \\ 1 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & -1 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \pmod{p},$$

$$L(n-2, n-1)L^{-1} \equiv \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -2 & -1 & -2 & -6 & -2 \\ & & & 1 & 1 & 1 \\ & & & 0 & 1 & 0 \\ & & & 0 & -2 & -1 \end{bmatrix} \pmod{p},$$

$$L(n-1, n)L^{-1} \equiv \begin{bmatrix} -1 & 0 & 0 & & & \\ 0 & -1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \end{bmatrix} \pmod{p}.$$

Arguing as in the preceding section, these submatrices arise in the case of the p-core $[p+1, 2]$ for $p \neq 2$ so that no reduction is possible and all six corresponding tableaux must belong to the same Σ of $[\lambda]$. It follows that we have proved the *irreducibility* of those components of $[\lambda]$ constructed according to the raising operator for $p \neq 2$, whether R is replaced by $(\rho)R$ or not.

5. The case $p = 2$. This case is different, in that $[p+1, 2]$ is *reducible* for $p = 2$, so that the argument with regard to the raising operator is inadequate in the sense that it does not always describe irreducibility. If we take the skew tableaux in the rearranged order

$$\begin{array}{ll} & \ldots (n-3)(n-2) \\ (n-1)n & \end{array} \qquad \begin{array}{ll} & \ldots (n-3)(n-1) \\ (n-2)n & \end{array} \qquad \begin{array}{ll} & \ldots (n-2)(n-1) \\ (n-3)n & \end{array}$$

$$\begin{array}{ll} & \ldots \quad (n-3)n \\ (n-2)(n-1) & \end{array} \qquad \begin{array}{ll} & \ldots \quad (n-2)n \\ (n-3)(n-1) & \end{array} \qquad \begin{array}{ll} & \ldots \quad (n-1)n \\ (n-3)(n-2) & \end{array}$$

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the matrices of **(3**, 8.26**)** become

$$L(n-3,n-2)L^{-1} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \pmod{2},$$

$$L(n-2,n-1)L^{-1} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ & & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 1 & 1 \end{bmatrix} \pmod{2},$$

$$L(n-1,n)L^{-1} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \pmod{2}.$$

It follows from **(3**, 8.27 and 8.28**)** that the L_R of **(3**, 8.26**)** can be modified as in § 4 for $p \neq 2$ but not for $p = 2$. This explains why the rearrangement of skew tableaux alone does not yield the matrix $L(n-2, n-1)L^{-1} \pmod{2}$ given above. It is not difficult to see that these matrices are equivalent to those of $[3, 1^2]$ reduced (mod 2) (cf. the reduction of [3, 2. 1] in **(3**, 8.35**)**).

The consequences of this reduction for $p = 2$, so far as the decomposition matrices are concerned, may be illustrated in the case of [5, 2] of S_7. Taking the tableaux in the order

12345; 67	12346 57	12356 47	12456 37	13456; 27
	12347 56	12357 46	12457 36	13457 26
12367; 45	12467 35	13467 25	12567 34	13567 24

with

$$L = \left[\begin{array}{c|cccc|cccc|c|cccc}
1 & & & & & & & & & \frac{1}{2} & & & & \\
\hline
 & 1 & -\frac{3}{4} & & & -\frac{3}{4} & \frac{9}{16} & & & & & & & \\
 & & 1 & & & & -\frac{3}{4} & & & & & & & \\
 & & & 1 & -\frac{1}{2} & & & -\frac{3}{4} & \frac{3}{8} & & & & & \\
 & & & & 1 & & & & -\frac{3}{4} & & & & & \\
\hline
 & & & & & 1 & -\frac{3}{4} & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
 & & & & & & & 1 & -\frac{1}{2} & & & & & \\
 & & & & & & & & 1 & & & & & \\
\hline
 & & & & & & & & & 1 & & & & \\
\hline
 & & & & & & & & & & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\
 & & & & & & & & & & & 1 & & -\frac{1}{2} \\
 & & & & & & & & & & & & 1 & -\frac{1}{2} \\
 & & & & & & & & & & & & & 1
\end{array}\right],$$

