Can. J. Math., Vol. XLI, No. 6, 1989, pp. 1021–1089

CLASSIFYING ALGEBRAS FOR THE K-THEORY OF σ-C^*-ALGEBRAS

N. CHRISTOPHER PHILLIPS

Introduction. In topology, the representable K-theory of a topological space X is defined by the formulas $RK^0(X) = [X, \mathbf{Z} \times BU]$ and $RK^1(X) = [X, U]$, where square brackets denote sets of homotopy classes of continuous maps,

$$U = \lim_{\rightarrow} U(n)$$

is the infinite unitary group, and BU is a classifying space for U. (Note that $\mathbf{Z} \times BU$ is homotopy equivalent to the space of Fredholm operators on a separable infinite-dimensional Hilbert space.) These sets of homotopy classes are made into abelian groups by using the H-group structures on $\mathbf{Z} \times BU$ and U. In this paper, we give analogous formulas for the representable K-theory for σ-C^*-algebras defined in [**20**]. That is, we produce σ-C^*-algebras P and U_{nc}, equipped with the appropriate analog of an H-group structure, such that there are natural isomorphisms of abelian groups

$$RK_0(A) \cong [P, A]_1 \quad \text{and} \quad RK_1(A) \cong [U_{nc}, A]_1$$

for unital σ-C^*-algebras A. Here $[A, B]_1$ denotes the set of unital homotopy classes of $*$-homorphisms from A to B. Thus, RK_* really is a representable functor in the sense of category theory. (Compare with the remarks in the introduction to [**20**].) A small variation on our results gives a proof, up to a minor technicality, of Conjecture 2.5.7 in the survey article [**21**].

As a byproduct of our proofs, we also obtain descriptions of $RK_0(A)$ and $RK_1(A)$ that are more closely related to the usual definitions of $K_0(A)$ and $K_1(A)$ for C^*-algebras A than the definitions given in [**20**]. We show that, for any σ-C^*-algebra A, the group $RK_0(A)$ is isomorphic to the group of homotopy classes of projections in $M_2((K \otimes A)^+)$ which differ from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $M_2(K \otimes A)$, with an operation derived from the direct sum operation. Also, we show that $RK_1(A)$ is naturally isomorphic to the group of path components in the unitary group of $(K \otimes A)^+$. Here, and throughout this paper, K is the algebra of compact operators on a separable infinite dimensional

Received October 6, 1988. This research was partially supported by an NSF postdoctoral fellowship.

Hilbert space, and the superscript + denotes the unitization. The examples in Section 4 of [**20**] suggest that it is unlikely that this description of $RK_0(A)$ can be much improved.

We should mention that Rosenberg has obtained in [**22**] a representation theorem for the K-theory of C^*-algebras which is analogous to what the topologists do for covariant functors. For example he proves in Theorem 4.1 of [**22**] that there is a natural isomorphism

$$K_1(A) \cong [C_0(\mathbf{R}), K \otimes A]$$

for any C^*-algebra A. Here the square brackets with no subscript denote homotopy classes of homomorphisms with no restrictions on units. This formula is also correct for σ-C^*-algebras A. Our approach, however, is the analog of what the topologists do for contravariant functors. Since K-theory is contravariant on spaces, and since the functor $X \longmapsto C(X)$ from spaces to (pro-) C^*-algebras is contravariant, our results are a more direct generalization of the formulas

$$RK^0(X) = [X, \mathbf{Z} \times BU] \quad \text{and} \quad RK^1(X) = [X, U].$$

This paper is organized as follows. In the first section, we define an adjoint to the functor $M_n \otimes -$ of tensoring with the $n \times n$ matrices, and we show how to use it to construct certain "noncommutative" H-groups. This is the method used to construct the classifying algebra P for RK_0. We define P and prove the isomorphism

$$RK_0(A) \cong [P, A]_1$$

in the second section. In the third section, we define (much more directly) the classifying algebra U_{nc}, and establish the isomorphism

$$RK_1(A) \cong [U_{nc}, A]_1.$$

In Section 4, we relate the construction of U_{nc} to that of P, and use this information to compute $RK_*(P)$ and $RK_*(U_{nc})$. These groups are much smaller than the corresponding groups $RK^*(\mathbf{Z} \times BU)$ and $RK^*(U)$, and the difference can be used to show that the algebras of continuous functions $C(\mathbf{Z} \times BU)$ and $C(U)$ cannot take the places of P and U_{nc}, even for the case in which A is a C^*-algebra. In the last section, we prove a version of our results in terms of pointed σ-C^*-algebras; this version is probably the closest to the topologists' approach, and leads to the formulation of a "noncommutative Bott periodicity" conjecture. We then discuss this conjecture and other related problems.

For general facts about σ-C^*-algebras and pro-C^*-algebras, we refer to [**19**] and [**21**]. In particular, the tensor products which appear above, and multiplier algebras, are as defined in Section 3 of [**19**]. We will also adopt the convention that all homomorphisms are assumed to preserve adjoints and to be continuous.

(Note that $*$-homomorphisms between σ-C^*-algebras are automatically continuous, by [**19**], Theorem 5.2.) If A is a unital pro-C^*-algebra, then $U(A)$ is the unitary group of $A, U_0(A)$ is the path component of the identity in $U(A)$, and $(U/U_0)(A)$ is $U(A)/U_0(A)$. Also, if X is a topological space, then $C(X,A)$ denotes the algebra of all continuous A-valued functions on X, with the topology of uniform convergence on compact subsets of X in each continuous C^*-seminorm on A. If X is compactly generated, then $C(X,A)$ is a pro-C^*-algebra which we identify with $C(X) \otimes A$. (Compare [**19**], Proposition 3.4.) If A is omitted, it is understood to be $\mathbf{C}$.

1. Homotopy dual groups and the adjoint functor to $M_n \otimes -$. The purpose of this section is to define, and prove some properties of, certain homotopy dual groups whose operation is derived from the direct sum opertation on matrices. The construction uses algebras $W_n(A)$ and $W_\infty(A)$, where $W_n(A)$ is defined by the property

$$\mathrm{Hom}(W_n(A), B) \cong \mathrm{Hom}(A, M_n(B)),$$

and where

$$W_\infty(A) = \lim_{\overleftarrow{n}} W_n(A).$$

Notation 1.1. For any unital pro-C^*-algebra A, denote by ϵ_A the homomorphism from $\mathbf{C}$ to A such that $\epsilon_A(1) = 1$, and denote by δ_A the homomorphism from $A *_{\mathbf{C}} A$ to A which sends both copies of A in the free product identically onto A. (For the definition of the free product of pro-C^*-algebras see Section 1.5 of [**21**], or, using an equivalent notion of pro-C^*-algebra, [**29**].) Finally, we denote by τ_A the flip homomorphism from $A *_{\mathbf{C}} A$ to $A *_{\mathbf{C}} A$ which sends each copy of A in the free product identically onto the other one.

Definition 1.2. (Compare [**29**], 2.1.) A *homotopy dual group* in the category of pro-C^*-algebras is a tuple (A, χ, μ, ι), where A is a unital pro-C^*-algebra, and where $\chi : A \longrightarrow \mathbf{C}, \mu : A \longrightarrow A *_{\mathbf{C}} A$, and $\iota : A \longrightarrow A$ are unital homomorphisms, such that there exist unital homotopies as follows:

$$(1) \qquad (\mu * id_A) \circ \mu \simeq (id_A * \mu) \circ \mu.$$

$$(2) \qquad \iota \circ \iota \simeq id_A.$$

$$(3) \qquad \delta_A \circ (\iota * id_A) \circ \mu \simeq \epsilon_A \circ \chi \simeq \delta_A \circ (id_A * \iota) \circ \mu.$$

$$(4) \qquad (\chi * id_A) \circ \mu \simeq id_A \simeq (id_A * \chi) \circ \mu.$$

In (4), we identify $\mathbf{C} *_{\mathbf{C}} A$ and $A *_{\mathbf{C}} \mathbf{C}$ with A in the obvious way. The homotopy dual group is called *abelian* if in addition there is a unital homotopy

$$(5) \qquad \tau_A \circ \mu \simeq \mu.$$

The only real difference between this definition and that of [**29**] is that we have everywhere replaced equality by unital homotopy. A homotopy dual group is the noncommutative analog of an H-group, as defined for example in Section 1.5 of [**26**], but without the base point. Indeed, if A in the definition is a commutative algebra, then it is the algebra of continuous functions on a quasitopological space [**25**] which is also an H-group. (Use the results of Section 2 of [**19**].) It follows that if A is a homotopy dual group, then the space $\mathrm{Hom}_1(A, \mathbf{C})$ of unital homomorphisms from A to $\mathbf{C}$ is a quasitopological H-group, with identity χ, multiplication $\varphi \cdot \psi = (\varphi * \psi) \circ \mu$, and inverse $\varphi^{-1} = \varphi \circ \iota$.

Just as in the topological category, we obtain a group structure on appropriate sets of homotopy classes. In the following proposition and throughout this paper, we will denote the (unital) homotopy class of a (unital) homomorphism $\varphi : A \longrightarrow B$ by $[\varphi]$. (We will usually mean the unital case, but it will always be clear from the context.)

PROPOSITION 1.3. *Let (A, χ, μ, ι) be a homotopy dual group. Then for each unital pro-C^*-algebra B, the set $[A, B]_1$ has a natural group structure, with identity $[\epsilon_B \circ \chi]$, multiplication $[\varphi][\psi] = [\delta_B \circ (\varphi * \psi) \circ \mu]$, and inverse $[\varphi]^{-1} = [\varphi \circ \iota]$. If (A, χ, μ, ι) is abelian then so is $[A, B]_1$.*

The proof is immediate, and is omitted.

We now set about constructing an important class of examples of homotopy dual groups.

