Can. J. Math., Vol. XXXII, No. 5, 1980, pp. 1266–1269

GROUP RINGS OVER $\mathbf{Z}_{(p)}$ WITH FC UNIT GROUPS

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Introduction. Let RG denote the group ring of a group G over a commutative ring R with unity. We recall that a group is said to be an FC-group if all its conjugacy classes are finite.

In [**6**], S. K. Sehgal and H. Zassenhaus gave necessary and sufficient conditions for $U(RG)$ to be an FC-group when R is either $\mathbf{Z}$, the ring of rational integers, or a field of characteristic 0.

One of the authors considered this problem for group rings over infinite fields of characteristic $p \neq 2$ in [**5**] and G. Cliffs and S. K. Sehgal [**1**] completed the study for arbitrary fields. Also, group rings of finite groups over commutative rings containing $\mathbf{Z}_{(p)}$, a localization of $\mathbf{Z}$ over a prime ideal (p) were studied in [**4**].

In this paper we prove the following:

THEOREM. *Let $R = \mathbf{Z}_{(p)}$. Then, the group of units of RG is an* FC-*group if and only if one of the following conditions hold:*

(i) *G is abelian.*

(ii) *G is an* FC *group whose torsion subgroup T is central and the subgroup T' of torsion units whose order is not divisible by p is either finite or has the form $T' = C.H$ where $C \cong \mathbf{Z}(q^{\infty})$ for a prime $q \neq p$, $[G, G] \subset C$ and H is finite.*

Proof of necessity. Let G be a non-abelian group such that $U(RG)$ is an FC-group. Then $U(\mathbf{Z}G)$ is also FC, thus G itself is FC and Theorem 1 of [**6**] together with Theorem 2 of [**4**] show that the torsion subgroup T of G satisfies one of the following conditions:

(T_1) T is central in G.

(T_2) T is abelian non central and for $x \in G$, $xtx^{-1} = x^{\delta(x)}$, $\delta(x) = \pm 1$, for all $t \in T$.

We wish to show first that in the present case T must always be central. This is a consequence of the following lemma.

LEMMA 1. *Let x, t be two elements in a group G such that $xtx^{-1} = t^{-1}$, with $o(t) = n \neq 2$. Then $U(RG)$ is not* FC.

Proof. Set $a = \operatorname{lcm}(p, n)$ and $S = \{x \in \mathbf{Z} | \, x \equiv 1 \pmod{a}\}$. Then the

Received September 7, 1978.

https://doi.org/10.4153/CJM-1980-095-8 Published online by Cambridge University Press

localization $R' = S^{-1}.\mathbf{Z}$ is such that $R' \subset R$ and no divisor of n is invertible in R'; thus, the units of finite order of $R'\langle t\rangle$ are trivial (see [**7**]).

The element $u = 1 + at + \ldots + a^{n-1}t^{n-1}$ is a unit in $R'\langle t\rangle$ whose inverse is $u^{-1} = (1 - a^n)^{-1}(1 - at)$. Now, an easy computation shows that

$$uxu^{-1} = vx \quad \text{where} \quad v = \left[\frac{1-a}{1-a^n}u - at^{n-1}\right] \in U(R'\langle t\rangle)$$

Since v is not trivial, it is of infinite order. Also, it is easy to see that

$$u^m x u^{-m} = v^m x, \text{ for all } m \in \mathbf{Z}.$$

Hence, x has infinitely many conjugates in RG, thus $U(RG)$ is not an FC-group.

To complete the proof of the necessity we shall now assume that G is not abelian. We shall denote by T' the set of all elements in T whose order is not divisible by p. We wish to show that if T' is infinite then $T' = C.H$ where $C = \mathbf{Z}(q^\infty)$ for a prime $q \neq p$, $[G, G] \subset C$ and H is finite.

LEMMA 2. *For each element $t = [x, g] \in [G, G]$, there is a finite set $H_t \subset T'$ such that for all $t' \in T'$, if $t' \notin H_t$, then $t \in \langle t'\rangle$.*

Proof. Assume, by contradiction, that there exists an infinite set $B \subset T'$ such that for all $t' \in B$ we have that $t \notin \langle t'\rangle$. We define a sequence $\{t_n\}$ of elements in B inductively in such a way that

$$t_n \notin \langle t, t_1, \ldots, t_{n-1}\rangle$$

and consider the idempotents

$$e_n = \frac{1}{s_n}(1 + t_n + \ldots + t_n^{s_n - 1})$$

where $s_n = 0(t_n)$.

Then, the elements $u_n = e_n x + (1 - e_n)$ are units in RG whose respective inverses are $u_n^{-1} = e_n x^{-1} + (1 - e_n)$. Now, consider the conjugates

$$g_n = u_n g u_n^{-1} = (e_n t + 1 - e_n)g.$$

It is easy to see that for $i > j$ $g_i = g_j$ if and only if $e_i(t-1) = e_j(t-1)$ and this cannot happen because of the choice of the elements $\{t_n\}$. Hence g has infinitely many conjugates in $U(RG)$, a contradiction.

