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GLEASON PARTS OF REAL FUNCTION ALGEBRAS

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Introduction. Although the theory of complex Banach algebras is by now classical, the first systematic exposition of the theory of real Banach algebras was given by Ingelstam [**5**] as late as 1965. More recently, further attention to real Banach algebras was paid in 1970 [**1**], where, among other things, the (real) standard algebras on finite open Klein surfaces were introduced. Generalizing these considerations, real uniform algebras were studied in [**7**] and [**6**].

In the present paper, an attempt is made to develop the theory of real function algebras (see Section 1 for the definition) along the lines of the complex function algebras. Although the real function algebras are not structurally different from the real uniform algebras introduced in [**7**], they are easier to deal with since their elements are actually (complex-valued) functions. In the first section, we define a real-part representing measure for an element ϕ of the carrier space Φ_A of a real function algebra A and prove that the statement

$$\sup\{|(\phi(f) - \psi(f))(\bar{\phi}(f) - \psi(f))|: f \in A, \quad \|f\| < 1\} < 4$$

is an equivalence relation for ϕ and ψ in Φ_A (Theorem 1.3). This relation decomposes Φ_A into 'Gleason parts' of A. It naturally descends to the maximal ideal space M_A of A. The theory of complex function algebras is thus obtained as a special case.

The second section deals with the complexification B of a real function algebra A. If α is the natural bijection from Φ_A to the space Φ_B of all non-zero complex homomorphisms of B, then it is shown that $\|\phi - \psi\| < 2$ if and only if $\|\alpha(\phi) - \alpha(\psi)\| < 2$ (Theorem 2.2). This enables us to relate the Gleason parts of A to those of its complexification B. Several examples of real function algebras are studied giving information about the parts of A in terms of the parts of B, and conversely. They include the (real) standard algebras on finite open Klein surfaces (Example 2.6) and certain real algebras obtained by requiring that the functions in a given complex function algebra U be real-valued at a finite number of points of the maximal ideal space of U and some continuous point derivations on U at these points be also real-valued (Example 2.7). These latter algebras were introduced and studied in [**7**]; our Theorems 2.8 and 2.9 complete this study.

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The presence of analytic structure in Φ_A and harmonic structure in M_A is discussed in the third section. Using well known results for the complexification B of A, sufficient conditions are given for the sets

$$W_\phi = \{\theta \in \Phi_A : \|\theta - \phi\| < 2\}$$

and

$$\bar{W}_\phi = \{\theta \in \Phi_A : \|\theta - \bar{\phi}\| < 2\}$$

to carry the structure of a connected finite open Riemann surface. Under these conditions, we show that the map $\tau_0(\theta) = \bar{\theta}$ from W_ϕ to $\bar{W}_\phi$ is anti-analytic and that the part in M_A containing $\phi^{-1}(0)$ becomes a connected finite open Riemann surface if $\|\phi - \bar{\phi}\| = 2$ and a connected finite Klein surface if $\|\phi - \bar{\phi}\| < 2$ in such a way that the real parts of functions in A are bounded harmonic functions on it. It would be interesting to find weaker conditions which would give harmonic structure in M_A without necessarily implying any analytic structure in Φ_A.

We shall denote the set of all real numbers by $\mathbf{R}$ and the set of all complex numbers by $\mathbf{C}$.

1. Characterizations of parts. Let X be a compact Hausdorff space. By $C(X)$ (respectively, $C_R(X)$) we denote the complex (respectively, real) Banach algebra of all continuous complex-valued (respectively, real-valued) functions on X, with the supremum norm.

Let $\tau : X \to X$ be a homeomorphism such that $\tau^2 = \tau \circ \tau$ is the identity map on X. Such a map will be called an *involution* on X. Let

$$C(X, \tau) = \{f \in C(X) : f(\tau(x)) = \bar{f}(x) \quad \text{for all } x \in X\}.$$

Then $C(X, \tau)$ is a real commutative Banach algebra with the identity 1. Also, it is not difficult to see that for any $x_1 \neq x_2$ in X, there is $f \in C(X, \tau)$ such that $f(x_1) \neq f(x_2)$; that is, $C(X, \tau)$ separates the points of X. (Unlike in the case of $C(X)$ and its complex subalgebras, we cannot assert the existence of an f in $C(X, \tau)$ with $f(x_1) = 0$ and $f(x_2) = 1$ whenever $x_1 \neq x_2$ in X.)

Definition. Let X be a compact Hausdorff space and τ an involution on X. A *real function algebra* on (X, τ) is a (real) subalgebra A of $C(x, \tau)$ that

(i) is uniformly closed in $C(X, \tau)$,
(ii) contains real constants, and
(iii) separates the points of X.

Note that every real function algebra is a *real uniform algebra* as defined in [**7**], that is, it is a real commutative Banach algebra with identity such that $\|f^2\| = \|f\|^2$ for every $f \in A$. Conversely, a real uniform algebra A can be viewed as a real function algebra as follows:

Consider the carrier space

$$\Phi_A = \{\phi : A \rightarrow \mathbf{C},\ \phi \text{ a non-zero real linear homomorphism of } A\}$$

of A. Let $\tau_0(\phi) = \bar{\phi}$ for $\phi \in \Phi_A$. Let X be any closed subset of Φ_A such that $\tau_0(X) = X$ and for every $f \in A$,

$$\sup\{|\phi(f)| : \phi \in \Phi_A\} = \sup\{|\phi(f)| : \phi \in X\}.$$

Then the Gelfand transforms of elements in A constitutes a real function algebra on (X, τ_0), which is isometrically isomorphic to A. (See [**5**], Section 4.)

We remark that a complex function algebra U on a compact Hausdorff space X is not, in general, a real function algebra on X with τ the identity map on X. But since a complex function algebra U is, in particular, a real uniform algebra, U can be regarded as a real function algebra on the disjoint union of two copies of X in the manner described above.

At this point, it is natural to ask when a real function algebra A on (X, τ) is the whole of $C(X, \tau)$. The following version of the Stone-Weierstrass theorem provides an answer.

PROPOSITION 1.1. *Let A be a real function algebra on (X, τ) such that $\bar{f} \in A$ for every $f \in A$. Then $A = C(X, \tau)$.*

Proof. Let $A_R = \{\operatorname{Re} f + \operatorname{Im} f : f \in A\}$. Then A_R is a real subalgebra of $C_R(X)$, since for $f, g \in A$, we have

$$(\operatorname{Re} f + \operatorname{Im} f)(\operatorname{Re} g + \operatorname{Im} g) = \operatorname{Re} h + \operatorname{Im} h$$

with

$$h = \tfrac{1}{2}(fg + f\bar{g} + \bar{f}g - \bar{f}\bar{g}) \in A.$$

Moreover, for $f \in C(X, \tau)$ and $x \in X$, $\operatorname{Re} f(\tau(x)) = \operatorname{Re} f(x)$ and $\operatorname{Im} f(\tau(x)) = -\operatorname{Im} f(x)$. Hence for $f \in C(X, \tau)$,

$$\begin{aligned}\|\operatorname{Re} f + \operatorname{Im} f\| &= \sup\{|\operatorname{Re} f(x) + \operatorname{Im} f(x)| : x \in X\} \\ &= \sup\{|\operatorname{Re} f(\tau(x)) + \operatorname{Im} f(\tau(x))| : x \in X\} \\ &= \sup\{|\operatorname{Re} f(x) - \operatorname{Im} f(x)| : x \in X\} \\ &= \|\operatorname{Re} f - \operatorname{Im} f\|.\end{aligned}$$

This implies that A_R is uniformly closed in $C_R(X)$. It is easy to see that A_R separates the points of X. Since A_R also contains real constants, the Stone-Weierstrass theorem shows that $A_R = C_R(X)$. Now let $f \in C(X, \tau)$. Then

$$\operatorname{Re} f + \operatorname{Im} f \in C_R(X) = A_R.$$

Hence there is $g \in A$ such that

$$\operatorname{Re} f + \operatorname{Im} f = \operatorname{Re} g + \operatorname{Im} g.$$

Since $f, g \in C(X, \tau)$, we see that $\operatorname{Re} f = \operatorname{Re} g$, $\operatorname{Im} f = \operatorname{Im} g$ and, in turn, $f = g$. Thus $A = C(X, \tau)$.