Definition 1.4. Let A be a C^*-algebra and let $n \geqq 1$. Then $W_n(A)$ is defined to be the universal C^*-algebra generated by the symbols $x_n(a, i, j)$ for $a \in A$ and $1 \leqq i, j \leqq n$, subject to certain relations. To state them, we introduce the symbols $x_n(a)$ for the $n \times n$ matrix $(x_n(a, i, j))_{i,j=1}^n$ and $x_n^*(a)$ for the $n \times n$ matrix $(x_n(a, j, i)^*)_{i,j=1}^n$. Then for every polynomial f in $2k$ noncommuting variables and having no constant term, and every $a_1, \ldots, a_k \in A$ such that

$$f(a_1, a_1^*, \ldots, a_k, a_k^*) = 0,$$

we impose the n^2 relations stating that the entries of the $n \times n$ matrix obtained by formal evaluation of the expression

$$f(x_n(a_1), x_n^*(a_1), \ldots, x_n(a_k), x_n^*(a_k))$$

are all zero. If furthermore $\varphi : A \longrightarrow B$ is a homomorphism of C^*-algebras, then we define

$$W_n(\varphi) : W_n(A) \longrightarrow W_n(B)$$

by setting

$$W_n(\varphi)(x_n(a, i, j)) = x_n(\varphi(a), i, j).$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

Finally, if B is a pro-C^*-algebra and $\psi : A \longrightarrow M_n(B)$ is a homomorphism, then we define a homomorphism

$$\bar{\psi} : W_n(A) \longrightarrow B$$

by setting

$$\bar{\psi}(x_n(a,i,j)) = \psi(a)_{i,j},$$

the ij entry of the $n \times n$ matrix $\psi(a)$.

The next proposition asserts that this definition makes sense, and that $A \longmapsto W_n(A)$ is an adjoint functor to $B \longmapsto M_n(B)$.

PROPOSITION 1.5. *The assignment $A \longmapsto W_n(A)$ is a functor from C^*-algebras to C^*-algebras. If A is a C^*-algebra and B is a pro-C^*-algebra, then the assignment $\psi \longmapsto \bar{\psi}$ defines isomorphisms of sets*

$$\operatorname{Hom}(A, M_n(B)) \cong \operatorname{Hom}(W_n(A), B) \quad \text{and} \quad [A, M_n(B)] \cong [W_n(A), B]$$

which are natural in both A and B.

Proof. $W_n(A)$ is a C^*-algebra since the relations defining it are admissible in the sense of Blackadar ([**4**], Section 1). Indeed, they imply that

$$\|x_n(a,i,j)\| \leqq \|a\|$$

in any representation. That $A \longmapsto W_n(A)$ defines a functor is now obvious. For the second part of the proposition, it suffices by the definition of an inverse limit to consider only the case in which B is a C^*-algebra. That $\psi \longmapsto \bar{\psi}$ defines a bijection between the sets of homomorphisms is now immediate from the definition of $W_n(A)$. Bijectivity on homotopy classes follows by looking at homomorphisms from A to $B \otimes C([0,1])$. Naturality is immediate.

Remarks 1.6. (1) W_n can be extended to a functor on the category of pro-C^*-algebras, with the same properties, by defining

$$W_n\left(\varprojlim A_d\right) = \varprojlim W_n(A_d).$$

We will not need this, so we omit the details.

(2) $W_n(A)$ is a generalization of the noncommutative Grassmannians and unitary groups of [**7**] and of the noncommutative classical Lie groups of [**29**]. In general, the abelianization of $W_n(A)$, when A is a C^*-algebra, is isomorphic to the set of all functions on the space of representations of A on $\mathbf{C}^n$ which are continuous and vanish on the zero representation. (Use the previous proposition with $B = \mathbf{C}$.)

(3) If $A = C^*(G, R)$ is a universal C^*-algebra on generators G and relations R of the form $f(a_1, \ldots, a_k^*) = 0$, with f as in Definition 1.4, then in the definition of $W_n(A)$ we only need the generators $x_n(a, i, j)$ for $a \in G$ and the relations

$$f(x_n(a_1), \ldots, x_n^*(a_k)) = 0$$

corresponding to the relations in R.

Definition 1.7. Let A be a C^*-algebra. Define a homomorphism

$$\pi_n : W_{n+1}(A) \longrightarrow W_n(A)$$

by

$$(*) \qquad \pi_n(x_{n+1}(a, i, j)) = \begin{cases} x_n(a, i, j) & i, j \leqq n \\ 0 & \text{otherwise.} \end{cases}$$

(To show that $(*)$ defines a homomorphism, one simply observes that

$$a \longmapsto \begin{pmatrix} x_n(a, 1, 1) & \ldots & x_n(a, 1, n) & 0 \\ \vdots & & \vdots & \vdots \\ x_n(a, n, 1) & \ldots & x_n(a, n, n) & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix} = x_n(a) \oplus 0$$

is a homomorphism from A to $M_{n+1}(W_n(A))$.) Then define $W_\infty(A)$ to be the σ-C^*-algebra $\varprojlim_n W_n(A)$, using the homomorphisms π_n.

W_∞ is not the adjoint to the functor $B \longmapsto K \otimes B$, but rather of the following "finite" version of this functor. The advantage of W_∞ is that it sends C^*-algebras to σ-C^*-algebras, while the adjoint of $B \longmapsto K \otimes B$, constructed in Proposition 5.8 below, presumably does not.

Definition 1.8. Let K_0 be the dense subalgebra $\bigcup_{k=1}^{\infty} M_k$ of K. For any pro-C^*-algebra B, denote by $K_0 \tilde{\otimes} B$ the algebra

$$\varprojlim_p \left[\bigcup_{k=1}^{\infty} M_k \otimes B_p \right],$$

as p runs through the set $S(B)$ of all continuous C^*-seminorms on B. That is, $K_0 \tilde{\otimes} B$ consists of all infinite matrices $b = (b_{ij})$ such that for every p, one has $p(b_{ij}) = 0$ for all but finitely many pairs (i, j). Give $K_0 \tilde{\otimes} B$ the relative topology that it inherits as a dense $*$-subalgebra of $K \otimes B$. A homotopy of homomorphisms from an algebra A to $K_0 \tilde{\otimes} B$ is defined to be an assignment $t \longmapsto \varphi_t$ such that, with the obvious identifications, the formula $\varphi(a)(t) = \varphi_t(a)$ defines a homomorphism from A to $K_0 \tilde{\otimes} (B \otimes C([0, 1]))$. Similarly, a homotopy $t \longmapsto a_t$ of elements in $K_0 \tilde{\otimes} B$ is required to define an element of $K_0 \tilde{\otimes} (B \otimes C([0, 1]))$, not just an element of $K \otimes B \otimes C([0, 1])$ which is sent to $K_0 \tilde{\otimes} B$ under the evaluation maps.

PROPOSITION 1.9. *Let A be a C^*-algebra and let B be a pro-C^*-algebra. Then there are natural isomorphisms of sets*

$$\mathrm{Hom}(W_\infty(A), B) \cong \mathrm{Hom}(A, K_0 \tilde{\otimes} B) \quad \textit{and} \quad [W_\infty(A), B] \cong [A, K_0 \tilde{\otimes} B].$$

Proof. The first statement follows from the definitions of an inverse limit and of $K_0 \tilde{\otimes} B$, and Proposition 1.5, using the fact that a homomorphism from $W_\infty(A)$ to a C^*-algebra (here B_p for some $p \in S(B)$) must factor through some $W_n(A)$. The second statement follows from the first by using $B \otimes C([0,1])$ in place of B.

We now show that, under favorable circumstances, the unitization $W_\infty(A)^+$ can be made into a homotopy dual group.

Definition 1.10. Let A be a C^*-algebra equipped with a homomorphism $\iota_0 : A \longrightarrow A$ such that $\iota_0 \circ \iota_0 \simeq id_A$ and such that

$$a \longmapsto \begin{pmatrix} \iota_0(a) & 0 \\ 0 & a \end{pmatrix}$$

is homotopic to the zero map from A to $M_2(A)$. We define homomorphisms $\chi : W_\infty(A)^+ \longrightarrow \mathbf{C}$, $\mu : W_\infty(A)^+ \longrightarrow W_\infty(A)^+ *_{\mathbf{C}} W_\infty(A)^+$, and $\iota : W_\infty(A)^+ \longrightarrow W_\infty(A)^+$, as follows. We set $\chi(x + \lambda \cdot 1) = \lambda$ for $x \in W_\infty(A)$ and $\lambda \in \mathbf{C}$, and we define

$$\iota = \varprojlim W_n(\iota_0)^+.$$

To define μ, we introduce the notation $x_\infty(a, i, j)$ for the element of $W_\infty(A)$ defined by the coherent sequence $(x_n(a, i, j))$, where $x_n(a, i, j)$ is taken to be zero if $n \leqq \max(i, j)$. (Thus $a \longmapsto (x_\infty(a, i, j))_{i,j=1}^\infty$ is the homomorphism from A to $K_0 \tilde{\otimes} W_\infty(A)$ corresponding under the previous proposition to $id_{W_\infty(A)}$.) We further denote the corresponding elements in the first and second copies of $W_\infty(A)$ inside $W_\infty(A) * W_\infty(A)$ by $x_\infty^{(1)}(a, i, j)$ and $x_\infty^{(2)}(a, i, j)$ respectively. Identifying $W_\infty(A)^+ *_{\mathbf{C}} W_\infty(A)^+$ with $(W_\infty(A) * W_\infty(A))^+$ in the obvious way, we let μ be the unitization of the homomorphism corresponding, under Proposition 1.9, to the homomorphism from A to $K_0 \tilde{\otimes} (W_\infty(A) * W_\infty(A))$ given by

$$(**)\quad a \longmapsto \left(\begin{array}{cc|cc|c} x_\infty^{(1)}(a,1,1) & 0 & x_\infty^{(1)}(a,1,2) & 0 & \cdots \\ 0 & x_\infty^{(2)}(a,1,1) & 0 & x_\infty^{(2)}(a,1,2) & \cdots \\ \hline x_\infty^{(1)}(a,2,1) & 0 & x_\infty^{(1)}(a,2,2) & 0 & \cdots \\ 0 & x_\infty^{(2)}(a,2,1) & 0 & x_\infty^{(2)}(a,2,2) & \cdots \\ \hline & \vdots & & \vdots & \ddots \end{array}\right).$$

We point out that μ can be defined directly by the formulas

$$\begin{aligned}
&\mu(1) = 1\\
&\mu(x_\infty(a, 2i-1, 2j-1)) = x_\infty^{(1)}(a, i, j)\\
&\mu(x_\infty(a, 2i, 2j)) = x_\infty^{(2)}(a, i, j)\\
&\mu(x_\infty(a, k, l)) = 0 \quad \text{for } k \neq l \bmod 2.
\end{aligned}$$

Furthermore, in the proof of the theorem below, we will usually write formulas of the type (**) by showing only the top left block and part of the next block to the right. This will save space; the proof of the homotopy associativity of μ will use expressions involving 4×4 blocks.

