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WEAKLY REGULAR ALGEBRAS, BOOLEAN ORTHOGONALITIES AND DIRECT PRODUCTS OF INTEGRAL DOMAINS

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1. Introduction. In this paper we consider representations of weakly regular algebras with permutable congruences and a Boolean orthogonality as direct products of orthoprime algebras. Our theorems generalize results of Abian [**2**; **3**] and Speed [**24**] which characterize direct products of integral domains, and results of Abian [**1**] and Chacron [**7**] which characterize direct products of division rings.

Throughout this paper, the set of elements of a finitary algebra, which is denoted by a capital script letter, will be denoted by the corresponding capital Latin letter. Every algebra is assumed to possess a distinguished element 0 which is a nullary operation and is called the *zero* of the algebra. If $\mathscr{A}$ is an algebra and θ is a congruence on $\mathscr{A}$, then $\ker \theta = \{x \in A : x \equiv 0(\theta)\}$ is the *kernel* of θ. A subset $J \subseteq A$ is an *ideal* if J is the kernel of some congruence on $\mathscr{A}$. The supremum in the lattice of ideals will be denoted by $+$. An algebra is *weakly regular* if the map $\theta \to \ker \theta$ is a lattice isomorphism from the lattice of congruences to the lattice of ideals, so each ideal J of a weakly regular algebra is the kernel of a unique congruence which will be denoted by $\theta(J)$.

2. Boolean orthogonalities and the Sussman order. Let $\mathscr{A}$ be an algebra and $\perp$ a binary relation on A. For $X \subseteq A$, $X^\perp = \{a \in A : a \perp x \text{ for all } x \in X\}$, $X^{\perp\perp} = (X^\perp)^\perp$, and $x^\perp = \{x\}^\perp$ for each $x \in A$. The set $\mathscr{B}(A) = \{X^\perp : X \subseteq A\}$ is the set of *polars*. The relation $\perp$ is a *Boolean orthogonality* if $x \perp 0$, $x \perp y \Rightarrow y \perp x$, $x \perp x \Rightarrow x = 0$, $x^{\perp\perp} \cap y^{\perp\perp} = \{0\} \Rightarrow x \perp y$, for all $x, y \in A$, and every polar is an ideal of $\mathscr{A}$. If $\perp$ is a Boolean orthogonality then $\mathscr{B}(A)$ is a Boolean algebra in which the infimum is set-theoretic intersection, the supremum is defined by $\vee X_\lambda = (\cap X_\lambda{}^\perp)^\perp$, and the complement of X is $X^\perp$. See [**10**; **11**] for the basic properties of Boolean orthogonalities.

Let $\mathscr{A}$ be a weakly regular algebra with a Boolean orthogonality. An ideal J of $\mathscr{A}$ is a $\perp$*-prime* $\perp$*-ideal* if $F^{\perp\perp} \subseteq J$ for all finite subsets $F \subseteq J$, and $x^{\perp\perp} \cap y^{\perp\perp} \subseteq J$ implies that $x \in J$ or $y \in J$ for all $x, y \in A$. The algebra $\mathscr{A}$ is *orthorepresentable* if it has a *separating set* $\mathbf{P}$ (that is, $\cap\{P : P \in \mathbf{P}\} = \{0\}$) of $\perp$-prime $\perp$-ideals. We note that if $\mathscr{B}(A)$ is atomic or if $\mathscr{D}(A) =$

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$\{F^{\perp\perp} : F \subseteq A,\ F \text{ finite}\}$ is a lattice (equivalently, a (distributive) sublattice of $\mathscr{B}(A)$), then $\mathscr{A}$ is orthorepresentable [**10**, 2.14; **11**, 2.1]. If $\mathbf{P}$ is a separating set of $\perp$-prime $\perp$-ideals of $\mathscr{A}$, define a relation $\leqq_{\mathbf{P}}$ on A by $x \leqq_{\mathbf{P}} y$ if and only if $x \in P$ or $x \equiv y(\theta(P))$ for all $P \in \mathbf{P}$. It is straightforward to check that $\leqq_{\mathbf{P}}$ is a partial order.