Definition. Let A be a real function algebra on (X, τ) and let ϕ be an element of the carrier space Φ_A of A. A *real-part representing measure* (*r.p.r. measure*) for ϕ is a regular Borel (positive) measure μ on X such that $\int_X \operatorname{Re} f d\mu = \operatorname{Re} \phi(f)$ for all $f \in A$ and $\mu(E) = \mu(\tau(E))$ for all Borel subsets E of X.

Let $\operatorname{Re} A = \{\operatorname{Re} f : f \in A\}$. Then $\operatorname{Re} A$ is a subspace of $C_R(X)$ and if we define $\operatorname{Re} \phi$ by $\operatorname{Re} \phi(\operatorname{Re} f) = \operatorname{Re} \phi(f)$, $f \in A$, then $\operatorname{Re} \phi$ is a bounded linear functional on $\operatorname{Re} A$. Hence by applying the Riesz representation theorem to any Hahn-Banach extension of $\operatorname{Re} \phi$ to $C_R(X)$, we can get a regular Borel measure μ_0 such that $\int_X \operatorname{Re} f d\mu_0 = \operatorname{Re} \phi(f)$ for all $f \in A$ and $\|\operatorname{Re} \phi\| = \|\mu_0\|$, where $\|\mu_0\|$ denotes the total variation of μ_0. We may define μ by

$$\mu(E) = \tfrac{1}{2}(\mu_0(E) + \mu_0(\tau(E)))$$

for every Borel subset E of X. Then μ is a r.p.r. measure for ϕ.

Note that, since ϕ is a homomorphism, we have

$$\|\phi\| = \sup\{|\phi(f)| : f \in A, \quad \|f\| < 1\} = 1,$$

so that $\|\operatorname{Re} \phi\| = 1$. Hence a r.p.r. measure for ϕ is a probability measure. Also, μ is a r.p.r. measure for ϕ if and only if it is a r.p.r. measure for $\bar{\phi}$.

We shall use the following notation throughout:

For $\phi, \psi \in \Phi_A$,

$$\|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \quad \|f\| < 1\},$$
$$\|\bar{\phi} - \psi\| = \sup\{|\bar{\phi}(f) - \psi(f)| : f \in A, \quad \|f\| < 1\}$$

and

$$\|(\phi - \psi)(\bar{\phi} - \psi)\| = \sup\{|(\phi(f) - \psi(f))(\bar{\phi}(f) - \psi(f))| : f \in A, \quad \|f\| < 1\}.$$

Since $\|\phi\| = \|\psi\| = 1$, we see that

$$\|\phi - \psi\| \leqq 2, \|\bar{\phi} - \psi\| \leqq 2 \quad \text{and} \quad \|(\phi - \psi)(\bar{\phi} - \psi)\| \leqq 4.$$

LEMMA 1.2. (Cf. Theorem 2.2, Chapter VI of [**4**].) *Let A be a real function algebra on (X, τ). Suppose $\phi, \psi \in \Phi_A$ such that $\|(\phi - \psi)(\bar{\phi} - \psi)\| = 4$. Then there are disjoint Borel subsets E_1 and E_2 of X such that $\tau(E_1) = E_1$, $\tau(E_2) = E_2$, every r.p.r. measure for ϕ is supported on E_1 and every r.p.r. measure for ψ is supported on E_2; in particular, any r.p.r. measure for ϕ and any r.p.r. measure for ψ are mutually singular.*

Proof. There exist $f_n \in A$ such that $\|f_n\| < 1$ for all n and

$$|(\phi - \psi)(\bar{\phi} - \psi)(f_n)| \to 4 \text{ as } n \to \infty.$$

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Passing to subsequences, if necessary, we may assume that $\phi(f_n) \to a$ and $\psi(f_n) \to b$ as $n \to \infty$, where $|a| \leqq 1$, $|b| \leqq 1$ and

$$|(a - b)(\bar{a} - b)| = 4 = |a - b|\,|\bar{a} - b|.$$

It can be easily seen that we must have $|a - b| = 2$ and $|\bar{a} - b| = 2$. Hence either $a = 1$ and $b = -1$, or $a = -1$ and $b = 1$. We assume the former. Therefore, we can choose f_n in A with $\|f_n\| < 1$ such that

$$|1 - \phi(f_n)| < 1/n^2$$

and

$$|1 + \psi(f_n)| < 1/n^2$$

for all n. Now let

$$E_1 = \{x \in X : \operatorname{Re} f_n(x) \to 1\}$$

and

$$E_2 = \{x \in X : \operatorname{Re} f_n(x) \to -1\}.$$

Then E_1 and E_2 are disjoint Borel subsets of X, and clearly, $\tau(E_1) = E_1$ and $\tau(E_2) = E_2$. Also, it follows, as in the proof of Theorem 2.2, Chapter VI of [**4**], that if μ is a r.p.r. measure for ϕ, then $f_n \to 1$ a.e.(μ). Hence $\operatorname{Re} f_n \to 1$ a.e.(μ), so that μ is supported on E_1. Similarly every r.p.r. measure for ψ is supported on E_2.

We shall now take up the question of partitioning the carrier space Φ_A of A with a view to seeking analytic structure in it. Just as in the complex case, the Blaschke factors will play a prominent role. But since A is only a real algebra, one has to consider a joint Blaschke factor as follows. For $f \in A$ with $\|f\| \leqq 1$ and $a \in \mathbf{C}$ with $|a| < 1$, let

$$\beta(f, a) = \frac{(f - a)}{(1 - \bar{a}f)}\,\frac{(f - \bar{a})}{(1 - af)} = \frac{f^2 - (a + \bar{a})f + |a|^2}{|a|^2 f^2 - (a + \bar{a})f + 1}.$$

Note that $\beta(f, a) \in A$ and $\|\beta(f, a)\| \leqq 1$.

THEOREM 1.3. *Let A be a real function algebra on (X, τ), and $\phi, \psi \in \Phi_A$. Then the following statements are equivalent:*

(i) $\|(\phi - \psi)(\bar{\phi} - \psi)\| < 4$.

(ii) $\|\phi - \psi\| < 2$ *or* $\|\bar{\phi} - \psi\| < 2$.

(iii) $\sup\{|\psi(f)| : f \in A, \|f\| < 1, \phi(f) = 0\} < 1$.

(iv) *If (f_n) is a sequence in A such that $\|f_n\| < 1$ for all n and $\phi(f_n) \to a$ as $n \to \infty$ with $|a| = 1$, then every convergent subsequence of $(\psi(f_n))$ converges to a or $\bar{a}$.*

(v) *If (f_n) is a sequence in A such that $\|f_n\| < 1$ for all n and $|\phi(f_n)| \to 1$ as $n \to \infty$, then $|\psi(f_n)| \to 1$ as $n \to \infty$.*

(vi) *There is* $k > 0$ *such that for all* $f \in A$ *with* $\operatorname{Re} f > 0$, *we have* $k^{-1} \operatorname{Re} \phi(f) \leqq \operatorname{Re} \psi(f) \leqq k \operatorname{Re} \phi(f)$.

(vii) *There are r.p.r. measures* μ *and* σ *on* X *for* ϕ *and* ψ *respectively such that* $k^{-1}\mu \leqq \sigma \leqq k\mu$ *for some* $k > 0$.

Proof. (i) implies (ii): Let (i) hold. If (ii) does not hold, then since $\|\phi - \psi\| = 2$ and $\|\phi\| = 1 = \|\psi\|$, there is a sequence (f_n) in A such that $\|f_n\| < 1$ for all n, $\phi(f_n) \to a$ and $\psi(f_n) \to -a$ as $n \to \infty$, where $|a| = 1$.