THEOREM 1.11. *With the hypotheses and notation of the previous definition,* $(W_\infty(A), \chi, \mu, \iota)$ *is an abelian homotopy dual group.*

The proof uses the following lemma, which will also be useful later.

LEMMA 1.12. *Any two homomorphisms from* $K_0 \oplus \cdots \oplus K_0$ *to* K_0*, sending rank one projections to rank one projections, are homotopic.*

The proof is easily derived from the ideas in the proof of Theorem 2.3 in [**3**], and is omitted. Note that every such homomorphism is continuous, because the domain is a union of C^*-algebras and the range is contained in a C^*-algebra.

Proof of Theorem 1.11. We must verify conditions (1) through (5) of Definition 1.2. We will use the notation $x_\infty(a, i, j)$ of Definition 1.10, as well as the obvious extension of the notation $x_\infty^{(l)}(a, i, j)$ to free products with more than two factors.

(1) Using Proposition 1.9 on homotopy classes, the relation

$$(\mu * id_A) \circ \mu \simeq (id_A * \mu) \circ \mu$$

will follow from the homotopy equivalence of the homomorphisms

$$\mu_1, \mu_2 : A \longrightarrow K_0 \tilde{\otimes} (W_\infty(A) * W_\infty(A) * W_\infty(A))$$

given by

$$\mu_1(a) = \left(\begin{array}{cccc|ccc}
x_\infty^{(1)}(a,1,1) & 0 & 0 & 0 & x_\infty^{(1)}(a,1,2) & 0 & \dots\\
0 & x_\infty^{(3)}(a,1,1) & 0 & x_\infty^{(3)}(a,1,2) & 0 & x_\infty^{(3)}(a,1,3) & \dots\\
0 & 0 & x_\infty^{(2)}(a,1,1) & 0 & 0 & 0 & \dots\\
0 & x_\infty^{(3)}(a,2,1) & 0 & x_\infty^{(3)}(a,2,2) & 0 & x_\infty^{(3)}(a,2,3) & \dots\\
\hline
 & \vdots & & & & \ddots &
\end{array}\right)$$

and

$$\mu_2(a) = \left(\begin{array}{cccc|ccc} x_\infty^{(1)}(a,1,1) & 0 & x_\infty^{(1)}(a,1,2) & 0 & x_\infty^{(1)}(a,1,3) & 0 & \dots \\ 0 & x_\infty^{(2)}(a,1,1) & 0 & 0 & 0 & x_\infty^{(2)}(a,1,2) & \dots \\ x_\infty^{(1)}(a,2,1) & 0 & x_\infty^{(1)}(a,2,2) & 0 & x_\infty^{(1)}(a,2,3) & 0 & \dots \\ 0 & 0 & 0 & x_\infty^{(3)}(a,1,1) & 0 & 0 & \dots \\ \hline & \vdots & & & & \ddots & \end{array}\right).$$

To simplify the notation, set $B = W_\infty(A) * W_\infty(A) * W_\infty(A)$. Further let

$$m_A : A \longrightarrow K_0 \tilde{\otimes} W_\infty(A)$$

be the map corresponding to $id_{W_\infty(A)}$ under Proposition 1.9, and let

$$\eta : K_0 \tilde{\otimes} W_\infty(A) \longrightarrow K_0 \tilde{\otimes} (B \oplus B \oplus B)$$

be the homomorphism given by

$$\eta(y \otimes x_\infty(a,i,j)) = (y \otimes x_\infty^{(1)}(a,i,j), y \otimes x_\infty^{(2)}(a,i,j), y \otimes x_\infty^{(3)}(a,i,j)).$$

Then one readily verifies that

$$\mu_i = (\varphi_i \otimes id_B) \circ \eta \circ m_A$$

for appropriate homomorphisms

$$\varphi_1, \varphi_2 : K_0 \oplus K_0 \oplus K_0 \longrightarrow K_0.$$

The homomorphisms φ_1 and φ_2 send rank one projections to rank one projections. So Lemma 1.12 implies that $\varphi_1 \simeq \varphi_2$, whence $\mu_1 \simeq \mu_2$, as desired.

(2) By assumption there is a homotopy $t \longmapsto \varphi_t$ from $\iota_0 \circ \iota_0$ to id_A. Then

$$t \longmapsto W_\infty(\varphi_t)^+$$

is a unital homotopy from $\iota \circ \iota$ to $id_{W_\infty(A)^+}$.

(3) The homomorphism

$$\delta_{W_\infty(A)^+} \circ (\iota * id_{W_\infty(A)^+}) \circ \mu$$

is the unitization of the homomorphism corresponding under Proposition 1.9 to the homomorphism

$$\eta : A \longrightarrow K_0 \tilde{\otimes} W_\infty(A)$$

given by

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

$$\eta(a) = \left(\begin{array}{cc|cc} x_\infty(\iota_0(a),1,1) & 0 & x_\infty(\iota_0(a),1,2) & \cdots \\ 0 & x_\infty(a,1,1) & 0 & \cdots \\ \hline & \vdots & & \ddots \end{array}\right).$$

It suffices to prove that η is homotopic to the zero map from A to $K_0 \tilde{\otimes} W_\infty(A)$, since the unitization of the corresponding map from $W_\infty(A)$ to $W_\infty(A)$ is $\epsilon_{W_\infty(A)^+} \circ \chi$. Let $t \longmapsto \varphi_t$ be the homotopy of maps from A to $M_2(A)$, with

$$\varphi_0(a) = \begin{pmatrix} \iota_0(a) & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad \varphi_1(a) = 0,$$

given by the assumption on ι_0. Then the required homotopy is $t \longmapsto \eta_t$, where

$$\eta_t(a) = \left(\begin{array}{cc|c} x_\infty(\varphi_t(a)_{11},1,1) & x_\infty(\varphi_t(a)_{12},1,1) & x_\infty(\varphi_t(a)_{11},1,2) \ \cdots \\ x_\infty(\varphi_t(a)_{21},1,1) & x_\infty(\varphi_t(a)_{22},1,1) & x_\infty(\varphi_t(a)_{21},1,2) \ \cdots \\ \hline & \vdots & \ddots \end{array}\right).$$

The proof that

$$\epsilon_{W_\infty(A)^+} \circ \chi \simeq \delta_{W_\infty(A)^+} \circ (id_{W_\infty(A)^+} * \iota) \circ \mu$$

is the same, except that one uses

$$a \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varphi_t(a) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in place of φ_t.

(4) Proposition 1.9 reduces the proof that

$$(\chi * id_{W_\infty(A)^+}) \circ \mu \simeq id_{W_\infty(A)^+}$$

to proving that

$$a \longmapsto \left(\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & \cdots \\ 0 & x_\infty(a,1,1) & 0 & x_\infty(a,1,2) & \cdots \\ \hline & \vdots & & \vdots & \ddots \end{array}\right).$$

is homotopic to

$$a \longmapsto \begin{pmatrix} x_\infty(a,1,1) & x_\infty(a,1,2) & \cdots \\ x_\infty(a,2,1) & x_\infty(a,2,2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

This can be proved using Lemma 1.12 in the same way as in the proof of (1).

(5) The homotopy $\tau_{W_\infty(A)^+} \circ \mu \simeq \mu$ is obtained in the same way as the homotopies in the proof of (1). We omit the details. This completes the proof.

We should note that the operation μ on $W_\infty(A)^+$ really just corresponds to the direct sum operation on maps to matrix algebras. (This statement will be made precise in Proposition 1.14 below.) The proof just given is slightly more complicated than the usual proof of, say, homotopy commutativity of direct sums, because of the need to deal with $\bigcup_1^\infty M_k$ all at once, rather then being able to consider the various matrix algebras separately.

LEMMA 1.13. *There is an isomorphism $M_2(K_0) \cong K_0$, and any two such isomorphisms are homotopic as homomorphisms.*

Proof. An isomorphism is given on matrix units by

$$e_{ij} \otimes e_{kl} \longmapsto e_{2(k-1)+i,2(l-1)+j} \quad \text{for } i,j \in \{1,2\} \text{ and } k,l \in \mathbf{N}.$$

The homotopy statement follows from Lemma 1.12.

PROPOSITION 1.14. *Let A and ι_0 be as in Definition 1.10, and let B be a unital pro-C^*-algebra. In $[A, K_0 \tilde{\otimes} B]$, let 0 be the class of the zero map, let $-[\varphi] = [\varphi \circ \iota_0]$, and let*

$$[\varphi] + [\psi] = [(\eta \otimes id_B) \circ (id_{K_0} \otimes \sigma) \circ (\varphi \oplus \psi)],$$

where η is as in the previous lemma, σ is the diagonal embedding of $B \oplus B$ in $M_2(B)$, and

$$(\varphi \oplus \psi)(a) = (\varphi(a), \psi(a)).$$

Then $[A, K_0 \tilde{\otimes} B]$ is an abelian group which is naturally isomorphic to $[W_\infty(A)^+, B]_1$ via the correspondence of Proposition 1.9.

Proof. Of course, we have

$$[W_\infty(A)^+, B]_1 \cong [W_\infty(A), B].$$

The proof consists of showing that the isomorphism of Proposition 1.9 converts the operations given above into those on $[W_\infty(A)^+, B]_1$. The details are easy and are omitted.

An important example of an algebra satisfying the conditions of Definition 1.10 is given in the next proposition.

PROPOSITION 1.15. *Let A be a C^*-algebra, let qA be the C^*-algebra of Section 1 of* [**8**], *and let $\tau : qA \longrightarrow qA$ be the involutive automorphism defined following Corollary 1.2 in* [**8**]. *Then the homomorphism*

$$x \longmapsto \begin{pmatrix} \tau(x) & 0 \\ 0 & x \end{pmatrix}$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

from qA to $M_2(qA)$ is homotopic to the zero map.

Proof. This result follows from the correspondence between homomorphisms from qA and quasihomomorphisms from A, as in [**8**]. It can also be seen directly: in the notation of [**8**], the required homotopy is the restriction to qA of the homotopy

$$\varphi_t : A * A \longrightarrow M_2(A * A)$$

given by

$$\varphi_t(\iota(a)) = \begin{pmatrix} \bar{\iota}(a) & 0 \\ 0 & \iota(a) \end{pmatrix}$$

and

$$\varphi_t(\bar{\iota}(a)) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \iota(a) & 0 \\ 0 & \bar{\iota}(a) \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}^*,$$

as t runs from 0 to $\pi/2$.