Part (i) of the following proposition is similar to [**13**, Theorem 1] and the main theorem in [**12**], so its proof will be omitted.

PROPOSITION 2.1. *Let $\mathscr{V}$ be a variety of weakly regular algebras.*

(i) *There is a finite set of binary polynomials $\{p_1, \ldots, p_n\}$ such that for any algebra $\mathscr{A}$ in $\mathscr{V}$, $p_i(a, a) = 0$ for all $a \in A$ and all $i = 1, \ldots, n$, and $p_i(a, b) = 0$ for all $i = 1, \ldots, n$ implies $a = b$ for all $a, b \in A$.*

(ii) *Let $\{p_1, \ldots, p_n\}$ be a set of polynomials satisfying* (i) *above. If $\mathscr{A}$ is an algebra in $\mathscr{V}$ which has a Boolean orthogonality and a separating set $\mathbf{P}$ of $\perp$-prime $\perp$-ideals, then*

$$a \leqq_{\mathbf{P}} b \text{ if and only if } a \perp p_i(a, b) \text{ for all } i = 1, \ldots, n.$$

Proof. To establish (ii) we observe that the following statements are equivalent for any two elements $a, b \in A : a \leqq_{\mathbf{P}} b$; for each $P \in \mathbf{P}$, either $a \in P$ or $a \equiv b(\theta(P))$; for each $P \in \mathbf{P}$, either $a \in P$ or $p_i(a, b) \in P$ for all $i = 1, \ldots, n$; for each $P \in \mathbf{P}$, $a^{\perp\perp} \cap p_i(a, b)^{\perp\perp} \subseteq P$ for all $i = 1, \ldots, n$; $a^{\perp\perp} \cap p_i(a, b)^{\perp\perp} = \{0\}$ for all $i = 1, \ldots, n$; $a \perp p_i(a, b)$ for all $i = 1, \ldots, n$.

It follows from Proposition 2.1 that the partial order $\leqq_{\mathbf{P}}$ depends only on the orthogonality and so, whenever $\mathscr{A}$ is orthorepresentable, we shall denote the partial order by $\leqq$. This order was first studied by Sussman [**25**] in connection with *reduced* ($x^2 = 0 \Rightarrow x = 0$) rings. More recently it has been studied in [**1**; **4**; **7** and **16**].

Example 2.2. A multi-operator group is an algebra $\langle A; +, -, 0, \Omega\rangle$ such that $\langle A; +, -, 0\rangle$ is a non-abelian group with neutral element 0 and $\langle A; \Omega\rangle$ is an algebra such that $\omega(0, \ldots, 0) = 0$ for all non-nullary operations $\omega \in \Omega$ [**17**, pp. 97–102]. The set consisting of the single polynomial $p(x, y) = y - x$ satisfies 2.1(i) for the variety of multi-operator groups. Thus, for any multi-operator group with a Boolean orthogonality which is orthorepresentable, the order is given by $x \leqq y$ if and only if $x \perp y - x$.

Examples of multi-operator groups with a Boolean orthogonality include reduced alternative rings ($x \perp y$ if and only if $xy = 0$) and abelian torsion groups ($x \perp y$ if and only if the order of x is relatively prime to the order of y). In the first case $\mathscr{D}(A)$ is a lattice and in the second $\mathscr{B}(A)$ is atomic, so in both cases the algebras are orthorepresentable.

Example 2.3. An integral residuated semilattice-ordered semigroup (briefly, an IRV semigroup) is an algebra $\langle A; \cdot, \vee, :, 1\rangle$ such that $\langle A; \cdot, 1\rangle$ is a commutative semigroup with identity 1, $\langle A; \vee, 1\rangle$ is an upper semilattice with largest element 1, $a(b \vee c) = ab \vee ac$ for all $a, b, c \in A$, and for any $a, b \in A$

there is a (necessarily unique) element $a : b \in A$ such that for any $c \in A$, $cb \leqq a$ if and only if $c \leqq a : b$. The variety of IRV semigroups is weakly regular and the set consisting of the single polynomial $p(x, y) = (x : y)(y : x)$ satisfies 2.1(i) [**6**].