$$L^{-1} = \left[\begin{array}{c|cccc|cccc|c|cccc}
1 & & & & & & & & & -\frac{1}{2} & & & & \\
\hline
 & 1 & \frac{3}{4} & & & \frac{3}{4} & \frac{9}{16} & & & & & & & \\
 & & 1 & & & & \frac{3}{4} & & & & & & & \\
 & & & 1 & \frac{1}{2} & & & \frac{3}{4} & \frac{3}{8} & & & & & \\
 & & & & 1 & & & & \frac{3}{4} & & & & & \\
\hline
 & & & & & 1 & \frac{3}{4} & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
 & & & & & & & 1 & \frac{1}{2} & & & & & \\
 & & & & & & & & 1 & & & & & \\
\hline
 & & & & & & & & & 1 & & & & \\
\hline
 & & & & & & & & & & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
 & & & & & & & & & & & 1 & & \frac{1}{2} \\
 & & & & & & & & & & & & 1 & \frac{1}{2} \\
 & & & & & & & & & & & & & 1
\end{array}\right],$$

$$(12) \equiv \left[\begin{array}{c|cccc|cccc|c|cccc}
1 & & & & & & & & & & & & & \\
\hline
 & 1 & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & \\
 & & & 1 & 1 & & & & & & & & & \\
 & & & & 1 & & & & & & & & & \\
\hline
 & & & & & 1 & & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
 & & & & & & & 1 & 1 & & & & & \\
 & & & & & & & & 1 & & & & & \\
\hline
 & & & & & & & & & 1 & & & & \\
\hline
 & & & & & & & & & & 1 & 1 & & \\
 & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & 1 & 1 \\
 & & & & & & & & & & & & & 1
\end{array}\right], \quad (23) \equiv \left[\begin{array}{c|cccc|cccc|c|cccc}
1 & & & & & & & & & & & & & \\
\hline
 & 1 & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & \\
 & & & 1 & 1 & & & & & & & & & \\
\hline
 & & & & & 1 & & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
 & & & & & & & 1 & & & & & & \\
 & & & & & & & 1 & 1 & & & & & \\
\hline
 & & & & & & & & & 1 & & & & \\
\hline
 & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 & 1 & & \\
 & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & 1 & 1
\end{array}\right],$$

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$$(34) \equiv \begin{bmatrix}
1 &&&&&&&&&&&&& 1 \\
& 1 & 1 & 0 & 1 &&&&&&&&& \\
&& 1 &&&&&&&&&&& \\
&& 1 & 1 & 1 &&&&&&&&& \\
&&&& 1 &&&&&&&&& \\
&&&&& 1 & 1 & 0 & 1 &&&&& \\
&&&&&& 1 &&&&&&& \\
&&&&&& 1 & 1 & 1 &&&&& \\
&&&&&&&& 1 &&&&& \\
&&&&&&&&& 1 &&&& \\
&&&&&&&&& 1 & 1 & 1 & 1 & 0 \\
&&&&&&&&&&& 1 & 0 & 1 \\
&&&&&&&&&&&& 1 & 1 \\
&&&&&&&&&&&&& 1
\end{bmatrix}, \quad (45) \equiv \begin{bmatrix}
1 &&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&& \\
& 1 & 1 &&&&&&&&&&& \\
&&& 1 &&&&&&&&&& \\
&&&& 1 &&&&&&&&& \\
&&&&& 1 &&&&&&&& \\
&&&&& 1 & 1 &&&&&&& \\
&&&&&&& 1 &&&&&& \\
&&&&&&&& 1 &&&&& \\
&&&&&&&&& 1 &&&& \\
&&&&&&&&&& 1 &&& \\
&&&&&&&&&&& 1 && \\
&&&&&&&&&& 1 && 1 & \\
&&&&&&&&&&& 1 && 1
\end{bmatrix},$$