The algebra P referred to in the introduction will be defined by

$$P = W_\infty(q\mathbf{C})^+.$$

We will need later the following explicit description of $q\mathbf{C}$.

PROPOSITION 1.16. *The free product $\mathbf{C} * \mathbf{C}$ is isomorphic to the C^*-algebra*

$$D = \left\{ a : [0, 1] \longrightarrow M_2 : a \textit{ is continuous, } a(0) \textit{ is diagonal, } \right.$$
$$\left. \textit{and } a(1) \in \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

via an isomorphism sending the identities of the first and second copies of $\mathbf{C}$ *respectively to the constant function*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the function

$$t \longmapsto \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

This isomorphism identifies $q\mathbf{C}$ with $\{a \in D : a(1) = 0\}$.

Proof. The first part is essentially contained in the remarks preceding Definition 9 of [**2**], where the unitization $(\mathbf{C} * \mathbf{C})^+$, which is isomorphic to $C^*(\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z})$, is calculated. (See also the calculation following Proposition V.1.4) of [**28**].) The identification of $q\mathbf{C}$ then follows by a calculation.

2. A classifying algebra for RK_0. The purpose of this section is to establish a natural isomorphism of abelian groups $RK_0(A) \cong [P, A]_1$ for unital σ-C^*-algebras A, where RK_0 is as defined in [**20**] and P is the σ-C^*-algebra and homotopy dual group given in the following definition.

Definition 2.1. $P = W_\infty(q\mathbf{C})^+$, with the homotopy dual group structure obtained from Definition 1.10 and Proposition 1.15.

The proof of this isomorphism will use the intermediate group $\bar{P}(A)$, which we now define. The isomorphism $\bar{P}(A) \cong RK_0(A)$ is the best analog we have in the σ-C^*-algebra case of the usual description of $K_0(A)$ in terms of projections in matrix algebras over A. The examples in Section 4 of [**20**] suggest that $\bar{P}(A)$ is about as good an analog as there is.

Definition 2.2. Let A be a σ-C^*-algebra. Then $\bar{P}(A)$ is defined to be the set of projections $p \in M_2((K \otimes A)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K \otimes A),$$

modulo homotopy within this set. We define addition in $\bar{P}(A)$ as follows. First, choose an isomorphism $\eta : M_2(K) \longrightarrow K$. Further let u be the unitary matrix

$$u = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix},$$

regarded as an element of $M_4((K \otimes A)^+)$. Then define

$$[p] + [q] = [(M_2(\eta) \otimes id_A)(u(p \oplus q)u^*)],$$

where as usual square brackets denote homotopy classes. Finally if $\varphi : A \longrightarrow B$ is a homomorphism of σ-C^*-algebras, define $\varphi_* : \bar{P}(A) \longrightarrow \bar{P}(B)$ by

$$\varphi_*([p]) = [M_2((id_K \otimes \varphi)^+)(p)].$$

PROPOSITION 2.3. *$\bar{P}$ is a functor from σ-C^*-algebras to abelian semigroups.*

Proof. Note that

$$(M_2(\eta) \otimes id_A)\left(u\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]u^*\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, in the definition of addition,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - (M_2(\eta) \otimes id_A)(u(p \oplus q)u^*)$$

is in fact in $M_2(K \otimes A)$. Thus the definition makes sense. The rest of the proposition is easy.

It will follow from the isomorphism $\bar{P}(A) \cong RK_0(A)$, proved in Proposition 2.8, that $\bar{P}(A)$ is actually a group.

We next prove a sequence of lemmas. Many of the proofs are by induction, and are done by writing a σ-C^*-algebra A as an inverse limit $\varprojlim A_n$. When this is done, we will denote the homomorphisms from A to A_n by κ_n and the homomorphisms from A_{n+1} to A_n by π_n. We will always assume that these maps are surjective, unless otherwise stated. We will also use the letters κ_n and π_n for homomorphisms derived from κ_n and π_n in standard ways, for example

$$\kappa_n : K \otimes A \otimes C([0,1]) \longrightarrow K \otimes A_n \otimes C([0,1])$$

or

$$\pi_n : M(K \otimes A_{n+1}) \longrightarrow M(K \otimes A_n).$$

(Note that κ_n and π_n do make sense on multiplier algebras, by [**19**], Proposition 3.14 (1).)

The next lemma is well known in the C^*-algebra case, which is the case we use most often.

LEMMA 2.4. *Let A be a unital σ-C^*-algebra, let I be a closed ideal in A, and let $t \longmapsto p_t$ be a continuous path of projections in A such that $p_t - p_0 \in I$ for all t. Then there exists a continuous path $t \longmapsto u_t$ of unitaries in A such that $u_0 = 1$ and such that $u_t p_0 u_t^* = p_t$ and $1 - u_t \in I$ for all t.*

Proof. Using Proposition 5.3 of [**19**], we can write $A = \varprojlim A_n$ in such a way that $I = \varprojlim I_n$ for closed ideals $I_n = \kappa_n(I)$ in A_n. We will now construct u_t by constructing inductively a coherent sequence of continuous paths $t \longmapsto u_t^{(n)}$ in A_n such that

$$u_t^{(n)} \kappa_n(p_0)(u_t^{(n)})^* = \kappa_n(p_t) \quad \text{and} \quad 1 - u_t^{(n)} \in I_n.$$

The construction of $u_t^{(1)}$ is standard: choose $0 = t(0) < t(1) < \cdots < t(k) = 1$ so that

$$\|\kappa_1(p_t) - \kappa_1(p_{t(i)})\| < \frac{1}{2} \quad \text{for } t \in [t(i), t(i+1)],$$

set

$$x_t^{(i)} = \kappa_1(p_t p_{t(i)} + (1-p_t)(1-p_{t(i)})) \quad \text{and} \quad v_t^{(i)} = x_t^{(i)}[(x_t^{(i)})^* x_t^{(i)}]^{-\frac{1}{2}}$$

for $t \in (t(i), t(i+1)]$, and set

$$u_t^{(1)} = v_t^{(i)} v_{t(i)}^{(i-1)} \cdot \cdots \cdot v_{t(1)}^{(0)},$$

again for $t \in (t(i), t(i+1)]$. It is easily checked that $1 - x_t^{(i)} \in I_1$ for all i and all $t \in (t(i), t(i+1)]$, from which it follows that $1 - u_t^{(1)} \in I_1$ for all t.

Now suppose $t \longmapsto u_t^{(n)}$ has been constructed. Using the argument of the previous paragraph, construct an arbitary continuous path $t \longmapsto w_t$ of unitaries in A_{n+1} such that $w_0 = 1$ and such that

$$w_t \kappa_{n+1}(p_0) w_t^* = \kappa_{n+1}(p_t) \quad \text{and} \quad 1 - w_t \in I_{n+1}$$

for all t. Let B_l be the C^*-algebra of A_l given by

$$B_l = \{a \in I_l + \mathbf{C} \cdot 1 : \quad a \text{ commutes with } \kappa_l(p_0)\}.$$

It is easily shown that $\pi_n(B_{n+1}) = B_n$. Now regard the path

$$t \longmapsto \pi_n(w_t^*) u_t^{(n)}$$

as an element of $B_n \otimes C([0, 1])$. Since its value at 0 is 1, it is in $U_0(B_n \otimes C([0, 1]))$. Therefore there is a continuous path $t \longmapsto z_t$ of unitaries in B_{n+1} such that

$$\pi_n(z_t) = \pi_n(w_t^*) u_t^{(n)}.$$

Write $z_t = a_t + \lambda_t \cdot 1$, where $a_t \in I_{n+1}$ and $\lambda_t \in \mathbf{C}$. The required path is then

$$u_t^{(n+1)} = w_t z_t z_0^* \lambda_0 \bar{\lambda}_t.$$

LEMMA 2.5. *Let A be a C^*-algebra, and let $\epsilon > 0$. Then:*
(1) *For any projection $p \in M_2((K \otimes A)^+)$ such that*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K \otimes A),$$

there is a homotopy of projections $t \longmapsto p_t$ from p to a projection p_1 such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_1 \in M_2(K_0 \tilde{\otimes} A),$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A) \quad \textit{and} \quad \|p_t - p\| < \epsilon \quad \textit{for all } t.$$

(2) *Let $t \longmapsto p_t$ be a homotopy of projections in $M_2((K \otimes A)^+)$ such that*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A) \quad \textit{for all } t,$$

and such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_i \in M_2(K_0 \tilde{\otimes} A) \quad \textit{for } i = 0, 1.$$

Then there is a continuous function $(t, s) \longmapsto p_{t,s}$ from $[0, 1]^2$ to projections in $M_2((K \otimes A)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_{t,s} \in M_2(K \otimes A),$$

$p_{t,0} = p_t$, $p_{0,s} = p_0$, $p_{1,s} = p_1$, *and* $\|p_{t,s} - p_t\| < \epsilon$ *for all* t, s, *and such that* $t \longmapsto p_{t,1}$ *is a homotopy from* p_0 *to* p_1 *of projections in* $M_2((K_0 \tilde{\otimes} A)^+)$, *differing from*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by elements of $M_2(K_0 \tilde{\otimes} A)$.

In this lemma, $K_0, K_0 \tilde{\otimes} A$, and homotopies in $K_0 \tilde{\otimes} A$ are as in Definition 1.8. Here, $K_0 \tilde{\otimes} A$ is of course just the algebraic tensor product

$$K_0 \otimes A = \bigcup_1^\infty (M_k \otimes A),$$

since A is a C^*-algebra.

Proof of Lemma 2.5. (1) Let $A_0 = A$ if A is unital, and let $A_0 = A^+$ if not. Let f_k be the identity of $M_2(M_k \otimes A_0)$, so that (f_k) is an increasing approximate identity of projections in $M_2(K \otimes A_0)$, and

$$M_2(K_0 \tilde{\otimes} A_0) = \bigcup f_k M_2(K \otimes A_0) f_k.$$

Choose

$$\delta \leqq \min\left(\frac{\epsilon}{2}, \frac{1}{4}\right),$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

and choose k so large that

$$\left\| f_k \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \right] f_k - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \right] \right\| < \delta.$$

Set

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + f_k \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \right] f_k,$$

and set $a_t = ta + (1-t)p$. Then

$$\|p - a_t\| < t\delta \leqq \frac{1}{4} \quad \text{for all } t \in [0, 1].$$

Furthermore,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - a_t \in M_2(K \otimes A) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - a_1 \in M_2(M_k \otimes A).$$

Let h be the function given by

$$h(\lambda) = 0 \quad \text{for } |\lambda| < \frac{1}{2} \quad \text{and} \quad h(\lambda) = 1 \quad \text{for } |\lambda - 1| < \frac{1}{2}.$$

Then h is defined and continuous on $sp(a_t)$. The desired path of projections is given by $t \longmapsto p_t = h(a_t)$. The verification of the required properties is immediate.