First assume that $a \neq \pm 1, \pm i$. For $n = 1, 2, \ldots$ and $0 < r < 1$, let $f_{n,r} = \beta(f_n, ra)$. Then $f_{n,r} \in A$ and $\|f_{n,r}\| < 1$ for all n, r. Also,

$$\phi(f_{n,r}) \to \frac{a - ra}{1 - r\bar{a}a} \cdot \frac{a - r\bar{a}}{1 - ra^2} = \frac{a^2 - r}{1 - ra^2}$$

as $n \to \infty$, which, in turn, tends to -1 as $r \to 1$. (Note that $a \neq \pm 1$.) On the other hand,

$$\psi(f_{n,r}) \to \frac{-a - ra}{1 + r\bar{a}a} \cdot \frac{-a - r\bar{a}}{1 + ra^2} = \frac{a^2 + r}{1 + ra^2}$$

as $n \to \infty$, which, in turn, tends to 1 as $r \to 1$. (Note that $a \neq \pm i$.) It now follows that

$$|(\phi - \psi)(\bar{\phi} - \psi)(f_{n,r})| \to 4$$

as $n \to \infty$ and $r \to 1$, which is a contradiction to (i).

Next, let $a = \pm 1$. Then

$$|(\phi - \psi)(\bar{\phi} - \psi)(f_n)| \to 4$$

as $n \to \infty$, which is also a contradiction to (i).

Finally, let $a = \pm i$. Since $\|\bar{\phi} - \psi\| = 2$, we may find $g_n \in A$ such that $\|g_n\| < 1$ for all n and $\bar{\phi}(g_n) \to b$, $\psi(g_n) \to -b$ as $n \to \infty$, where $|b| = 1$. The cases $b \neq \pm 1, \pm i$ and $b = \pm 1$ can be treated as above. If $b = \pm i$, then let $h_n = f_n g_n$. Since $a = \pm i$, we can easily verify that

$$|(\phi - \psi)(\bar{\phi} - \psi)(h_n)| \to 4$$

as $n \to \infty$. Note that $h_n \in A$ and $\|h_n\| < 1$. This again contradicts (i).

(ii) implies (iii): Suppose $\|\phi - \psi\| = 2c$, where $0 \leqq c < 1$. If (iii) does not hold, there exists a sequence (f_n) in A such that $\|f_n\| < 1$, $\phi(f_n) = 0$ for all n and $\psi(f_n) \to a$ as $n \to \infty$, where $|a| = 1$.

First, suppose $a \neq \pm 1$. Define for $n = 1, 2, \ldots$ and $0 < r < 1$, $f_{n,r} = \beta(f_n, ra)$. Then $f_{n,r} \in A$ and $\|f_{n,r}\| < 1$ for all n, r. Also, $\phi(f_{n,r}) = r^2 \to 1$ as $r \to 1$, and as we have earlier seen, $\psi(f_{n,r}) \to -1$ as $n \to \infty$ and $r \to 1$. This contradicts (ii).

Now, let $a = \pm 1$. Considering $(-f_n)$ in place of (f_n), if necessary, we can assume that $a = 1$. For $n = 1, 2, \ldots,$ define

$$g_n = \frac{c - f_n}{1 - cf_n}.$$

Then $g_n \in A$ and $\|g_n\| < 1$ for all n. Hence

$$|\phi(g_n) - \psi(g_n)| \leqq 2c,$$

i.e.,

$$\left| c - \frac{c - \psi(f_n)}{1 - c\psi(f_n)} \right| \leqq 2c$$

for all n. Letting $n \to \infty$, we see that $1 + c \leqq 2c$, or $1 \leqq c$, which is a contradiction to our assumption $c < 1$.

Similar proof holds if $\|\bar{\phi} - \psi\| = 2c$ with $0 \leqq c < 1$.

(iii) implies (iv): Let (f_n) be a sequence in A such that $\|f_n\| < 1$ for all n and $\phi(f_n) \to a$ as $n \to \infty$, where $|a| = 1$. If a subsequence of $(\psi(f_n))$ (which we shall denote by $(\psi(f_n))$ only) converges to b, where b is different from a and $\bar{a}$, then let $g_n = \beta(f_n, \phi(f_n))$. Now, $g_n \in A$, $\|g_n\| < 1$ and $\phi(g_n) = 0$ for all n, while

$$\psi(g_n) \to \frac{b - a}{1 - \bar{a}b} \cdot \frac{b - \bar{a}}{1 - ab}$$

as $n \to \infty$, since $b \neq a, \bar{a}$. Thus, $|\psi(g_n)| \to 1$ as $n \to \infty$, which contradicts (iii).

(iv) implies (v): The proof follows by passing to subsequences sufficiently many times.

(v) implies (vi): If (vi) does not hold, then there is a sequence (k_n) in A with $\operatorname{Re} k_n > 0$ for all n such that either

$$\frac{\operatorname{Re} \psi(k_n)}{\operatorname{Re} \phi(k_n)} \to \infty \quad \text{or}$$

$$\frac{\operatorname{Re} \phi(k_n)}{\operatorname{Re} \psi(k_n)} \to \infty$$

as $n \to \infty$. In the first case, we may find a sequence (α_n) of positive real numbers such that $\alpha_n \to 0$ but

$$\alpha_n \frac{\operatorname{Re} \psi(k_n)}{\operatorname{Re} \phi(k_n)} \to \infty,$$

and let

$$f_n = \frac{\alpha_n k_n}{\operatorname{Re} \phi(k_n)}.$$

In the second case, we may find a sequence (β_n) of positive real numbers such that $\beta_n \to 0$ but

$$\beta_n \frac{\operatorname{Re} \phi(k_n)}{\operatorname{Re} \psi(k_n)} \to \infty,$$

and let

$$f_n = \frac{\beta_n k_n}{\mathrm{Re}\,\psi(k_n)}.$$

Thus, there is a sequence (f_n) in A with $\mathrm{Re} f_n > 0$ for all n such that either (1) $\mathrm{Re}\,\phi(f_n) \to 0$, $\mathrm{Re}\,\psi(f_n) \to \infty$ as $n \to \infty$, or (2) $\mathrm{Re}\,\phi(f_n) \to \infty$, $\mathrm{Re}\,\psi(f_n) \to 0$ as $n \to \infty$. Define $g_n = \exp(-f_n)$. Then $g_n \in A$, $\|g_n\| < 1$,

$$|\phi(g_n)| = \exp(-\mathrm{Re}\,\phi(f_n)) \quad \text{and} \quad |\psi(g_n)| = \exp(-\mathrm{Re}\,\psi(f_n)).$$

In case alternative (1) holds, we have $|\phi(g_n)| \to 1$ with $|\psi(g_n)| \to 0$ as $n \to \infty$, in contradiction to (v). In case alternative (2) holds, let $h_n = \beta(g_n, \psi(g_n))$. Then $\psi(h_n) = 0$ and $|\phi(h_n)| \to 1$ as $n \to \infty$, again in contradiction to (v).

(vi) implies (vii): Let (vi) hold. Since $\mathrm{Re}\,\phi$ and $\mathrm{Re}\,\psi$ are continuous linear functionals on $\mathrm{Re}\,A = \{\mathrm{Re} f : f \in A\}$ and since

$$\|\mathrm{Re}\,\phi\| = \mathrm{Re}\,\phi(1) = 1 = \mathrm{Re}\,\psi(1) = \|\mathrm{Re}\,\psi\|,$$

there exist representing measures μ_0 for $\mathrm{Re}\,\phi$ and σ_0 for $\mathrm{Re}\,\psi$ such that $k^{-1}\mu_0 \leqq \sigma_0 \leqq k\mu_0$ (Theorem 2.6.5 of [**3**]). Define, for each Borel subset E of X,

$$\mu(E) = \tfrac{1}{2}[\mu_0(E) + \mu_0(\tau(E))]$$

and

$$\phi(E) = \tfrac{1}{2}[\sigma_0(E) + \sigma_0(\tau(E))].$$

Then μ and σ are r.p.r. measures for ϕ and ψ respectively such that $k^{-1}\mu \leqq \sigma \leqq k\mu$.

(vii) implies (i): This follows from Lemma 1.2.

Definition. For a real function algebra A on (X, τ) and $\phi, \psi \in \Phi_A$, define $\phi \sim \psi$ if any one of the equivalent statements (i)–(vii) of Theorem 1.3 holds. From the statement (vi), it can be seen that $\sim$ is an equivalence relation on Φ_A. We call the equivalence classes under $\sim$ the *Gleason parts of A*. We shall denote the equivalence class in Φ_A containing ϕ by $Q_A(\phi)$.

The above definition of Gleason parts is analogous to the usual definition of Gleason parts for a complex function algebra. (See, e.g., p. 142 of [**4**]). In this case, the equivalence of statements (iii), (v), (vi) and (vii) is well known. (See, e.g., Theorems 1.1 and 2.1, Chapter VI of [**4**].)