Let $\mathscr{A}$ be an IRV semigroup. The element 1 is the zero of $\mathscr{A}$, and a subset $J \subseteq A$ is an ideal if and only if it is a *filter* (that is, J is a subsemigroup of $\langle A; \cdot, 1\rangle$ and, for any $a, b \in A$, $b \geqq a \in J$ implies that $b \in J$) [**6**; **19**]. It is straightforward to verify that the relation $\perp$ defined by $x \perp y$ if and only if $x \vee y = 1$ is a Boolean orthogonality on $\mathscr{A}$. Also, $\mathscr{A}$ is orthorepresentable, because $\mathscr{A}$ has a separating set of minimal prime filters [**8**, Section 2 and Theorem 3.2] and all minimal prime filters are $\perp$-prime $\perp$-ideals. Thus, the order on $\mathscr{A}$ is given by $x \leqq y$ if and only if $x \perp (x : y)(y : x)$.

An algebra $\mathscr{A}$ with a Boolean orthogonality is *orthoprime* if $x \perp y$ implies that $x = 0$ or $y = 0$, for all $x, y \in A$; and $\mathscr{A}$ is *orthocomplete* if $\mathscr{A}$ is orthorepresentable and every subset $S \subseteq A$ such that $x \perp y$ for all $x, y \in S$, $x \neq y$, has a least upper bound with respect to the order $\leqq$.

If $\{\mathscr{A}_\lambda : \lambda \in \Lambda\}$ is a set of algebras with a Boolean orthogonality the *product orthogonality* on $\prod \mathscr{A}_\lambda$ is defined by $x \perp y$ if and only if $x(\lambda) \perp y(\lambda)$ for all $\lambda \in \Lambda$.

In the next theorem, which is a specialization of [**11**, Theorem 2.7], we assume that the algebra under discussion has permutable congruences. In general, weakly regular algebras need not have permutable congruences [**23**], but multi-operator groups do have permutable congruences and so do IRV semigroups [**9**, Proposition 3.5].

THEOREM 2.5. *A weakly regular algebra $\mathscr{A}$ with permutable congruences and a Boolean orthogonality is the direct product (endowed with the product orthogonality) of a set $\{\mathscr{A}_\lambda : \lambda \in \Lambda\}$ of orthoprime algebras if and only if $\mathscr{B}(A)$ is atomic, $\mathscr{A}$ is orthocomplete, and $M + M^\perp = A$ for all maximal polars M. In this case we may take $\{\mathscr{A}_\lambda : \lambda \in \Lambda\}$ to be $\{\mathscr{A}/\theta(M) : M$ is a maximal polar$\}$.*

Speed [**24**, Theorem 4.3] has shown that an (associative) commutative ring with identity is the direct product of integral domains with identity if and only if it is a *Baer ring* (a commutative ring in which annihilators of single elements are generated by idempotents) which is orthocomplete (the orthogonality is defined by $x \perp y$ if and only if $xy = 0$) and whose Boolean algebra of idempotents is atomic. This is a consequence of Theorem 2.5 because the Boolean algebra of polars is isomorphic to the Boolean algebra of idempotents, and since maximal polars are generated by idempotents they are direct summands. Abian's theorem for commutative reduced associative rings [**2**], and for reduced alternative rings [**3**], can also be derived from Theorem 2.5.

An associative ring A is *biregular* if for each $x \in A$, (x) (the principal two-sided ideal generated by x) is generated by a central idempotent. Biregular rings are semiprime and the relation $\perp$ defined by $x \perp y$ if and only if

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$(x)(y) = 0$ is a Boolean orthogonality on every semiprime associative ring. The next corollary follows by an argument similar to the one used above to derive Speed's theorem.