(2) Regard the homotopy $t \longmapsto p_t$ as an element $p \in M_2((K \otimes A \otimes C([0, 1]))^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K \otimes A \otimes C([0, 1])).$$

Then apply the argument of (1) to p, taking care to choose k so large that $p_0, p_1 \in M_2((M_k \otimes A)^+)$. The result is a continuous function $(t, s) \longmapsto p_{t,s}$ satisfying all the required properties. (Note that the construction of part (1) yields $p_{0,s} = p_0$ and $p_{1,s} = p_1$ for all s, because $p_0, p_1 \in M_2((M_k \otimes A)^+)$. The remaining properties follow directly from (1).)

LEMMA 2.6. *Let A be a σ-C^*-algebra, and let $\bar{P}_0(A)$ be the set of projections $p \in M_2((K_0 \tilde{\otimes} A)^+)$ such that*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K_0 \tilde{\otimes} A),$$

modulo homotopy within this set. (Homotopy is interpreted as in Definition 1.8.*) Then the obvious map from $\bar{P}_0(A)$ to $\bar{P}(A)$ is bijective.*

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

Proof. We must prove the following two statements.

(1) Every projection $p \in M_2((K \otimes A)^+)$ with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K \otimes A)$$

is homotopic, through such projections, to a projection $q \in M_2((K_0 \tilde{\otimes} A)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q \in M_2(K_0 \tilde{\otimes} A).$$

(2) If two projections in $M_2((K_0 \tilde{\otimes} A)^+)$, both differing from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by elements of $M_2(K_0 \tilde{\otimes} A)$, are homotopic through projections in $M_2((K \otimes A)^+)$ differing from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by elements of $M_2(K \otimes A)$, then the homotopy can be taken to be an element of $M_2([K_0 \tilde{\otimes} (A \otimes C([0, 1]))]^+)$ differing from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $M_2(K_0 \tilde{\otimes} (A \otimes C([0, 1])))$.

It is convenient to prove both statements at once, where, for the proof of the second one, we assume that $A = B \otimes C([0, 1])$ and prove it for B rather than for A. In that case, we will of course regard A and $M_2((K_0 \tilde{\otimes} A)^+)$ as being certain algebras of continuous functions on $[0, 1]$. We will denote the corresponding evaluation maps by $a \longmapsto a(s)$, for $s \in [0, 1]$, and use subscript notation for all homotopies which take values in A.

Let $A = \varprojlim A_n$. (In case (2), we take $A_n = B_n \otimes C([0, 1])$ where $B = \varprojlim B_n$.) We will construct a continuous path $t \longmapsto p_t$ of projections, for $t \in [0, \infty)$, and an increasing sequence of integers $n \longmapsto k(n)$, such that the following properties hold.

(H1) $p_0 = p$, the given projection in $M_2((K \otimes A)^+)$.

(H2) $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A) \quad \text{for all } t.$$

(H3) $\kappa_n\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - p_n\right) \in M_2(M_{k(n)} \otimes A)$ for all n.

(H4) $\kappa_n(p_t) = \kappa_n(p_n)$ for $t \geqq z$.

In the case (2), p is a homotopy with $p(0), p(1) \in M_2((K_0 \tilde{\otimes} B)^+)$. Thus there is an increasing sequence $n \longmapsto k_0(n)$ of integers such that

$$\kappa_n(p(0)), \kappa_n(p(1)) \in M_2((M_{k_0(n)} \otimes B_n)^+) \quad \text{for all } n.$$

We then impose the following additional conditions.

(H5) $k(n) \geqq k_0(n)$ for all n.

(H6) $p_t(0) = p(0)$ and $p_t(1) = p(1)$ for all t.

If the conditions (H1)–(H4) are satisfied, then $p_\infty = \lim_{t\to\infty} p_t$ clearly exists, lies in $M_2((K_0 \tilde{\otimes} A)^+)$, and satisfies

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - p_\infty \in M_2(K_0 \tilde{\otimes} A).$$

Therefore $t \longmapsto p_t$, for $t \in [0, \infty]$, defines the homotopy required for (1). For (2), assuming also (H5) and (H6), it follows that p_∞ is a homotopy in $M_2((K_0 \tilde{\otimes} B)^+)$, from $p(0)$ to $p(1)$, which satisfies the required conditions. We therefore need only show that conditions (H1) through (H4), or, for (2), through (H6), can be satisfied.

We begin by applying the previous lemma (part (1) in case (1) and part (2) in case (2)) to $\kappa_1(p)$, with $\epsilon = 1/4$, obtaining a continuous path $t \longmapsto q_t$ of projections in $M_2((K \otimes A_1)^+)$, defined for $t \in [0, 1]$. Thus

$$q_1 \in M_2((M_{k(1)} \otimes A_1)^+) \quad \text{for some } k(1),$$

which we take larger than $k_0(1)$ in case (2). Using the C^*-algebra case of Lemma 2.4, choose a continuous path $t \longmapsto v_t$ of unitaries in $M_2(K \otimes A_1)^+$ such that $v_0 = 1$ and $v_t q_0 v_t^* = q_t$. Then

$$v \in U_0(M_2(K \otimes A_1)^+ \otimes C([0, 1])),$$

so by Lemma 1.11 of [**20**], there is

$$u \in U_0(M_2(K \otimes A)^+ \otimes C([0, 1]))$$

such that $\kappa_1(u) = v$. We view u as a continuous path $t \longmapsto u_t$, and we may clearly require that $u_0 = 1$. In case (2), we have $q_t(0) = q_0(0)$ and $q_t(1) = q_0(1)$ for $t \in [0, 1]$, so that Lemma 2.4 gives us unitaries in $[M_2(K \otimes B_1 \otimes C_0((0, 1)))]^+$. Thus we may replace A_1 by $B_1 \otimes C_0((0, 1))$ and A by $B \otimes C_0((0, 1))$. Then the resulting unitaries u_t satisfy in addition $u_t(0), u_t(1) \in \mathbf{C} \cdot 1$ for $t \in [0,1]$. We now set $p_t = u_t p u_t^*$ for $t \in [0,1]$. Clearly

$$\kappa_1(p_t) = q_t \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A).$$

Furthermore, in case (2), we have $p_t(0) = p(0)$ and $p_t(1) = p(1)$ by the choice of u.

We now assume inductively that p_t has been defined for $t \in [0, n]$, and that $k(m)$ has been chosen for $m \leqq n$, in such a way that the properties above hold on $[0, n]$ and for $m \leqq n$. We apply the previous lemma to $\kappa_{n+1}(p_n)$, with $\epsilon = 1/20$, again using part (1) in case (1) and part (2) in case (2). We obtain a continuous path $t \longmapsto q_t$ of projections and $M_2((K \otimes A_{n+1})^+)$, which we take to be defined for $t \in [n, n+1]$, with

$$q_n = \kappa_{n+1}(p_n) \quad \text{and} \quad q_{n+1} \in M_2((M_{k(n+1)} \otimes A_{n+1})^+) \quad \text{for some } k(n+1).$$

We can certainly choose $k(n+1) \geqq k(n)$, and, in case (2), $\geqq k_0(n+1)$. To obtain p_t for $t \in [n, n+1]$, we will first modify q_t so that $\pi_n(q_t) = \kappa_n(p_n)$, and then lift q_t up to $M_2((K \otimes A)^+)$ as was done on $[0, 1]$.

To do the modification, set

$$w_t = a_t(a_t^* a_t)^{-\frac{1}{2}}, \quad \text{where } a_t = \pi_n(q_t q_n + (1 - q_t)(1 - q_n)).$$

Thus $t \longmapsto w_t$ is a continuous path of unitaries in $M_2(K \otimes A_n)^+$, and it is easily seen to satisfy the following properties:

(U1) $w_n = 1$.

(U2) $w_t^* \pi_n(q_n) w_t = \pi_n(q_t)$.

(U3) $w_{n+1} \in M_2(M_{k(n+1)} \otimes A_n)^+$.

(U4) $\|w_t - 1\| \leqq \frac{1}{2}$ for all t.

(The estimate (U4) follows because

$$\|\pi_n(q_t) - \pi_n(q_n)\| < \frac{1}{20} \quad \text{implies} \quad \|a_t - 1\| < \frac{1}{10},$$

whence $\|a_t(a_t^* a_t)^{-\frac{1}{2}} - 1\| < 3/8$.) Furthermore, in case (2) we also have:

(U5) $w_t(0) = w_t(1) = 1$ for all t.

For $l = n$ or $n+1$, we now let D_l be the C^*-algebra of all continuous functions $t \longmapsto x_t$ from $[n, n+1]$ to $M_2(K \otimes A_l)^+$ such that

$$x_n \in \mathbf{C} \cdot 1, \quad x_{n+1} \in M_2(M_{k(n+1)} \otimes A_l)^+,$$

and, in case (2), $x_t(0)$ and $x_t(1)$ are in $\mathbf{C}\cdot 1$ and do not depend on t. Then w is a unitary element of D_n which by (U4) satisfies $\|w-1\| \leqq 1/2$. Therefore $w \in U_0(D_n)$. Clearly

$$\pi_n : D_{n+1} \longrightarrow D_n$$

is surjective, so there is $x \in U_0(D_{n+1})$ such that $\pi_n(x) = w$. Then x is a continuous path of unitaries in $M_2(K \otimes A_{n+1})^+$ with

$$x_n \in \mathbf{C}\cdot 1, \quad x_{n+1} \in M_2(M_{k(n+1)} \otimes A_{n+1})^+,$$

and, in case (2), $x_t(0), x_t(1) \in \mathbf{C}\cdot 1$ for all t. Our modified path of projections is then

$$t \longmapsto x_t q_t x_t^*.$$

We have

$$x_n q_n x_n^* = q_n$$

because $x_n \in \mathbf{C}\cdot 1$, we have

$$\pi_n(x_t q_t x_t^*) = \kappa_n(p_n)$$

by (U2) and because $\pi_n(q_n) = \kappa_n(p_n)$, and we still have

$$x_{n+1} q_{n+1} x_{n+1}^* \in M_2((M_{k(n+1)} \otimes A_{n+1})^+).$$

Also, of course, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - x_t q_t x_t^* \in M_2(K \otimes A_{n+1}) \quad \text{for all } t.$$

Finally, in case (2) we have

$$(x_t q_t x_t^*)(i) = q_t(i) = q_n(i) \quad \text{for } i = 0, 1,$$

because $x_t(i) \in \mathbf{C}\cdot 1$.