Let M_A be the maximal ideal space of a real uniform algebra A. Then for each $f \in A$, $\mathrm{Re}\,\hat{f}$ and $|\hat{f}|$ are well defined real-valued functions on M_A. It was shown in Proposition 1.1 of [**6**] that the smallest topology on M_A, making $\mathrm{Re}\,\hat{f}$ continuous for all $f \in A$, is the same as the smallest topology on M_A, making $|\hat{f}|$ continuous for all $f \in A$. Let $T: \Phi_A \to M_A$ be defined by $T(\phi) = \phi^{-1}(0)$.

COROLLARY 1.4. *Let A be a real uniform algebra and $x, y \in M_A$. Then the following are equivalent:*

(i) $\sup\{|\hat{f}|(y): f \in A, \|f\| < 1, |\hat{f}|(x) = 0\} < 1$.

(ii) *If (f_n) is a sequence in A such that $\|f_n\| < 1$ for all n and $|\hat{f}_n|(x) \to 1$ as $n \to \infty$, then $|\hat{f}_n|(y) \to 1$ as $n \to \infty$.*

(iii) *There is a constant $k > 0$ such that for all $f \in A$ with* $\operatorname{Re} \hat{f} > 0$, *we have*

$$k^{-1} \operatorname{Re} \hat{f}(x) \leqq \operatorname{Re} \hat{f}(y) \leqq k \operatorname{Re} \hat{f}(x).$$

(iv) *There exist regular Borel probability measures $\hat{\mu}$ and $\hat{\sigma}$ on the Silov boundary S_A of A and a constant $k > 0$ such that*

$$\int_{S_A} \operatorname{Re} \hat{f}\, d\hat{\mu} = \operatorname{Re} \hat{f}(x) \quad \textit{and} \quad \int_{S_A} \operatorname{Re} \hat{f}\, d\hat{\sigma} = \operatorname{Re} \hat{f}(y)$$

for all $f \in A$ and $k^{-1}\hat{\mu} \leqq \hat{\sigma} \leqq k\hat{\mu}$.

Proof. Let $\phi, \psi \in \Phi_A$ be such that $x = \phi^{-1}(0)$, $y = \psi^{-1}(0)$. Since for every $f \in A$, $\operatorname{Re} \hat{f}(x) = \operatorname{Re} \phi(f)$, $\operatorname{Re} \hat{f}(y) = \operatorname{Re} \psi(f)$, $|\hat{f}|(y) = |\phi(f)|$ and $|\hat{f}|(y) = |\psi(f)|$, the equivalence of statements (i), (ii) and (iii) follows from the equivalence of statements of (iii), (v) and (vi) of Theorem 1.3. Next, (iv) obviously implies (iii), while (iii) implies (iv) by Theorem 2.6.5 of [**3**].

For a real uniform algebra A and $x, y \in M_A$, we see that statement (iii) of Corollary 1.4 defines an equivalence relation on M_A. The equivalence class of x in M_A under this relation will be denoted by $P_A(x)$.

It is then clear that $T(Q_A(\phi)) = P_A(T(\phi))$ for every $\phi \in \Phi_A$.

Remark 1.5. It is easy to see that the well known characterizations of Gleason parts of a complex uniform algebra U (given, for example, in Theorems 2.1 and 2.2, Chapter VI of [**4**]) can be derived from Theorem 1.3 and Corollary 1.4 by regarding U as a real uniform algebra. In addition, we have the following result for a complex uniform algebra U.

Let ϕ and ψ belong to the same part of U. If $f_n \in U$, $\|f_n\| < 1$ for all n and $\phi(f_n) \to a$ as $n \to \infty$ with $|a| = 1$, then every convergent subsequence of $(\psi(f_n))$ converges to a. This follows from (iv) of Theorem 1.3, since $\phi(\bar{a}f_n) = \bar{a}\phi(f_n) \to 1$ whenever $\phi(f_n) \to a$ with $|a| = 1$, so that $\psi(\bar{a}f_n) = \bar{a}\psi(f_n) \to 1$, or $\psi(f_n) \to a$ as $n \to \infty$.

2. Parts of the complexification. Let A be a real function algebra on (X, τ). Define

$$B = \{f + ig: f, g \in A\}.$$

It can be easily seen that for $f, g \in A$,

$$\|f + ig\| = \|f - ig\|$$

so that

$$\|f\|, \|g\| \leqq \|f + ig\| \leqq \|f\| + \|g\|.$$

This shows that B is uniformly closed in $C(X)$. B is thus a complex function algebra on X and can be regarded as the complexification of A. As usual, the maximal ideal space M_B of B will be identified with the space Φ_B of all non-zero complex homomorphisms of B. Define $\alpha: \Phi_A \to \Phi_B$ by

$$\alpha(\phi)(f + ig) = \phi(f) + i\phi(g)$$

for $\phi \in \Phi_A$ and $f, g \in A$. Then α is a bijection, and $\alpha(\phi)|_A = \phi$. Also, since for $\phi, \psi \in \Phi_A$ and $f, g \in A$,

$$|\alpha(\phi)(f + ig) - \alpha(\psi)(f + ig)| = |\alpha(\bar{\phi})(f - ig) - \alpha(\bar{\psi})(f - ig)|,$$

we see that

$$\|\alpha(\phi) - \alpha(\psi)\| = \|\alpha(\bar{\phi}) - \alpha(\bar{\psi})\|.$$

Let $P_B(\alpha(\phi))$ denote the Gleason part of $\alpha(\phi)$ in Φ_B. Then it follows that $\alpha(\psi) \in P_B(\alpha(\phi))$ if and only if $\alpha(\bar{\psi}) \in P_B(\alpha(\bar{\phi}))$.

LEMMA 2.1. *For* $\phi \in \Phi_A$,

$$\alpha(Q_A(\phi)) = P_B(\alpha(\phi)) \cup P_B(\alpha(\bar{\phi})).$$

Proof. If $\alpha(\psi) \in P_B(\alpha(\phi)) \cup P_B(\alpha(\bar{\phi}))$, then

$$\|\alpha(\phi) - \alpha(\psi)\| < 2 \quad \text{or} \quad \|\alpha(\bar{\phi}) - \alpha(\psi)\| < 2.$$

Hence $\|\phi - \psi\| < 2$ or $\|\bar{\phi} - \psi\| < 2$; i.e., $\psi \in Q_A(\phi)$ by (ii) of Theorem 1.3.

Now assume that $\alpha(\psi) \notin P_B(\alpha(\phi)) \cup P_B(\alpha(\bar{\phi}))$. We shall show that $\psi \notin Q_A(\phi)$.

First we assert that there is a sequence $(u_n + iv_n)$ in B such that

$$\|u_n + iv_n\| < 1,$$
$$\alpha(\phi)(u_n + iv_n) = 0 = \alpha(\phi)(u_n - iv_n) \quad \text{for all } n \quad \text{and}$$
$$|\alpha(\psi)(u_n + iv_n)| \to 1 \quad \text{as } n \to \infty.$$

Since $\alpha(\psi) \notin P_B(\alpha(\phi))$, there is a sequence $(f_n + ig_n)$ in B, such that

$$\|f_n + ig_n\| < 1,$$
$$\alpha(\phi)(f_n + ig_n) = 0 \quad \text{for all } n \quad \text{and}$$
$$|\alpha(\psi)(f_n + ig_n)| \to 1 \quad \text{as } n \to \infty.$$

Similarly, since $\alpha(\psi) \notin P_B(\alpha(\bar{\phi}))$, there is a sequence $(h_n + ik_n)$ in B such that

$$\|h_n + ik_n\| < 1,$$
$$\alpha(\bar{\phi})(h_n + ik_n) = 0 \quad \text{for all } n \quad \text{and}$$
$$|\alpha(\psi)(h_n + ik_n)| \to 1 \quad \text{as } n \to \infty.$$

Now let

$$u_n + iv_n = (f_n + ig_n)(h_n + ik_n).$$

Then

$$u_n - iv_n = (f_n - ig_n)(h_n - ik_n),$$

and

$$\|u_n + iv_n\| \leqq \|f_n + ig_n\| \, \|h_n + ik_n\| < 1.$$

Also,

$$\alpha(\phi)(u_n + iv_n) = 0,$$

since

$$\alpha(\phi)(f_n + ig_n) = 0,$$

and

$$\alpha(\phi)(u_n - iv_n) = 0,$$

since

$$\alpha(\phi)(h_n - ik_n) = \overline{\alpha(\bar{\phi})(h_n + ik_n)} = 0.$$

Lastly,

$$|\alpha(\psi)(u_n + iv_n)| = |\alpha(\psi)(f_n + ig_n)| \, |\alpha(\psi)(h_n + ik_n)| \to 1$$

as $n \to \infty$. This proves our assertion.

