Direction-Cosines of the Axes of the Conicoid
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1.$$

In *Mathematical Notes*, No. 20 (April 1916), there is a note on the above; I add a form of the equations of these axes which I have not seen in a text-book, and which is perhaps worth recording.

If on transformation of axes

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy \equiv a'X^2+b'Y^2+c'Z^2,$$

then $$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy \equiv (a'-\lambda)X^2+(b'-\lambda)Y^2+(c'-\lambda)Z^2.$$

The right-hand side resolves into factors if $\lambda=a'$ or b' or c',

$\therefore$ the left-hand side does the same for $\lambda=a'$ or b' or c',

$\therefore$ a', b', c' are the roots of

$$(a-\lambda)(b-\lambda)(c-\lambda)+2fgh-f^2(a-\lambda)-g^2(b-\lambda)-h^2(c-\lambda)=0.$$

When $\lambda=a'$, $a'X^2+b'Y^2+c'Z^2=0$ is the equation of two planes which intersect in the X-axis, on which the conicoid intercepts a length $\dfrac{2}{\sqrt{a'}}$.

$\therefore$ $\phi(x,y,z)\equiv(a-a')x^2+(b-a')y^2+(c-a')z^2+2fyz+2gzx+2hxy=0$ is the equation of two planes intersecting in the axis of length $\dfrac{2}{\sqrt{a'}}$.

The intersection of these planes is given by any two of the equations

$$\frac{\partial\phi}{\partial x}=0,\quad \frac{\partial\phi}{\partial y}=0 \text{ and } \frac{\partial\phi}{\partial z}=0,$$

$$i.e.\quad (a-a')x+hy+gz=0,\quad hx+(b-a')y+fz=0,\quad gx+fy+(c-a')z=0.$$

From the first and second, any point on the axis satisfies

$$\frac{x}{hf-bg+ga'}=\frac{y}{gh-af+fa'},$$

and from the first and third, the point satisfies

$$\frac{x}{fg-ch+ha'}=\frac{z}{gh-af+fa'},$$

$$i.e.\quad \frac{x}{G+ga'}=\frac{y}{F+fa'} \quad\text{and}\quad \frac{x}{H+ha'}=\frac{z}{F+fa'},$$

where F, G, H are the customary minors.

Therefore the equations of this axis are

$$x(F+fa')=y(G+ga')=z(H+ha').$$

Similar equations hold for the other axes, with b' and c' instead of a'.

LAWRENCE CRAWFORD.

Proofs of some Inequalities and Limits.

In his article in No. 20, Professor Gibson gives proofs of the inequalities $1-na<(1-a)^n<\dfrac{1}{1+na}$ with certain restrictions as to the values of n and a. The mode of proof, avoiding as it does the use of the Binomial Theorem, is comparatively short and simple. The following mode of proof is, I think, still simpler, as it does not involve the use of Mathematical Induction.

If n is a positive integer and a positive, we have

$$\frac{(1+a)^n-1}{(1+a)-1}=(1+a)^{n-1}+(1+a)^{n-2}+(1+a)^{n-3}+\dots+(1+a)+1,$$
$$>n,$$
$$\therefore\quad (1+a)^n-1>na,$$
$$\therefore\quad (1+a)^n>1+na. \quad\dots\dots\dots\dots\dots\dots\dots (1)$$

Again, n being a positive integer and a a positive proper fraction, we have

$$\frac{1-(1-a)^n}{1-(1-a)}=1+(1-a)+(1-a)^2+\dots+(1-a)^{n-1},$$
$$<n,$$
$$\therefore\quad 1-(1-a)^n<na,$$
$$\therefore\quad (1-a)^n>1-na. \quad\dots\dots\dots\dots\dots\dots\dots (2)$$

Then, since $(1-a)(1+a)=1-a^2$

$$<1,$$
$$\therefore\quad 1-a<\frac{1}{1+a},$$
$$\therefore\quad (1-a)^n<\frac{1}{(1+a)^n},$$
$$\therefore\quad \text{by (1)},\ <\frac{1}{1+na}. \quad\dots\dots\dots\dots\dots\dots\dots (3)$$

https://doi.org/10.1017/S1757748900001675 Published online by Cambridge University Press