$$(56) \equiv \begin{bmatrix}
1 &&&&&&&&&&&&& \\
1 & 1 & 1 &&& 1 &&&&&&&& \\
&& 1 &&&& 1 &&&&&&& \\
&&& 1 &&&& 1 &&&&& 1 & \\
&&&& 1 &&&& 1 &&&&& 1 \\
&&&&& 1 & 1 &&&&&&& \\
&&&&&& 1 &&&&&&& \\
&&&&&&& 1 &&&&&& \\
&&&&&&&& 1 &&&&& \\
&&&&&& 1 &&& 1 &&&& \\
&&&&&&& 1 &&& 1 && 1 & \\
&&&&&&&& 1 &&& 1 && 1 \\
&&&&&&&&&&&& 1 & \\
&&&&&&&&&&&&& 1
\end{bmatrix}, \quad (67) \equiv \begin{bmatrix}
1 &&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&& \\
&& 1 &&&&&&&&&&& \\
&&& 1 &&&&&&&&&& \\
&&&& 1 &&&&&&&&& \\
& 1 &&&& 1 &&&&&&&& \\
&& 1 &&&& 1 &&&&&&& \\
&&& 1 &&&& 1 &&&&&& \\
&&&& 1 &&&& 1 &&&&& \\
&&&&&&&&& 1 &&&& \\
&&&&&&&&&& 1 &&& \\
&&&&&&&&&&& 1 && \\
&&&&&&&&&&&& 1 & \\
&&&&&&&&&&&&& 1
\end{bmatrix}.$$

Note that the admitted permutation $P = 0\,1\,0\,1\,0\,1\,0$ is associated with *each* of the standard tableaux

$$t_1: \begin{matrix} 12345 \\ 67 \end{matrix} \qquad\qquad t_2: \begin{matrix} 12367 \\ 45 \end{matrix}$$

so, arguing as in the case $p \neq 2$, we would have expected that the identity component[2] would have appeared *twice* in the reduction of $[5, 2]$ (mod 2). That this does not happen is apparent from the "interferences" in the upper right corners of the matrices representing (34) and (56).

[2]Note that the permutation P associated with the identity representation is admitted by the tableaux under consideration here *only* if $p = 2$.

To make the difficulty clear, the rows associated with the skew tableaux

$$\begin{array}{llll}
\quad\;\; 0 \quad\;\; 1 \quad\;\; 0 & & & \\
\ldots(n-3)(n-2)(n-1) & \ldots(n-3)(n-2)n & \ldots(n-3)(n-1)n & \ldots(n-2)(n-1)n \\
0 & & & \\
n & (n-1) & (n-2) & (n-3)
\end{array}$$

have been indicated in the matrix representing (56) and by constructing the appropriate L we have for $p = 2$

$$L(n-3, n-2)L^{-1} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad L(n-2, n-1)L^{-1} \equiv \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L(n-1, n)L^{-1} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\text{mod } 2).$$

But these are just the matrices of [4, 1], which is irreducible (mod 2). Thus the "interference" in (56) is irremovable and it can be verified that no choice of a in **(3**, 8.26**)** will remove the interference in (34) while keeping (56) 2-integral. Thus [5, 2] is irreducible (mod 2).

Professor R. Brauer has pointed out to me that this irreducibility of [5, 2] and the corresponding reduction of [5, 1^2] as indicated in the revised Table 2–7 below can be deduced from the characters by restricting S_7 to the subgroup $\{(1234567), (235)(476)\}$ of order 21.

It follows from this discussion that irreducibility is dependent only on the form of $L(n-2, n-1)L^{-1}$ and $L(n-1, n)L^{-1}$ for $p \not\equiv 2$ so that the step-by-step construction of the decomposition matrices is valid, it is a *surface* phenomenon. On the other hand, if $p = 2$, irreducibility may depend on the form of $L(34)L^{-1}$ as in the case of [5, 2], so that the Tables 2–7 to 2–10 of **(3)** are incorrect.[3]

6. Conclusions. In **(3**, 6.58**)** it was proved that the number of modularly irreducible representations in any p-block **B** is equal to the number of p-regular [λ]'s in **B**; these [λ]'s are called *head* diagrams in **B**. In **(3**, 7.52**)** we saw that these head diagrams must yield the *heads* of the columns of the decomposition matrix of **B** and so each head must contain the corresponding modularly irreducible component exactly once.

[3]Cf. Osima **(2**, p. 520**)** where the decomposition matrix for S_7 with $p = 2$ is also incorrect.