Now, let $s_n = \alpha(\psi)(u_n + iv_n)$. Then $|s_n| < 1$. Define

$$h_n = \frac{u_n + iv_n - s_n}{1 - \bar{s}_n(u_n + iv_n)} \cdot \frac{u_n - iv_n - \bar{s}_n}{1 - s_n(u_n - iv_n)}.$$

Then $h_n \in A$, $\|h_n\| < 1$, $\psi(h_n) = \alpha(\psi)(h_n) = 0$ for all n and

$$\phi(h_n) = \alpha(\phi)(h_n) = |s_n|^2 \to 1 \quad \text{as } n \to \infty.$$

Hence $\psi \notin Q_A(\phi)$ by (iii) of Theorem 1.3.

We are now in a position to discuss the relationship between the parts of A and those of its complexification B.

THEOREM 2.2. (*a*) *Let* $\phi, \psi \in \Phi_A$. *Then* $\|\phi - \psi\| < 2$ *if and only if*

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$\|\alpha(\phi) - \alpha(\psi)\| < 2$. *In particular*, $\|\phi - \psi\| < 2$ *is an equivalence relation on* Φ_A.

(*b*) *Let* $\phi \in \Phi_A$. *Then*

$$P_B(\alpha(\phi)) = \{\alpha(\psi) : \psi \in \Phi_A, \quad \|\phi - \psi\| < 2\}.$$

Proof. (a) If $\|\alpha(\phi) - \alpha(\psi)\| < 2$, then

$$\|\phi - \psi\| \leqq \|\alpha(\phi) - \alpha(\psi)\| < 2.$$

Now let $\|\phi - \psi\| < 2$, and assume for a moment that $\|\alpha(\phi) - \alpha(\psi)\| = 2$. Then there exists a sequence $(f_n + ig_n)$ in B such that $\|f_n + ig_n\| < 1$ for all n and

$$\alpha(\phi)(f_n + ig_n) \to 1, \quad \alpha(\psi)(f_n + ig_n) \to -1 \quad \text{as } n \to \infty.$$

Since $\|\phi - \psi\| < 2$, Lemma 2.1 shows that

$$\alpha(\psi) \in P_B(\alpha(\phi)) \cup P_B(\alpha(\bar{\phi})).$$

But $\|\alpha(\phi) - \alpha(\psi)\| = 2$, so that

$$\alpha(\psi) \in P_B(\alpha(\bar{\phi})) \quad \text{and} \quad \alpha(\bar{\psi}) \in P_B(\alpha(\phi)).$$

By passing to a subsequence, we can assume by Remark 1.5 that

$$\alpha(\bar{\phi})(f_n + ig_n) \to -1 \quad \text{and} \quad \alpha(\bar{\psi})(f_n + ig_n) \to 1 \quad \text{as } n \to \infty.$$

Thus, we have

$$\phi(f_n) + i\phi(g_n) \to 1, \quad \psi(f_n) + i\psi(g_n) \to -1,$$
$$\phi(f_n) - i\phi(g_n) \to -1 \quad \text{and} \quad \psi(f_n) - i\psi(g_n) \to 1$$

as $n \to \infty$. Hence $\phi(g_n) \to -i$ and $\psi(g_n) \to i$ as $n \to \infty$. This contradicts $\|\phi - \psi\| < 2$, since $\|g_n\| \leqq \|f_n + ig_n\| < 1$ for all n.

Since $\|\alpha(\phi) - \alpha(\psi)\| < 2$ is an equivalence relation on Φ_B, we now see that $\|\phi - \psi\| < 2$ is an equivalence relation on Φ_A.

$$\text{(b) } P_B(\alpha(\phi)) = \{\alpha(\psi) : \psi \in \Phi_A, \quad \|\alpha(\psi) - \alpha(\phi)\| < 2\}$$
$$= \{\alpha(\psi) : \psi \in \Phi_A, \quad \|\phi - \psi\| < 2\},$$

by (a) above.

COROLLARY 2.3. *Let* $\phi \in \Phi_A$ *such that* $\|\phi - \bar{\phi}\| = 2$. *Then* $\|\psi - \bar{\psi}\| = 2$ *for all* $\psi \in Q_A(\phi)$.

Proof. Let $\psi \in Q_A(\phi)$, and assume for a moment that $\|\psi - \bar{\psi}\| < 2$. Then by (ii) of Theorem 1.3, $\|\phi - \psi\| < 2$ or $\|\bar{\phi} - \psi\| < 2$. If $\|\phi - \psi\| < 2$, then clearly, $\|\bar{\phi} - \bar{\psi}\| < 2$. This together with $\|\psi - \bar{\psi}\| < 2$ implies that $\|\phi - \bar{\phi}\| < 2$ by the transitivity guaranteed in Theorem 2.2(a). Hence $\|\phi - \psi\|$ cannot be less than 2. A similar argument shows that $\|\bar{\phi} - \psi\|$ cannot be less than 2. Thus, $\|\psi - \bar{\psi}\| = 2$.

COROLLARY 2.4. *Let* $\phi \in \Phi_A$. *Then*

(a) $\|\phi - \bar{\phi}\| < 2$ *if and only if* $Q_A(\phi) = \{\psi \in \Phi_A : \|\phi - \psi\| < 2\}$.

In this case, $P_B(\alpha(\phi)) = \alpha(Q_A(\phi)) = P_B(\alpha(\bar{\phi}))$.

(b) $\|\phi - \bar{\phi}\| = 2$ *if and only if* $Q_A(\phi)$ *is the disjoint union of*

$$\{\psi \in \Phi_A : \|\phi - \psi\| < 2\} \quad \textit{and} \quad \{\psi \in \Phi_A : \|\bar{\phi} - \psi\| < 2\}.$$

In this case, $\alpha(Q_A(\phi))$ *is the disjoint union of* $P_B(\alpha(\phi))$ *and* $P_B(\alpha(\bar{\phi}))$.

Proof. (a) Let $\|\phi - \bar{\phi}\| < 2$, and $\psi \in Q_A(\phi)$. By (ii) of Theorem 1.3, $\|\phi - \psi\| < 2$ or $\|\bar{\phi} - \psi\| < 2$. If $\|\bar{\phi} - \psi\| < 2$, then by the transitivity guaranteed in Theorem 2.2(a), $\|\phi - \bar{\phi}\| < 2$ implies that $\|\phi - \psi\| < 2$. Hence

$$Q_A(\phi) = \{\psi \in \Phi_A : \|\phi - \psi\| < 2\}.$$

The converse is obvious since $\bar{\phi} \in Q_A(\phi)$ always.

(b) follows by (ii) of Theorem 1.3, Corollary 2.3 and Theorem 2.2(a).

Using the above results, we shall now compute the parts of some real function algebras. A part of a real function algebra having more than two points will be called a *nontrivial part*. We may recall that a part of a complex function algebra is said to be nontrivial if it has more than one point.

Example 2.5. Let Y be a compact subset of $\mathbf{C}$ which is symmetric about the x-axis (that is, $\bar{z} \in Y$ for all $z \in Y$) and whose complement in $\mathbf{C}$ has a finite number of components. Let X be the boundary of Y. Let A be the algebra of all functions on X which are uniform limits of sequences of rational functions p/q, where p and q are polynomials with real coefficients and q has no zeros on Y. Then A is a real function algebra on (X, τ), where $\tau : X \to X$ is defined by $\tau(z) = \bar{z}$. The complexification $B = \{f + ig : f, g \in A\}$ of A is the algebra of all functions on X which are limits of sequences of rational functions with poles off Y. The nontrivial Gleason parts of B are the components of the interior of Y, whereas each point on X is a trivial part. (See Theorem 4.4, Chapter VI of [**4**].) Hence, by Lemma 2.1, we see that for any component E of the interior of Y, $E \cup \bar{E}$ is a nontrivial part of A, where $\bar{E} = \{\bar{z} : z \in E\}$, $\{z, \bar{z}\}$ are two-point parts for $z \in X - \mathbf{R}$ and the points in $X \cap \mathbf{R}$ are one-point parts. In particular, if X is the unit circle, then the open unit disc is a nontrivial part, $\{z, \bar{z}\}$ are two-point parts for $|z| = 1$, $z \neq \pm 1$, while $\{1\}$ and $\{-1\}$ are one-point parts.