COROLLARY 2.6. *An associative ring is the direct product of simple rings with identity if and only if it is an orthocomplete biregular ring whose Boolean algebra of central idempotents is atomic.*

3. Direct products of division rings, division near-rings and alternative division rings. An (associative) *near-ring* is an algebra $\langle N; +, -, 0, \cdot\rangle$ which is a multioperator group with $\Omega = \{\cdot\}$ such that $\langle N; \cdot\rangle$ is a semigroup and $x(y + z) = xy + xz$ and $0x = 0$ for all $x, y \in N$. It follows that $x0 = 0$ and $x(-y) = -(xy)$ for all $x, y \in N$. An element r of a near-ring $\mathscr{N}$ is *right distributive* if $(m + n)r = mr + nr$ for all $m, n \in N$.

A non-empty set J of elements of a near-ring $\mathscr{N}$ is an ideal if and only if it is a normal subgroup of $\langle N; +, -, 0\rangle$ and for every $k \in J$ and $m, n \in N$, $nk \in J$ and $(m + k)n - mn \in J$ [**5**]. If I and J are ideals of a near-ring $\mathscr{N}$, then $I + J$ (the supremum of I and J in the lattice of ideals) is equal to $\{x + y : x \in I \text{ and } y \in J\}$.

Let $\mathscr{N}$ be a reduced near-ring. The relation $\perp$ defined by $x \perp y$ if and only if $xy = 0$ is a Boolean orthogonality on $\mathscr{N}$ and $x^{\perp\perp} \cap y^{\perp\perp} = (xy)^{\perp\perp}$ for all $x, y \in N$. It follows that $\mathscr{D}(N)$ is a lattice and so $\mathscr{N}$ is orthorepresentable. A *hyperatom* is an element $0 \neq a \in N$ such that for each $n \in N$ with $an \neq 0$ there is an element $t \in N$ satisfying $a(nt) = a$, and such that there is an $r \in N$ for which $ar \neq 0$ and is right distributive. The near-ring $\mathscr{N}$ is *hyperatomic* if for each $0 \neq n \in N$ there is a hyperatom $a \in N$ such that $a \leqq n$. If $\mathscr{N}$ has a multiplicative identity and each non-zero $n \in N$ is invertible, then $\mathscr{N}$ is a *division near-ring.*

PROPOSITION 3.1. *Let $\mathscr{N}$ be a reduced near-ring.*

(i) *If $a \in N$ is a hyperatom, then $a^{\perp\perp}$ is an atom in $\mathscr{B}(N)$, $a^{\perp\perp}$ is a division near-ring, and $a^{\perp} + a^{\perp\perp} = N$.*

(ii) *If $\mathscr{N}$ is hyperatomic, then $\mathscr{B}(N)$ is atomic and each atom in $\mathscr{B}(N)$ is of the form $a^{\perp\perp}$ for some hyperatom a.*

Proof. Just as in [**15**, Lemma 1], we see that if a product is equal to 0, then any rearrangement of the product will also be 0.

(i) Let $a \in N$ be a hyperatom and $u, v \in a^{\perp\perp}$ be non-zero. Then $au \neq 0$ and $av \neq 0$ so there are $t', t'' \in N$ such that $aut' = a = avt''$. Thus $aut'avt'' = a^2 \neq 0$ and so $uv \neq 0$. It follows immediately that $a^{\perp\perp}$ is an atom in $\mathscr{B}(N)$. Since a is a hyperatom there is a $t \in N$ such that $a^2t = a$. Then $a(ata - a) = 0$ and so $ata = a$. Thus $e = at$ is an idempotent and consequently e is central in N and an identity for $a^{\perp\perp}$. Let $0 \neq b \in a^{\perp\perp}$. Then $b = eb$ and so there is an $s \in N$ such that $bs = a$. It follows that $bN = a^{\perp\perp}$. Hence $b(a^{\perp\perp}) = a^{\perp\perp}$ because if $x \in a^{\perp\perp}$, then $x = bn$ for some $n \in N$ and so $x = exe = e(bn)e = (eb)(ne) = b(ne) \in b(a^{\perp\perp})$.

