BULLETIN **13** (2), 1970

A UNITARY RELATION BETWEEN A MATRIX AND ITS TRANSPOSE

BY
WILLIAM R. GORDON

It is well known that if A is an $n \times n$ complex matrix and A^T is its transpose, then there is an invertible $n \times n$ complex matrix S such that $A^T = S^{-1}AS$. In this note we wish to point out another simple relation between A and A^T.

THEOREM 1. *If A is an $n \times n$ complex matrix and A^T is its transpose then there are unitary $n \times n$ complex matrices U and V such that $A^T = UAV$.*

We shall deduce Theorem 1 from Theorem 2 below. Before stating Theorem 2, we remind the reader that the singular values of an $m \times n$ complex matrix A are the nonnegative square roots of the characteristic values of the matrix A^*A (in this note A^* denotes the conjugate transpose of A).

THEOREM 2. *Two $n \times n$ complex matrices A and B have the same singular values if and only if there are unitary $n \times n$ complex matrices U and V such that $B = UAV$.*

Proof. If there are unitary U and V such that $B = UAV$, then $B^*B = V^*(A^*A)V$ and so B^*B and A^*A being similar have the same characteristic values and hence A and B have the same singular values.

In [1] Carl Eckart and Gale Young showed that if C is an $m \times n$ complex matrix then there are unitary $m \times m$ and $n \times n$ complex matrices U and V, respectively, such that the matrix $UCV = D = (d_{ij})$ satisfies:

$d_{ij} = 0$ for $i \neq j$
d_{ii} is real and nonnegative for $i = 1, \ldots, \min\{m, n\}$.

It is easily checked that the d_{ii} are the singular values of C.

Thus if the $n \times n$ complex matrices A and B have the same singular values $d_1, \ldots, d_n$ and if D denotes the diagonal matrix diag $(d_1, \ldots, d_n)$, then it follows from the result of Eckart and Young that there are unitary $n \times n$ complex matrices U_1, V_1, U_2, V_2 such that $U_1AV_1 = D = U_2BV_2$. Thus $B = UAV$ where U and V are the unitary matrices $U_2^*U_1$ and $V_1V_2^*$ respectively.

Theorem 1 now follows immediately from Theorem 2 since it is easily checked that an $n \times n$ complex matrix and its transpose have the same singular values.

We point out that Theorem 1 can be deduced directly from [1]. However, Theorem 2 seems to us to be interesting in itself.

REFERENCE

1. C. Eckart and G. Young, *A principal axis transformation for non-hermitian matrices*. Bull. Amer. Math. Soc. **45** (1939), 118–121.

UNIVERSITY OF VICTORIA,
VICTORIA, BRITISH COLUMBIA