We now do the lifting. This argument is the same as the argument used in the initialization part of the induction, to define p_t for $t \in [0,1]$, and we omit the details. The result is a path $t \longmapsto p_t$, for $t \in [n, n+1]$, of projections in $M_2((K \otimes A)^+)$, with p_n as already given,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A), \quad \kappa_{n+1}(p_t) = x_t q_t x_t^*,$$

and, in case (2), $p_t(0) = p(0)$ and $p_t(1) = p(1)$ for $t \in [n, n+1]$. This completes the induction argument, so the desired path $t \longmapsto p_t$ for $t \in [0, \infty]$ has been shown to exist.

LEMMA 2.7. *Let A be a unital σ-C^*-algebra and let $p \in M_2((K \otimes A)^+)$ be a projection such that*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K \otimes A).$$

Then there is a unitary $u \in M_2(M(K \otimes A))$ such that

$$upu^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $M(K \otimes A)$ is the multiplier algebra, defined in Section 3 of [**19**]. We regard $(K \otimes A)^+$ as a subalgebra of it in the obvious way.

Proof of Lemma 2.7. By the previous lemma, p is homotopic to a projection $q \in M_2((K_0 \tilde{\otimes} A)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q \in M_2(K_0 \tilde{\otimes} A),$$

and, by Lemma 2.4, p is unitarily equivalent to q. So we can assume that $p \in M_2((K_0 \tilde{\otimes} A)^+)$.

We now identify $M_2(M(K \otimes A))$ with the algebra of bounded operators on the Hilbert A-module $l^2(A) \oplus l^2(A)$. (Compare [**19**], Section 4.) It then suffices to find an isomorphism

$$v : p(l^2(A) \oplus l^2(A)) \longrightarrow l^2(A).$$

Indeed, the same argument will then produce an isomorphism

$$w : (1 - p)(l^2(A) \oplus l^2(A)) \longrightarrow l^2(A),$$

and we define

$$u : l^2(A) \oplus l^2(A) \longrightarrow l^2(A) \oplus l^2(A)$$

by

$$u\xi = (vp\xi, w(1 - p)\xi).$$

We will construct inductively a coherent sequence of isomorphisms

$$v_n : \kappa_n(p)[l^2(A_n) \oplus l^2(A_n)] \longrightarrow l^2(A_n).$$

Choose an increasing sequence $n \longmapsto k(n)$ of integers such that

$$\kappa_n(p) \in M_2((M_{k(n)} \otimes A_n)^+).$$

Let $f \in K \otimes A_1$ be the identity of $M_{k(1)} \otimes A_1$. Then

$$\begin{pmatrix} 1-f & 0 \\ 0 & 0 \end{pmatrix} \leqq \kappa_1(p)$$

because

$$\kappa_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p\right) \in M_2(M_{k(1)} \otimes A_1).$$

Therefore

$$(*) \qquad \begin{aligned} &\kappa_1(p)[l^2(A_1) \oplus l^2(A_1))] \\ &\cong (1-f)l^2(A_1) \oplus \left(\kappa_1(p) - \begin{pmatrix} 1-f & 0 \\ 0 & 0 \end{pmatrix}\right)[l^2(A_1) \oplus l^2(A_1)] \\ &\cong l^2(A_1). \end{aligned}$$

Here the last isomorphism follows from the stabilization theorem [**16**], because

$$(1-f)l^2(A_1) \cong l^2(A_1).$$

Let v_1 be an isomorphism as in $(*)$.

Now suppose we are given an isomorphism

$$v_n : \kappa_n(p)[l^2(A_n) \oplus l^2(A_n)] \longrightarrow l^2(A_n).$$

Use the reasoning of the previous paragraph to produce any isomorphism

$$w : \kappa_{n+1}(p)[l^2(A_{n+1}) \oplus l^2(A_{n+1})] \longrightarrow l^2(A_{n+1}).$$

Following our conventions, $\pi_n : A_{n+1} \longrightarrow A_n$ defines a map

$$\begin{aligned} \pi_n : L(\kappa_{n+1}(p)[l^2(A_{n+1}) \oplus l^2(A_{n+1})], l^2(A_{n+1})) \\ \longrightarrow L(\kappa_n(p)[l^2(A_n) \oplus l^2(A_n)], l^2(A_n)). \end{aligned}$$

Now $v_n\pi_n(w)^*$ is a unitary in $L(l^2(A_n)) \cong M(K \otimes A_n)$. Since $K \otimes A_{n+1}$ has a countable approximate identity, the map

$$\pi_n : M(K \otimes A_{n+1}) \longrightarrow M(K \otimes A_n)$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

is surjective ([**17**], Theorem 10). Since $U(M(K \otimes A_n))$ is connected ([**15**], Theorem 2.5), there is $z \in U(M(K \otimes A_{n+1}))$ such that

$$\pi_n(z) = v_n \pi_n(w)^*.$$

Now set $v_{n+1} = zw$, so that

$$v_{n+1} : \kappa_{n+1}(p)[l^2(A_{n+1}) \oplus l^2(A_{n+1})] \longrightarrow l^2(A_{n+1})$$

is an isomorphism satisfying $\pi_n(v_{n+1}) = v_n$. This completes the induction step, and the proof.

PROPOSITION 2.8. *If A is a σ-C^*-algebra, then $\bar{P}(A)$ is naturally isomorphic to $RK_0(A)$ as defined in* [**20**].

Proof. We first consider the case in which A is unital. Recall that

$$RK_0(A) = (U/U_0)(Q(A)),$$

where

$$Q(A) = M(K \otimes A)/(K \otimes A).$$

We define $\Phi : \bar{P}(A) \longrightarrow RK_0(A)$ as follows. If $p \in M_2((K \otimes A)^+)$ is a projection such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K \otimes A),$$

then by the previous lemma there is a unitary

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_2(M(K \otimes A))$$

such that

$$upu^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The relations

$$(*) \qquad u \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u^* - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = u \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \right) u^* \in M_2(K \otimes A)$$

and

$$(**) \qquad u^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K \otimes A)$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

imply that

$$u_{11}u_{11}^* - 1, u_{11}^*u_{11} - 1 \in K \otimes A.$$

Therefore, with $\pi : M(K \otimes A) \longrightarrow Q(A)$ being the quotient map, $\pi(u_{11})$ is unitary, and thus defines a class $\Phi([p]) \in RK_0(A)$.

We now show that Φ is well defined. First, if v is some other unitary such that

$$vpv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then vu^* commutes with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore has the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

for $x, y \in UM(K \otimes A)$. Using the fact ([**20**], Lemma 1.9) that $UM(K \otimes A)$ is path connected, we can therefore produce a continuous path

$$t \longmapsto u_t = \begin{pmatrix} (u_t)_{11} & (u_t)_{12} \\ (u_t)_{21} & (u_t)_{22} \end{pmatrix}$$

in $UM_2(M(K \otimes A))$ such that $u_0 = u, u_1 = v$, and

$$u_t p u_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } t.$$

Then $t \longmapsto \pi((u_t)_{11})$ is a continuous path from $\pi(u_{11})$ to $\pi(v_{11})$ in $UQ(A)$, so that

$$[\pi(u_{11})] = [\pi(v_{11})].$$

So $\Phi([p])$ does not depend on the choice of u. Now let $t \longmapsto p_t$ be a homotopy of projections in $M_2((K \otimes A)^+)$ with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A) \quad \text{for all } t.$$

Regard this path as a projection in $M_2((K \otimes A \otimes C([0, 1]))^+)$, and using the previous lemma choose a unitary

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_2(M(K \otimes A \otimes C([0, 1])))$$

such that

$$upu^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

By Proposition 3.17 of [**19**], u can be regarded as a (strictly continuous) function $t \longmapsto u_t$ from $[0, 1]$ to $M_2(M(K \otimes A))$. It follows from Corollary 1.13 of [**20**] that

$$[\pi((u_0)_{11})] = [\pi((u_1)_{11})]$$

in $RK_0(A)$. Therefore

$$\Phi([p_0]) = [\pi((u_0)_{11})] = [\pi((u_1)_{11})] = \Phi([p_1]),$$

as desired. This completes the proof that Φ is well defined.

Next we prove surjectivity. Let $u \in UQ(A)$. Choose $x \in M(K \otimes A)$ such that $\pi(x) = u$. Then $x^*x - 1, xx^* - 1 \in K \otimes A$. Let $f : [0, \infty) \longrightarrow \mathbf{R}$ be the function $f(\lambda) = 1$ for $\lambda \leqq 1$ and $f(\lambda) = \lambda^{-\frac{1}{2}}$ for $\lambda \geqq 1$. Set $y = xf(x^*x)$, so that $\pi(y) = u, y^*y \leqq 1$, and $yy^* \leqq 1$. Now set

$$r = \begin{pmatrix} y & (1 - yy^*)^{\frac{1}{2}} \\ -(1 - y^*y)^{\frac{1}{2}} & y^* \end{pmatrix} \in UM_2(M(K \otimes A)),$$

and set

$$p = r^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} r.$$

Then one can check that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p = \begin{pmatrix} 1 - y^*y & -y^*(1 - yy^*)^{\frac{1}{2}} \\ -(1 - yy^*)^{\frac{1}{2}} y & yy^* - 1 \end{pmatrix} \in M_2(K \otimes A).$$

Therefore $[p] \in \bar{P}(A)$ and $\Phi([p]) = [\pi(r_{11})] = [\pi(y)] = [u]$. So Φ is surjective.