Example 2.6 (Standard algebras on Klein surface). Let Y be a compact nonorientable Klein surface with nonempty boundary ∂Y. Then Y admits an orienting double (X, p, τ), where X is a compact Riemann surface with boundary ∂X, $p : X \to Y$ is a two to one covering map such

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that $p(\partial X) = \partial Y$, $p^{-1}(\partial Y) = \partial X$ and $\tau : X \to X$ is an antianalytic involution of X such that $p \circ \tau = p$. (For definitions, see p. 40 of [**2**].) Let $B = \{f \in C(\partial X) : f$ admits an analytic extension to the interior X^0 of $X\}$. Let $A = \{h \in B : h(x) = \overline{h(\tau(x))}$ for all $x \in X\}$. Then A is a real function algebra on $(\partial X, \tau)$ and B is the complexification of A. The only nontrivial Gleason part of B is X^0 and each point in ∂X is a trivial part. (This follows from Theorem 3 of [**8**] and Corollary IV.3 of [**9**].) Hence, by Lemma 2.1, a nontrivial Gleason part of A is X^0 and $\{x, \tau(x)\}$ are two-point parts for $x \in \partial X$. As a concrete example, let Y be a Mobius strip. Then $X = \{z \in \mathbf{C}, r \leqq |z| \leqq r^{-1}\}$ for some r with $0 < r < 1$ and $\tau(z) = -1/\bar{z}$. Now $X^0 = \{z \in \mathbf{C}, r < |z| < r^{-1}\}$ is a nontrivial Gleason part of A, and $\{z, -1/\bar{z}\}$ are two-point parts for $|z| = r$.

So far we have considered examples of real function algebras whose complexifications are well known complex function algebras. Now, we consider an example of the opposite nature.

Example 2.7. (Cf. Section 2 of [**7**].) Let U be a complex function algebra with the maximal ideal space Z. Let $\{z_1, \ldots, z_q\}$ be a specified finite subset of q points in Z and let D_k be a continuous point derivation of U at z_k for each k. Let $A_q = \{f \in U : f(z_k)$ and $D_k(f)$ are real for $1 \leqq k \leqq q\}$. Then A_q is a real uniform algebra. Let Y be its maximal ideal space. The restriction map $j : Z \to Y$ given by $j(z) = z \cap A_q$ is one-one and onto (Proposition 2.2 of [**7**]). Hence we can and shall identify Y with Z.

THEOREM 2.8. *For every* $z \in Z$, $P_{A_q}(z) = P_U(z)$.

Proof. If $z' \in P_U(z)$, then

$$\begin{aligned} &\sup\{|f(z)| : f \in A_q, \quad \|f\| < 1, \quad f(z') = 0\} \\ &\leqq \sup\{|f(z)| : f \in U, \quad \|f\| < 1, \quad f(z') = 0\} < 1. \end{aligned}$$

Hence $z' \in P_{A_q}(z)$ by (i) of Corollary 1.4. Thus,

$$P_U(z) \subset P_{A_q}(z).$$

Now, consider $z' \notin P_U(z)$. Then by renaming the z_j's, if necessary, we can assume that the first p points belong to $P_U(z)$ while $z_{p+1}, \ldots, z_q$ do not belong to $P_U(z)$, where $0 \leqq p \leqq q$. Then there exist sequences $(f_{n,k})$, $p + 1 \leqq k \leqq q$, and a sequence (f_n') in U such that $\|f_{n,k}\|$, $\|f_n'\| < 1$, $f_{n,k}(z_k) = f_n'(z') = 0$ for all n and $k = p + 1, \ldots, q$, and $|f_{n,k}(z)| \to 1$, $p + 1 \leqq k \leqq q$, $|f_n'(z)| \to 1$ as $n \to \infty$.

Let $f_n = f_{n,p+1}^2 \ldots f_{n,q}^2 f_n'$. Then $\|f_n\| < 1$, $f_n(z_k) = f_n(z') = 0$ for all n and $k = p + 1, \ldots, q$ and $|f_n(z)| \to 1$ as $n \to \infty$. Also,

$$D_k(f_n^2) = 2f_n(z_k)D_k(f_n) = 0$$

for all n and $k = p + 1, \ldots, q$.

Now, if $p = 0$, then $f_n^2 \in A_q$, $\|f_n^2\| < 1$, $f_n^2(z') = 0$ for all n and $|f_n^2(z)| \to 1$ as $n \to \infty$, and it follows that $z' \notin P_{A_q}(z)$.

If $0 < p \leqq q$, we construct functions $h_n \in A_q$ such that $\|h_n\| < 1$, $h_n(z') = 0$ for all n and $|h_n(z)| \to 1$ as $n \to \infty$. Let $f_n^2(z_j) = \alpha_{n,j}$ for $j = 1, 2, \ldots, p$. Then $|\alpha_{n,j}| < 1$ and $|\alpha_{n,j}| \to 1$ as $n \to \infty$, because $|f_n^2(z)| \to 1$ as $n \to \infty$, and $z_j \in P_U(z)$ for $1 \leqq j \leqq p$.

Define

$$g_{n,j} = \bar{\alpha}_{n,j} \frac{f_n^2 - \alpha_{n,j}}{1 - \bar{\alpha}_{n,j} f_n^2}$$

for all n and $j = 1, 2, \ldots, p$. Then for $p + 1 \leqq k \leqq q$,

$$\begin{aligned} D_k(g_{n,j}) &= \bar{\alpha}_{n,j}[(f_n^2 - \alpha_{n,j})(z_k) D_k((1 - \bar{\alpha}_{n,j} f_n^2)^{-1}) \\ &\qquad + D_k((f_n^2 - \alpha_{n,j}))(1 - \bar{\alpha}_{n,j} f_n^2)^{-1}(z_k)] \\ &= \bar{\alpha}_{n,j}[-(f_n^2(z_k) - \alpha_{n,j})(1 - \bar{\alpha}_{n,j} f_n^2(z_k))^{-2}(D_k(1) \\ &\quad - \bar{\alpha}_{n,j} D_k(f_n^2)) + (D_k(f_n^2) - \alpha_{n,j} D_k(1))(1 - \bar{\alpha}_{n,j} f_n^2(z_k))^{-1}] = 0, \end{aligned}$$

because $D_k(1) = D_k(f_n^2) = 0$ for all n and $p + 1 \leqq k \leqq q$.

Let $g_n = g_{n,1} \ldots g_{n,p}$. Then $\|g_n\| < 1$ for all n. Let $1 \leqq j \leqq p$. Then $g_n(z_j) = 0$, and hence

$$D_j(g_n^2) = 2 g_n(z_j) D_j(g_n) = 0 \quad \text{for all } n.$$

Let $p + 1 \leqq k \leqq q$. Since $D_k(g_{n,j}) = 0$ for all n and $j = 1, \ldots, p$, it follows that

$$D_k(g_n) = 0 = D_k(g_n^2).$$

Thus, we see that $g_n^2(z_j) = 0$ for all n and $j = 1, \ldots, p$ and $D_j(g_n^2) = 0$ for all n and $j = 1, \ldots, q$. Next, for all n and $k = p + 1, \ldots, q$,

$$\begin{aligned} g_n(z_k) &= g_n(z') \\ &= (-1)^p |\alpha_{n,1}|^2 \ldots |\alpha_{n,p}|^2 \\ &= \alpha_n, \text{ say.} \end{aligned}$$

Then $|\alpha_n| < 1$, α_n is real and $|\alpha_n| \to 1$ as $n \to \infty$. Let

$$h_n = \frac{g_n^2 - \alpha_n^2}{1 - \alpha_n^2 g_n^2}.$$

Then for all n, $h_n \in U$, $\|h_n\| < 1$ and

$$h_n(z_j) = \begin{cases} -\alpha_n^2 & \text{for } j = 1, 2, \ldots, p \\ 0 & \text{for } j = p + 1, \ldots, q. \end{cases}$$

Also, since $D_j(g_n^2) = 0$ for $j = 1, \ldots, q$, it follows as before that $D_j(h_n) = 0$ for $j = 1, \ldots, q$. Hence $h_n \in A_q$ for all n. Note that $h_n(z') = 0$, and $|h_n(z)| \to 1$ as $n \to \infty$, because $|h_n(z_1)| = \alpha_n^2 \to 1$ as

$n \to \infty$ and $z_1 \in P_U(z)$. Hence $z' \notin P_{A_q}(z)$. This proves $P_{A_q}(z) \subset P_U(z)$. Hence $P_U(z) = P_{A_q}(z)$.