Now we can finish our argument:

Since T' is central, we can always find a commutator α of prime order $q \neq p$. Applying Lemma 2 to this element it follows readily that the q'-part of T' must be finite, and the q-part infinite.

Setting $C = \{t' \in T' | \alpha \in \langle t'\rangle\}$ we see that C is an infinite abelian group which is torsion and indecomposable. By a result of Kulikov (see [**3**, 27.4]) it must be $C = \mathbf{Z}(q^\infty)$. Also ,$T' = C \times H$ where H is finite.

Proof of sufficiency. If G satisfies (i), $U(RG)$ is trivially an FC-group. So, we shall assume G satisfies (ii) and consider first the case where T' is infinite.

Since T is abelian, the p-Sylow subgroup of T, which we shall denote by T_p, is a direct factor. Writing $T = T_p \times T'$ we have that $RT = ST'$ where $S = RT_p$. We remark that S is a commutative ring and, by the theorem in [**2**], it contains no non-trivial idempotents.

We wish to show that each conjugate class in $U(RG)$ is finite. Let Σ be a transversal of T in G, which we may choose such that if $x \in \Sigma$ then $x^{-1} \in \Sigma$. Since T is central, it will be enough to prove that conjugate classes of elements in Σ are finite.

For each positive integer m we shall denote by Q_m the subgroup of C of order q^m, so that $Q_m \times H$ is an increasing chain of subgroups, whose union is T'.

As in [**6**, Lemma 2.4] we see that any unit v in RG can be written in the form:

$$v = \sum_h \alpha(h)h \quad \text{with} \quad \alpha(h) \in ST', \quad h \in \Sigma$$

where $\alpha(h)\alpha(h') = 0$ whenever $h \neq h'$.

Given v, we pick m such that $Q_m \times H$ contains the supports of all $\alpha(h)$ in the above expression of v (note that, since $\alpha(h) \in ST'$ we consider supp $(\alpha(h)) \subset T'$). Let $\{e_i\}$ be a complete set of primitive orthogonal idempotents in $\mathbf{Q}(Q_m \times H)$. Since $p \nmid |Q_m \times H|$ all these idempotents belong to $R(Q_m \times H)$.

Writing v^{-1} in the form $v^{-1} = \sum \beta(h)h^{-1}$ we obtain

$$\sum_h \alpha(h)\beta(h) = 1 \quad \text{and} \quad e_i\alpha(h) = e_i\alpha(h)^2\beta(h).$$

If we choose $h \in \Sigma$ such that $e_i\alpha(h) \neq 0$ it is easy to see that the element $e_i\alpha(h)\beta(h)$ is an idempotent in $S(Q_m \times H)e_i$. Since this ring contains no non-trivial idempotents, it follows that

$$e = \alpha(h)\beta(h) = e_i$$

and, for each i, there is only one element $h_i \in \Sigma$ such that $e_i\alpha(h_i) \neq 0$. Hence, setting $\alpha_i = e_i\alpha(h_i)$, we have

$$v = \sum_i \alpha_i h_i, \quad v^{-1} = \sum_i \alpha_i^{-1} h_i^{-1} \quad \text{and} \quad \alpha_i\alpha_i^{-1} = e_i.$$

Since G is an FC-group, the set of all commutators of the form $[h, g]$, $h \in G$, g fixed in G, is finite; thus we may find a finite group Q in C such that Q contains all these commutators.

For a given unit $v = \sum \alpha(h)h$ in RG, we may now choose a finite subgroup $Q_v \times H$ of T which contains both Q_m and Q.

https://doi.org/10.4153/CJM-1980-095-8 Published online by Cambridge University Press

If we consider $\mathbf{Q}Q$ as included in $\mathbf{Q}(Q_v)$, it is easy to see that

$$\mathbf{Q}Q_v \cong \mathbf{Q}Q \oplus (\bigoplus_i K_i)$$

where each K_i is a cyclotomic extension of $\mathbf{Q}$. Also if we denote by f_i the idempotents such that $K_i = (\mathbf{Q}Q_v)f_i$ and set

$$e = (\textstyle\sum_{x \in Q} x)/|Q|$$

for all f_i we have that $ef_i = f_i$.

This means that we may index the idempotents of $Q(Q_v \times H)$ in such a way that

$$e_i e = e_i \text{ if } 1 \leqq i \leqq t$$
$$e_i e = 0 \text{ if } t + 1 \leqq i \leqq r$$

where $e_{t+1}, \ldots, e_r$ are fixed, independently of v.

Now, we are ready to complete the proof. We have:

$$[v, g] = \sum_i [h_i, g]e_i = \sum_{i=1}^{t} [h_i, g]ee_i + \sum_{i=t+1}^{r} [h_i, g]e_i.$$

Since $[h_i, g] \in Q$, we see that $[h_i, g]e = e$, and so

$$[v, g] = e + \sum_{i=t+1}^{r} [h_i, g]e_i.$$

Consequently, the set of commutators $[v, g]$, $v \in U(RG)$, is finite.

Finally, we note that in the case where T' is finite the previous argument may be repeated in a much simpler form, because it is possible to use a fixed family of idempotents.

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