We next prove that Φ is injective. Let $p, q \in M_2((K \otimes A)^+)$, let

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

be unitaries in $M_2(M(K \otimes A))$ such that

$$upu^* = vqv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and suppose that $[\pi(u_{11})] = [\pi(v_{11})]$ in $RK_0(A)$. This is equivalent to

$$\pi(u_{11})\pi(v_{11})^* \in U_0Q(A).$$

The relations $(*)$ and $(**)$ also imply that $\pi(u_{12}) = \pi(u_{21}) = 0$ and that $\pi(u_{22})$ is unitary. Since by Lemma 1.9 of [**20**], the group $UM_2(M(K \otimes A))$ is path connected, it follows that

$$\begin{pmatrix} \pi(u_{11}) & 0 \\ 0 & \pi(u_{22}) \end{pmatrix} \in U_0M_2(Q(A)).$$

Therefore

$$\begin{pmatrix} \pi(u_{11})\pi(u_{22})^* & 0 \\ 0 & 1 \end{pmatrix} \in U_0M_2(Q(A)).$$

By the stability isomorphism of [**20**], Theorem 3.4 (1), the map

$$z \longmapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

defines an isomorphism from $(U/U_0)(Q(A))$ to $(U/U_0)(M_2(Q(A)))$. Consequently

$$\pi(u_{11})\pi(u_{22})^* \in U_0Q(A).$$

Similarly

$$\pi(v_{11})\pi(v_{22})^* \in U_0Q(A).$$

Combining these two equations with

$$\pi(u_{11})\pi(v_{11})^* \in U_0Q(A)$$

gives

$$\pi(u_{22})\pi(v_{22})^* \in U_0Q(A).$$

Using Lemma 1.11 of [**20**], choose $y, z \in UM(K \otimes A)$ such that

$$\pi(y) = \pi(u_{11})\pi(v_{11})^* \quad \text{and} \quad \pi(z) = \pi(u_{22})\pi(v_{22})^*.$$

Now replace v by

$$\begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} v.$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

Then we still have

$$vqv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and in addition we now have $\pi(v_{11}) = \pi(u_{11})$ and $\pi(v_{22}) = \pi(u_{22})$. Since in any case $\pi(u_{12}) = \pi(u_{21}) = \pi(v_{12}) = \pi(v_{21}) = 0$, it follows that $u - v \in M_2(K \otimes A)$, whence $v^*u \in M_2(K \otimes A)^+$.

We now claim that there is a continuous path $t \longmapsto x_t$ in $U(M_2(K \otimes A)^+)$ such that $x_0 = v^*u$ and x_1 commutes with p. For the purposes of proving this claim we may assume, using the previous lemma, that

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let H be a separable infinite dimensional Hilbert space, so that $K = K(H)$, and choose a $*$-strong operator continuous path $t \longmapsto c_t$ of isometries on $H \oplus H$ such that

$$c_0 = 1 \quad \text{and} \quad c_1 c_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $t \longmapsto c_t$ can be regarded as a strictly continuous path of isometries in $M_2(M(K \otimes A))$. To obtain the desired path $t \longmapsto x_t$, write

$$v^*u = \lambda + a \quad \text{for } \lambda \in \mathbf{C} \text{ and } a \in M_2(K \otimes A).$$

Then set

$$x_t = c_t v^* u c_t^* + \lambda(1 - c_t c_t^*) = \lambda + c_t a c_t^*.$$

From the first expression, we see that x_t is unitary, $x_0 = v^*u$, and x_1 commutes with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From the second expression, we see that $x_t \in M_2(K \otimes A)^+$ and that $t \longmapsto x_t$ is continuous for the seminorm topology on $M_2(K \otimes A)^+$. This proves the claim.

Define $p_t = x_t p x_t^*$. Then $t \longmapsto p_t$ is a continuous path of projections in $M_2((K \otimes A)^+)$, each differing from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $M_2(K \otimes A)$, such that $p_0 = q$ and $p_1 = p$. Therfore $[p] = [q]$, and injectivity of Φ has been proved.

It remains to prove that Φ is a semigroup homomorphism. It suffices to prove that if $u, v \in UQ(A)$, then the class of

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

corresponds, under the isomorphism $M_2(Q(A)) \cong Q(A)$ coming from an isomorphism $M_2(K) \cong K$, to the class of uv. This has been essentially done in the proof that $RK_0(A)$ is abelian, [**20**], Proposition 2.2. This completes the proof for unital A.

We now do the case in which A is not unital. Note that the map $RK_0(A) \longrightarrow RK_0(A^+)$ induces an isomorphism of $RK_0(A)$ with the kernel of $RK_0(A^+) \longrightarrow RK_0(\mathbf{C})$. Therefore it suffices to prove that the map $\bar{P}(A) \longrightarrow \bar{P}(A^+)$ induces an isomorphism (of semigroups) of $\bar{P}(A)$ with the kernel of $\bar{P}(A^+) \longrightarrow \bar{P}(\mathbf{C})$. To do this, we must prove three things: that the composite

$$\bar{P}(A) \longrightarrow \bar{P}(A^+) \longrightarrow \bar{P}(\mathbf{C})$$

is zero, that

$$\bar{P}(A) \longrightarrow \mathrm{Ker}[\bar{P}(A^+) \longrightarrow \bar{P}(\mathbf{C})]$$

is surjective, and also that this map is injective.

The first statement is trivial. To prove the second, let $[p]$ be a class in $\bar{P}(A^+)$, let q be the image of p in $M_2(K^+)$ coming from $A^+ \longrightarrow \mathbf{C}$, and assume that $[q] = 0$. Then there is a continuous path $t \longmapsto q_t$ of projections in $M_2(K^+)$, differing from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by elements of $M_2(K)$, such that

$$q_0 = q \quad \text{and} \quad q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 2.4 provides a unitary path $t \longmapsto u_t$ in $M_2(K^+)$ such that

$$u_t q_t u_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad u_1 = 1.$$

Regard $t \longmapsto u_t$ as a path in $M_2(K \otimes A^+)^+$ via $\mathbf{C} \longrightarrow A^+$, and set $p_t = u_t p u_t^*$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A^+) \quad \text{for all } t,$$

and $p_1 = p$. So $[p_0] = [p]$, and one checks that p_0 is in fact $M_2((K \otimes A)^+)$, with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_0 \in M_2(K \otimes A).$$

This proves surjectivity.

For injectivity, let $t \longmapsto p_t$ be a homotopy of projections in $M_2((K \otimes A^+)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_t \in M_2(K \otimes A^+) \quad \text{for all } t$$

and such that

$$p_0, p_1 \in M_2((K \otimes A)^+).$$

Let q_t be the image of p_t in $M_2(K^+)$, obtained from the map $A^+ \longrightarrow \mathbf{C}$. By Lemma 2.4, there is a path $t \longmapsto u_t$ in $U(M_2(K)^+)$ such that

$$u_t q_t u_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_0 = 1, \quad \text{and} \quad 1 - u_t \in M_2(K).$$

Since u_1 commutes with

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we can write $u_1 = v_1 \oplus w_1$ with $v_1, w_1 \in U(K^+)$ and $1 - v_1, 1 - w_1 \in K$. Since $\{z \in U(K^+) : 1 - z \in K\}$ is connected, we can find paths $t \longmapsto v_t, w_t$ in this group such that $v_0 = w_0 = 1$ and v_1, w_1 are as above. Replacing u_t by $(v_t \oplus w_t)^* u_t$, we can assume $u_1 = 1$. Now regard $t \longmapsto u_t$ as a path in $U(M_2(K \otimes A^+)^+)$ via the map $\mathbf{C} \longrightarrow A^+$. Then $t \longmapsto u_t p_t u_t^*$ is a homotopy of projections in $M_2((K \otimes A)^+)$, and one checks that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - u_t p_t u_t^* \in M_2(K \otimes A).$$

Since $u_0 = u_1 = 1$, this path shows that $[p_0] = [p_1]$ in $\bar{P}(A)$, and injectivity is proved. This completes the proof of the proposition.

COROLLARY 2.9. *For any σ-C^*-algebra A, the semigroup $\bar{P}(A)$ is an abelian group, with identity*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

LEMMA 2.10. *Let A be a σ-C^*-algebra which has a countable approximate identity. Then there is a natural isomorphism of abelian groups*

$$\bar{P}_0(A) \cong [q\mathbf{C}, K_0 \tilde{\otimes} A].$$

Here $\bar{P}_0(A)$ is defined as in the statement of Lemma 2.6, *and the group structure on $[q\mathbf{C}, K_0 \tilde{\otimes} A]$ is as in Proposition* 1.14.

Note that in Proposition 1.14 we only proved that $[q\mathbf{C}, K_0 \tilde{\otimes} A]$ is an abelian group if A is unital. However, it will follow from this lemma that this result also holds when A has a countable approximate identity.

The proof requires the following generalization of a lemma from [**20**].

LEMMA 2.11. *Let A be a σ-C^*-algebra with a countable approximate identity. Then the unitary group $UM(K \otimes A)$ is path connected.*

Proof. In the proof of Lemma 1.9 of [**20**], replace all references to Theorem 2.5 of [**15**] by references to [**9**].

Proof of Lemma 2.10. We define

$$\Psi : \bar{P}_0(A) \longrightarrow [q\mathbf{C}, K_0 \tilde{\otimes} A]$$

as follows. Let $p \in M_2((K_0 \tilde{\otimes} A)^+)$ be a projection such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K_0 \tilde{\otimes} A).$$

Write $A = \varprojlim A_n$. Then there is an increasing sequence $n \longmapsto k(n)$ of positive integers such that

$$\kappa_n(p) \in M_2((M_{k(n)} \otimes A_n)^+).$$

Consequently there are homomorphisms

$$\varphi_n : \mathbf{C} * \mathbf{C} \longrightarrow M_2((M_{k(n)} \otimes A_n)^+)$$

such that the images under φ_n of the generating projections e_0 and f_0 of $\mathbf{C} * \mathbf{C}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } p$$

respectively. It follows from the definition of $q\mathbf{C}$ that

$$\varphi_n(q\mathbf{C}) \subset M_2(M_{k(n)} \otimes A_n).$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

It is easily seen that we obtain a homomorphism

$$\varphi : q\mathbf{C} \longrightarrow M_2(K_0 \tilde{\otimes} A)$$

such that $\kappa_n \circ \varphi$ is φ_n followed by the inclusion of $M_2(M_{k(n)} \otimes A_n)$ in $M_2(K_0 \tilde{\otimes} A_n)$, for every n. Chose an isomorphism

$$\eta : M_2(K_0) \longrightarrow K_0$$

as in Lemma 1.13, and define

$$\Psi([p]) = [(\eta \otimes id_A) \circ \varphi].$$

This class clearly does not depend on the choice of the numbers $k(n)$, and, by Lemma 1.13, it also does not depend on η. Because of our definition of homotopy in $K_0 \tilde{\otimes} A$ (see Definition 1.8), it is easily seen that $\Psi([p])$ depends only on the homotopy class of $[p]$. So Ψ is well defined.