Now, let B be the complexification of A_q. It was shown in Proposition 2.3 of [**7**] that the maximal ideal space X of B is homeomorphic to two copies of the maximal ideal space Z of U pasted together at $\{z_1, \dots, z_q\}$. We now show that a similar situation exists for the Gleason parts of B and U.

Let $cx^* = T \circ \alpha^{-1}$ so that the following diagram commutes:

$$\begin{tikzcd} \Phi_{A_q} \arrow[r, "\alpha"] \arrow[d, "T"'] & X \arrow[dl, "cx^*"] \\ Z & \end{tikzcd}$$

THEOREM 2.9. *Let $x \in X$ and $z = cx^*(x)$.*

(i) *If $P_U(z) \cap \{z_1, \dots, z_q\}$ is empty, then $P_B(x)$ is homeomorphic to $P_U(z)$.*

(ii) *If $P_U(z) \cap \{z_1, \dots, z_q\}$ is nonempty, then $P_B(x)$ is homeomorphic to two copies of $P_U(z)$ identified as follows: if $z_j \in P_U(z)$, let the two points over it be identified.*

Proof. Let $\phi \in \Phi_{A_q}$ with $\alpha(\phi) = x$. Then $T(\phi) = z$, and $\phi(f) = f(z)$ for all $f \in A_q$, or $\phi(f) = \overline{f(z)}$ for all $f \in A_q$. We can assume without loss of generality that $\phi(f) = f(z)$ for all $f \in A_q$. Now, by Theorem 2.8,

$$P_U(z) = P_{A_q}(z) = P_{A_q}(T(\phi)),$$

while

$$P_{A_q}(T(\phi)) = T(Q_{A_q}(\phi)).$$

But, by Lemma 2.1,

$$Q_{A_q}(\phi) = \alpha^{-1}(P_B(\alpha(\phi)) \cup P_B(\alpha(\bar{\phi}))).$$

Hence we see that

$$P_U(z) = cx^*(P_B(\alpha(\phi)) \cup P_B(\alpha(\bar{\phi}))).$$

(i) Let $z_j \notin P_U(z)$ for all $j = 1, \dots, q$. Then there exists a sequence of functions (f_n) in U such that $\|f_n\| < 1$, $f_n(z_j) = 0$ for all n and $j = 1, \dots, q$ and $f_n(z) \to 1$ as $n \to \infty$. Then it follows that for $j = 1, \dots, q$ and $n = 1, 2, \dots,$

$$if_n^2(z_j) = 0 = D_j(if_n^2).$$

Hence $if_n^2 \in A_q$ for all n. Also, $\|if_n^2\| < 1$ and $if_n^2(z) \to i$. Hence $\phi(if_n^2) \to i$ and $\bar{\phi}(if_n^2) \to -i$. This shows that $\|\phi - \bar{\phi}\| = 2$. By Theorem 2.2(a), $\|\alpha(\phi) - \alpha(\bar{\phi})\| = 2$ so that $P_B(\alpha(\phi)) \cap P_B(\alpha(\bar{\phi}))$ is empty.

Thus, $cx^* |_{P_B(\alpha(\phi))}$ is one to one. Also, $cx^*(P_B(\alpha(\phi)) = P_U(z)$. Since cx^* is continuous and open (Cf. Addendum in [**6**]), it follows that $P_B(x)$ is homeomorphic to $P_U(z)$ under cx^*.

(ii) Let $z_j \in P_U(z)$ for some $j = 1, \ldots, q$. If $\psi \in \Phi_{A_q}$ is such that $\psi(f) = f(z_j)$ for $f \in A_q$, then $\psi = \bar{\psi}$, so that $0 = \|\psi - \bar{\psi}\| < 2$. Since $\psi \in Q_{A_q}(\phi)$, it follows by Corollary 2.3 that $\|\phi - \bar{\phi}\| < 2$. Hence by Corollary 2.4(a), $P_B(\alpha(\phi)) = P_B(\alpha(\bar{\phi}))$. Now, as in the proof of Proposition 2.3 of [**7**], there is a continuous section s of cx^* over $P_U(z)$ such that $P_B(x)$ is the union of $s(P_U(z))$ and $\{\alpha(\bar{\psi}):\alpha(\psi) \in s(P_U(z))\}$. Hence we conclude that $P_B(x)$ is homeomorphic to two copies of $P_U(z)$ pasted together at those z_j's which belong to $P_U(z)$.

3. Parts, analyticity and harmonicity. Let A be a real function algebra and $B = \{f + ig : f, g \in A\}$ its complexification. In this section, we employ a well known result (see, e.g., p. 161 of [**4**]) about the existence of an analytic structure in Φ_B to obtain a similar result for Φ_A. This, in turn, implies the presence of harmonic structure in M_A in the form of a connected finite Klein surface, which can be orientable or nonorientable (Examples 3.4, 3.5). This section heavily uses concepts appearing in the monograph [**2**].

THEOREM 3.1. *Let A be a real function algebra on (X, τ) and $\phi \in \Phi_A$. Suppose that there is a $\psi \neq \phi$ in Φ_A such that $\|\phi - \psi\| < 2$. Also, assume that the linear span of the set of regular Borel probability measures μ on X satisfying $\int_X f d\mu = \phi(f)$ for all $f \in A$ is finite dimensional, and that there exists a unique regular Borel probability measure σ on X satisfying*

$$\log |\phi(f) + i\phi(g)| = \int_X \log |f + ig| d\sigma$$

for all pairs of functions $f, g \in A$ with $f^2 + g^2$ invertible in A. Let

$$W = \{\theta \in \Phi_A : \|\theta - \phi\| < 2\} \quad \text{and}$$
$$\bar{W} = \{\theta \in \Phi_A : \|\theta - \bar{\phi}\| < 2\}.$$

Then W and $\bar{W}$ can be given the structures of connected finite open Riemann surfaces in such a way that for every $f \in A$, $\hat{f}$ is a bounded holomorphic function in W as well as on $\bar{W}$. Moreover, with respect to these structures, the map $\tau_0 : W \to \bar{W}$ given by $\tau_0(\theta) = \bar{\theta}$ is antianalytic.

In particular, if there is a unique probability measure μ on X satisfying $\int_X f d\mu = \phi(f)$ for all $f \in A$, then W and $\bar{W}$ can, in fact, be given the structure of an open unit disc in **C**.

Proof. Since $\|\phi - \psi\| < 2$, $P = P_B(\alpha(\phi))$ is nontrivial by Theorem 2.2(b). The assumed conditions imply that the set of representing measures for $\alpha(\phi)$ is finite dimensional and $\alpha(\phi)$ has a unique logmodular measure. (For definitions, see p. 31 and p. 110 of [**4**].) Hence by Theorem 7.5, Chapter VI of [**4**], P can be given the structure of a connected finite open Riemann surface such that $\widehat{(f+ig)}$ is a bounded holomorphic function for every $f+ig \in B$. Hence $W = \alpha^{-1}(P)$ can be given the structure of a connected finite open Riemann surface such that for every $f \in A$, $\hat{f}$ is a bounded holomorphic function on W.