Since the definition of hyperatom forces the existence of a right distributive element in $a^{\perp\perp}$ we may apply a theorem of Ligh [**18**, Theorem 2.3] to see that $a^{\perp\perp}$ is a division near-ring.

Let $x \in N$. Then $a(x - xe) = ax + a(-xe) = ax - axe = ax - ax = 0$ and so $x - xe \in a^{\perp}$. Thus $x = (x - xe) + (xe) \in a^{\perp} + a^{\perp\perp}$.

(ii) Assume that $\mathscr{N}$ is hyperatomic and $C \in \mathscr{B}(N)$. Suppose there is a non-zero $x \in \bigcap \{a^{\perp} : a \in C$ and a is a hyperatom$\} \cap C$. Let b be a hyperatom such that $b \leqq x$. Then $b^{\perp\perp} \subseteq x^{\perp\perp}$ and so $b \in C$. Thus, $b^{\perp\perp} \subseteq x^{\perp\perp} \subseteq b^{\perp}$ which is a contradiction. It follows that $C = \bigvee \{a^{\perp\perp} : a \in C$ and a is a hyperatom$\}$ and this establishes (ii).

Let $\mathscr{N}$ be a hyperatomic near-ring and M a maximal polar in $\mathscr{B}(N)$. One easily checks that $M = a^{\perp}$, where $a^{\perp\perp}$ is an atom in $\mathscr{B}(N)$. From 3.1 (ii) we may assume that a is a hyperatom and so by 3.1(i) $\mathscr{N}/M \cong a^{\perp\perp}$ is a division near-ring. One easily verifies that a direct product of division near-rings is hyperatomic, and so from Theorem 2.5 we obtain the following result.

THEOREM 3.2. *A near-ring is the direct product of division near-rings if and only if it is reduced, hyperatomic and orthocomplete.*

Using results of Rjabuhin [**21**; **22**] (the relevant facts can also be found in [**15**, General Remarks and Lemma 1]) we see that the above arguments apply equally well to alternative rings, so we obtain the following result of Myung and Jimenez [**20**]: *An alternative ring is the direct product of alternative division rings if and only if it is reduced, hyperatomic and orthocomplete.*

B. H. Neumann has proved that the additive group of a division near-ring is commutative (for references see [**18**]) and Fröhlich [**14**] has shown that a near-ring $\mathscr{N}$ is a ring if and only if its additive group is commutative and it is *distributively generated* (that is, there is a subsemigroup S of $\langle N; \cdot \rangle$ consisting of right distributive elements which generates the group $\langle N; +, -, 0\rangle$).

COROLLARY 3.3. *A near-ring is the direct product of division rings if and only if it is reduced, hyperatomic, orthocomplete, and distributively generated.*

The results of Abian [**1**] and Chacron [**7**], which are ring-theoretic versions of Theorem 3.2, follow from Theorem 3.2 and can, of course, be obtained from Theorem 2.5 without reference to near-rings.

Further characterizations of direct products of division rings can be derived from Theorem 3.2, we illustrate this in the following corollary. An associative ring A is *strongly regular* if, for each $x \in A$, there is a $y \in A$ such that $x^2y = x$. A strongly regular ring is reduced and so the relation $\perp$ defined by $x \perp y$ if and only if $xy = 0$ is a Boolean orthogonality on every strongly regular ring. It is straightforward to check that if the Boolean algebra of central idempotents of a strongly regular ring A is atomic, then A is hyperatomic.

COROLLARY 3.4. *An associative ring is the direct product of division rings if and only if it is an orthocomplete strongly regular ring whose Boolean algebra of central idempotents is atomic.*

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