What we have shown here is that the sets of permutations B_i referred to in **(3**, VII and VIII**)** do correspond to irreducible modular components for $p \neq 2$, so that it is possible to speak of a *characteristic permutation* as in **(3**, 7.54**)**.

In the case $p = 2$, however, the columns M_i can still be determined by means of a properly chosen P but it is no longer correct to say that they yield the decomposition matrix of the block. *The raising operator provides a necessary but not a sufficient criterion of irreducibility or indecomposability.*

We have seen the correction which must be made in **(3**, Table 2–7**)**. Similarly, in the block with core [1] of Table 2–8, twice the column M_3 should be subtracted from M_1, so that we have a "wave effect" penetrating deeper and deeper into the decomposition matrices as n increases. The correct Tables 2–7 and 2–8 are printed below along with a revision of 2–9, which is believed to be correct; no attempt has been made to correct Table **(3**, 2–10**)**.

2–7

$\beta =$	1	14	20	68		
[7]	1					
[5, 2]	0	1				
[5, 1^2]	1	1				
[4, 2, 1]	1	1	1			
[3^2, 1]	1	0	1			
[3, 2^2]	1	0	1			
[3, 2, 1^2]	1	1	1			core [1]
[3, 1^4]	1	1				
[2^2, 1^3]	0	1				
[1^7]	1					
[6, 1]				1		
[4, 3]				1	1	
[4, 1^3]				2	1	core [2, 1]
[2^3, 1]				1	1	
[2, 1^5]				1		

2–8

$\beta =$	1	6	14	8	40	64	
[8]	1						
[7, 1]	1	1					
[6, 2]	0	1	1				
[6, 1^2]	1	1	1				
[5, 3]	0	1	1	1			
[5, 1^3]	1	2	1	1			
[4^2]	0	1	0	1			
[4, 3, 1]	2	1	1	1	1		
[4, 2^2]	2	0	1	0	1		
[4, 2, 1^2]	2	2	2	1	1		core [Ø]
[4, 1^4]	1	2	1	1	0		
[3^2, 2]	2	0	0	0	1		
[3^2, 1^2]	2	0	1	0	1		
[3, 2^2. 1]	2	1	1	1	1		
[2^4]	0	1	0	1			
[2^3, 1^2]	0	1	1	1			
[3, 1^5]	1	1	1				
[2^2, 1^4]	0	1	1				
[2, 1^6]	1	1					
[1^8]	1						
[5, 2, 1]						1	core [3, 2, 1]
[3, 2, 1^3]						1	

2–9

$\beta =$	1	26	78	16	40	8	48	160
[9]	1							
[7, 2]	1	1						
[7, 1^2]	2	1						
[6, 2, 1]	1	1	1					
[5, 4]	0	1	0	1				
[5, 3, 1]	2	1	1	1	1			
[5, 2^2]	2	0	1	0	1			
[5, 2, 1^2]	3	2	1	1	1			
[5, 1^4]	2	2	0	1	0		core	
[4^2, 1]	2	1	0	1	1		[1]	
[4, 2, 1^3]	3	2	1	1	1			
[3^3]	2	0	0	0	1			
[3^2, 1^3]	2	0	1	0	1			
[3, 2^3]	2	1	0	1	1			
[3, 2^2, 1^2]	2	1	1	1	1			
[3, 2, 1^4]	1	1	1	0				
[3, 1^6]	2	1		0				
[2^4, 1]	0	1		1				
[2^2, 1^5]	1	1						
[1^9]	1							
[8, 1]						1		
[6, 3]						0	1	
[6, 1^3]						1	1	
[4, 3, 2]						1	0	1
[4, 3, 1^2]						1	1	1
[4, 2^2, 1]			core			1	1	1
[4, 1^5]			[2, 1]			1	1	0
[3^2, 2, 1]						1	0	1
[2^3, 1^3]						0	1	
[2, 1^7]						1		

The representations [5, 2], [5, 1^2], [4, 2, 1], [3^2, 1]; [7, 1], [6, 2], [4^2], [3^2, 2]; [7, 2] have all been written out and reduced (mod 2) in conformity with the above tables. The only difficulty arises in constructing the transforming matrix $L = L_R L_n \bar{L}$ to accord with the appropriate raising operator, but the labour involved increases rapidly as n increases.

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