Note that for any $\alpha(\psi) \in \Phi_B$, σ is a representing (respectively, logmodular) measure for $\alpha(\psi)$ if and only if $\bar{\sigma}$ is a representing (respectively, logmodular) measure for $\alpha(\bar{\psi})$, where $\bar{\sigma}$ is defined by

$$\bar{\sigma}(E) = \sigma(\{\alpha(\bar{\theta}) : \alpha(\theta) \in E\}).$$

This shows that $\alpha(\bar{\phi})$ also has a unique logmodular measure and the dimension of the set of representing measures for $\alpha(\bar{\phi})$ is the same as that for $\alpha(\phi)$; hence it is finite.

Thus, $\bar{W} = \{\theta \in \Phi_A : \|\theta - \bar{\phi}\| < 2\}$ can be given the structure of a finite open Riemann surface in exactly the same fashion as above.

To prove that the map $\tau_0 : W \to \bar{W}$ is antianalytic, let $U = (U_j, \alpha_j)_{j \in J}$ and $V = (V_k, \beta_k)_{k \in K}$ be analytic atlases over W and $\bar{W}$. (For definition, see p. 5 of [**2**].) Let $\theta \in W$ be such that $\theta \in U_j$ and $\tau_0(\theta) \in V_k$. We can find a bounded holomorphic function F on $\bar{W}$ whose ramification index at $\tau_0(\theta)$ is one. (For definition, see p. 27 of [**2**].) By Theorem 7.5, Chapter VI of [**4**], there exists a sequence $(f_n + ig_n)$ in B such that $\widehat{(f_n + ig_n)}$ converges to F uniformly on compact subsets of $\bar{W}$. Since $\hat{f}_n$ and $\hat{g}_n$ are bounded holomorphic functions on $\bar{W}$, $\widehat{(f_n + ig_n)} \circ \tau_0 = \bar{\hat{f}}_n + i\bar{\hat{g}}_n$ is antianalytic on W for each n. Hence $F \circ \tau_0$ is antianalytic. Now, let

$$f = \beta_k \circ \tau_0 \circ \alpha_j^{-1}, \quad g = F \circ \beta_k^{-1} \quad \text{and}$$
$$h = F \circ \tau_0 \circ \alpha_j^{-1} = g \circ f.$$

Then g is analytic and h is antianalytic. Let $w = f(z)$. Then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial z}$$

by Lemma 1.1.2 of [**2**]. Since h is antianalytic, $\partial h/\partial z = 0$ and since g is analytic $\partial g/\partial \bar{w} = 0$. Thus,

$$\frac{\partial g}{\partial w}\frac{\partial f}{\partial z} = 0.$$

But $\partial g/\partial w \neq 0$ as the ramification index of F at $\tau_0(\theta)$ is 1. Hence $\partial f/\partial z = 0$; that is, $\beta_k \circ \tau_0 \circ \alpha_j^{-1}$ is antianalytic. Hence τ_0 is antianalytic.

Finally, if $\alpha(\phi)$ has a unique representing measure on X, then $P = P_B(\alpha(\phi))$ can be given the structure of an open unit disc. (See Theorem 7.2, Chapter VI of [**4**].) Hence W and $\bar{W}$ can also be given the structure of an open unit disc.

Remark 3.2. The conditions of Theorem 3.1 are satisfied if the Dirichlet-deficiency and the Imaginary Dirichlet-deficiency of A are finite, and the Arens-Singer deficiency and the Inverse Arens-Singer deficiency are zero. (See [**7**] for definitions and examples.)

COROLLARY 3.3. *Assume that the hypotheses in Theorem* 3.1 *hold.*

(i) *If* $\|\phi - \bar{\phi}\| = 2$, *then* $Q_A(\phi)$ *is the disjoint union of the two connected finite open Riemann surfaces* W *and* $\bar{W}$. *Also,* $V = P_A(T(\phi)) \subset M_A$ *can be given the structure of a connected finite open Riemann surface in such a way that* $\mathrm{Re}\,\hat{f}$ *is a bounded harmonic function on* V *for every* $f \in A$.

(ii) *If* $\|\phi - \bar{\phi}\| < 2$, *then* $V = P_A(T(\phi)) \subset M_A$ *can be given the structure of a finite connected Klien surface without boundary in such a way that for every* $f \in A$, $\mathrm{Re}\,\hat{f}$ *is a bounded harmonic function on* V. *(For definition, see p.* 6 *of* [**2**].*)* W *is canonically isomorphic to the complex double* V_c *of* V. *(For definition, see p.* 40 *of* [**2**].*) If* τ_0 *has no fixed points in* W *(that is, there is no* $\psi \in W$ *such that* $\psi = \bar{\psi}$*), then* V *is nonorientable and* $W = V_c$ *is also the orienting double* V_0 *of* V.

Proof. (i) If $\|\phi - \bar{\phi}\| = 2$, then by Corollary 2.4(b), $Q_A(\phi)$ is the disjoint union of the connected finite open Riemann surfaces

$$W = \{\theta \in \Phi_A : \|\theta - \phi\| < 2\} \quad \text{and}$$
$$\bar{W} = \{\theta \in \Phi_A : \|\theta - \bar{\phi}\| < 2\}.$$

Since $T|_W$ is one to one and onto $V = P_A(T(\phi))$, V is also a connected finite open Riemann surface.

(ii) Let now $\|\phi - \bar{\phi}\| < 2$. Then by Corollary 2.4(a), $W = \bar{W}$. Now, τ_0 is an antianalytic involution on W. The quotient space W/τ_0 can be identified with V via the quotient map T. By Theorem 1.8.4 of [**2**], V has a unique dianalytic structure such that the map T of W onto V is a morphism of Klien surfaces. (For definition, see p. 17 of [**2**]. Theorem 1.8.4 of [**2**] is proved for a group G of automorphisms on a Klein surface which act discontinuously on it. In the present case, W can be regarded as a Klein surface, and $G = \{i_0, \tau_0\}$, where i_0 denotes the identity map on W.) Since T is a morphism, V is finite and connected. It also follows that $\mathrm{Re}\,\hat{f}$, $f \in A$, is a bounded harmonic function on V. By Proposition 1.9.1 of [**2**], W is canonically isomorphic to the complex double V_c of V. Scanning carefully through the proof of Theorem 1.8.4 of [**2**], it can be seen that only fixed points of τ_0 are sent to the boundary of V by T (as W has no boundary points). Hence if τ_0 does not have any fixed point, then the boundary of V is empty. That V is nonorientable follows from Lemma 1.6.3 of [**2**]. In this case, the orienting double V_0 of V is the same as the complex double V_c of V which is isomorphic to W.

Finally, we give two examples to show that if the involution τ_0 does have fixed points, then the Klein surface V can be either orientable or nonorientable.

Example 3.4. Let A be the algebra of continuous functions on the closed unit disc which are analytic in the open unit disc and are real on the real axis. Let ϕ be the evaluation functional at 0. Then $W = Q_A(\phi)$ is the open unit disc and $\tau_0 : W \to W$ is given by $\tau_0(z) = \bar{z}$. The set F of fixed points of τ_0 is the open interval $(-1, 1)$ and $P_A(0) = V = W/\tau_0$ is

$$\{z : |z| < 1, \quad \mathrm{Im}\, z \geqq 0\},$$

which is orientable.

Example 3.5. Let $w = 1 + i$ and $L = \{n + mw : n, m \text{ integers}\}$. Since $\bar{w} = 1 - i = 2 - w \in L$, the map $z \to \bar{z}$ descends to the complex torus $S = \mathbf{C}/L$. Let $\tau : S \to S$ be the map induced by $z \to -i\bar{z}$. Then τ is anti-analytic. The set F of fixed points of τ is given by $F = \{tw : t \in \mathbf{R}\}$. Hence F is a circle in S. Let D be an open disc in S such that $\bar{D} \cap F$ is empty. Then $X = S - \{D \cup \tau(D)\}$ is a compact bordered Riemann surface and $\tau|_S$ is an antianalytic involution on X. Let ∂X be the boundary of X and A be the real function algebra on $(\partial X, \tau)$ as described in Example 2.6. Let ϕ be the evaluation functional at an interior point of X. Then $W = Q_A(\phi) = X^0$ and $\tau_0 = \tau|_{X^0}$. In this case, $P_A(T(\phi)) = V = W/\tau_0$ is a Mobius strip with one disc removed, which is nonorientable. (Cf. Example 1.6.3, Proposition 1.9.1 and Corollary 1.9.3 of [**2**].)

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