We now prove that Ψ is surjective. Let $\varphi_0 : q\mathbf{C} \longrightarrow K_0 \tilde{\otimes} A$ be a homomorphism. Let $\varphi = M_2(\varphi_0^+)$ be the corresponding homomorphism from $M_2(q\mathbf{C}^+)$ to $M_2((K_0 \tilde{\otimes} A)^+)$. Using Proposition 1.16, we can identify $M_2(q\mathbf{C}^+)$ with the algebra of all continuous functions $a : [0, 1] \longrightarrow M_4$ such that, when 4×4 matrices are regarded as 2×2 block matrices with 2×2 matrices as entries, the entries of $a(0)$ are all diagonal and the entries of $a(1)$ are all scalar multiples of the identity. Let $e,f \in M_2(q\mathbf{C}^+)$ be the projections given by

$$e(t) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{and}$$

$$f(t) = \begin{pmatrix} t & 0 & 0 & -\sqrt{t(1-t)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{t(1-t)} & 0 & 0 & 1-t \end{pmatrix}$$

for $t \in [0, 1]$. Then

$$\varphi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $p = \varphi(f)$ is a projection in $M_2((K \tilde{\otimes} A)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p \in M_2(K_0 \tilde{\otimes} A).$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

We claim that $\Psi([p]) = [\varphi_0]$. First, we define $u_s \in UM_2(M(q\mathbf{C}))$, for $s \in [0, \pi/2]$, by

$$u_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(s) & 0 & -\sin(s) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(s) & 0 & \cos(s) \end{pmatrix},$$

regarded as a constant function on $[0, 1]$. Further identify e_0, f_0 with the projections

$$t \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad t \longmapsto \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},$$

as in Proposition 1.16. Next, set

$$e^{(s)} = u_s \left(e_0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) u_s^* \quad \text{and} \quad f^{(s)} = u_s \left(f_0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) u_s^*$$

for $s \in [0, \pi/2]$. Then $e^{(s)}$ and $f^{(s)}$ are projections in $M_2(M(q\mathbf{C}))$ such that $e^{(s)} - f^{(s)} \in M_2(q\mathbf{C})$, and which vary continuously with s. Therefore there is a homotopy $s \longmapsto \gamma_s$ of homomorphisms from $q\mathbf{C}$ to $M_2(q\mathbf{C})$ such that

$$\gamma_s(e_0 - f_0) = e^{(s)} - f^{(s)} \quad \text{and} \quad \gamma_s(e_0(e_0 - f_0)) = e^{(s)}(e^{(s)} - f^{(s)}).$$

(Recall that $q\mathbf{C}$ is generated as a C^*-algebra by $e_0 - f_0$ and $e_0(e_0 - f_0)$.) Now

$$\gamma_0(x) = x \oplus 0$$

while

$$\gamma_{\pi/2}(e_0 - f_0) = e - f \quad \text{and} \quad \gamma_{\pi/2}(e_0(e_0 - f_0)) = e(e - f).$$

If $\psi : q\mathbf{C} \longrightarrow M_2(K_0 \tilde{\otimes} A)$ is the homomorphism determined by p as in the definition of Ψ, so that $\Psi([p]) = [(\eta \otimes id_A) \circ \psi]$, then it follows that $\psi = \varphi \circ \gamma_{\pi/2}$. Therefore $(\eta \otimes id_A) \circ \psi$ is homotopic to $(\eta \otimes id_A) \circ \varphi \circ \gamma_0$, which is given by the formula

$$x \longmapsto (\eta \otimes id_A)(\varphi_0(x) \oplus 0).$$

By Lemma 1.12, this map is homotopic to φ_0. So $\Psi([p]) = [\varphi_0]$, and surjectivity is proved.

Next, we prove that Ψ is injective. Thus, let $p_0, p_1 \in M_2((K_0 \tilde{\otimes} A)^+)$ be projections such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_i \in M_2(K_0 \tilde{\otimes} A),$$

and suppose that $\Psi([p_0]) = \Psi([p_1])$. This means that the corresponding maps

$$\varphi_i : q\mathbf{C} \longrightarrow M_2(K_0 \tilde{\otimes} A)$$

are homotopic, via, say, a homotopy $t \longmapsto \varphi_t$. For simplicity write $B = A \otimes C([0, 1])$. Then we can regard $t \longmapsto \varphi_t$ as a homomorphism

$$\varphi : q\mathbf{C} \longrightarrow M_2(K_0 \tilde{\otimes} B).$$

Just as if B were a C^*-algebra, φ determines a quasihomomorphism

$$(*) \qquad (\alpha, \bar{\alpha}) : \mathbf{C} \rightrightarrows E \triangleright J \longrightarrow M_2(K \otimes B),$$

as in Section 1 of [**8**]. Indeed, $(*)$ is obtained as the inverse limit of the quasi-homomorphisms

$$(\alpha_n, \bar{\alpha}_n) : \mathbf{C} \longrightarrow E_n \triangleright J_n \longrightarrow M_2(K \otimes B_n)$$

determined by $\kappa_n \circ \varphi : q\mathbf{C} \longrightarrow M_2(K \otimes B_n)$. Identify $M_2(K \otimes B)$ with the algebra of compact module morphisms of the right Hilbert B-module $H = l^2(B) \oplus l^2(B)$. Since B has a countable approximate identity and J is separable (being generated as a σ-C^*-algebra by $\alpha(1) - \bar{\alpha}(1)$ and $\alpha(1)(\alpha(1) - \bar{\alpha}(1))$), the Hilbert B-module $\overline{JH}$ is countably generated. The stabilization theorem ([**19**], Theorem 5.12) therefore yields isomorphisms

$$v_1 : l^2(B) \oplus \alpha(1)\overline{JH} \longrightarrow l^2(B) \quad \text{and}$$

$$v_2 : (1 - \alpha(1))\overline{JH} \oplus l^2(B) \longrightarrow l^2(B).$$

(Note that $\alpha(1)$ acts on $\overline{JH}$ because $\alpha(1)$ can be regarded as a multiplier of J. The same holds, of course, for $\bar{\alpha}(1)$.) Then $u = v_1 \oplus v_2$ is an isomorphism

$$u : l^2(B) \oplus \overline{JH} \oplus l^2(B) \longrightarrow l^2(B) \oplus l^2(B)$$

such that

$$u(1 \oplus \alpha(1) \oplus 0)u^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $q = u(1 \oplus \bar{\alpha}(1) \oplus 0)u^*$ satisfies

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q \in M_2(K \otimes B).$$

Thus, q can be regarded as a homotopy $t \longmapsto q_t$ of projections in $M_2((K \otimes A)^+)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q_t \in M_2(K \otimes A).$$

Thus $[q_0] = [q_1]$ in $\bar{P}(A)$.

We now show that $[p_i] = [q_i]$ in $\bar{P}(A)$. It will then follow that $[p_0] = [p_1]$ in $\bar{P}(A)$, and so by Lemma 2.6 also in $\bar{P}_0(A)$. This will complete the proof of the injectivity of Ψ. Let J_i be the image of J under the homomorphism

$$ev_i : M_2(K \otimes B) \longrightarrow M_2(K \otimes A)$$

given by evaluation at i. Thus, J_i is the sub-σ-C^*-algebra of $M_2(K \otimes A)$ generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_i \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - p_i \right].$$

Furthermore,

$$\overline{J_i(l^2(A) \oplus l^2(A))} = (ev_i)_*(\overline{JH}).$$

(See Section 4 of [**19**].) Define

$$E_1 = \left\{ \xi : [0,1] \longrightarrow l^2(A) : \xi \text{ is continuous and } \xi(1) \in \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \overline{J_i(l^2(A) \oplus l^2(A))} \right\}$$

and

$$E_2 = \left\{ \xi : [0,1] \longrightarrow l^2(A) : \xi \text{ is continuous and } \xi(1) \in \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \overline{J_i(l^2(A) \oplus l^2(A))} \right\}.$$

It is easily checked that the definitions of E_1 and E_2 make sense when $l^2(A) \oplus 0$ and $0 \oplus l^2(A)$ are identified with $l^2(A)$ in the obvious way, because

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are multipliers of J_i. Furthermore, E_1 and E_2 are countably generated Hilbert B-modules. Using the stabilization theorem as before, we obtain an isomorphism

$$w : l^2(B) \oplus E_1 \oplus E_2 \oplus l^2(B) \longrightarrow l^2(B) \oplus l^2(B)$$

such that

$$w \left(1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \right) w^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

and $r = w(1 \oplus p_i \oplus 0)w^*$ is a projection satisfying

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - r \in M_2(K \otimes B).$$

(Here we regard

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_i$$

as the operators on $E_1 \oplus E_2$ given by the corresponding constant functions.) Now $(ev_1)_*(w)$ and $(ev_i)_*(u)$ are both isomorphisms of

$$l^2(A) \oplus \overline{J_i(l^2(A) \oplus l^2(A))} \oplus l^2(A)$$

with $l^2(A) \oplus l^2(A)$ which send

$$1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0$$

to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$UL(l^2(A)) \cong UM(K \otimes A)$$

by [**19**], Theorem 4.2(6) and Remark 4.8, it is path connected by Lemma 2.11. Therefore there is a homotopy of such isomorphisms, also sending

$$1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0$$

to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

from $(ev_1)_*(w)$ to $(ev_i)_*(u)$. This yields a homotopy from $r_1 = (ev_1)_*(r)$ to q_i, so that $[r_1] = [q_i]$ in $\bar{P}(A)$. Furthermore $[r_1] = [r_0]$ in $\bar{P}(A)$, using the homotopy r. An argument similar to the one just given shows that

$$[r_0] = [p_i] + \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Since

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

is the identity in $\bar{P}(A)$ by Corollary 2.9, we obtain $[p_i] = [q_i]$. This completes the proof that Ψ is injective.

It remains only to prove that Ψ is a group homomorphism. Of course, addition in $\bar{P}_0(A)$ is defined the same way as in $\bar{P}(A)$, except using an isomorphism $M_2(K_0) \cong K_0$ (from Lemma 1.13) instead of an isomorphism $M_2(K) \cong K$. We may disregard the isomorphism $M_2(K_0) \cong K_0$ used in the definition of $\Psi([p])$, and thus take the codomain of Ψ to be $[q\mathbf{C}, M_2(K_0 \tilde{\otimes} A)]$. Addition in it is given by

$$[\varphi] + [\psi] = [(\eta_0 \otimes id_A) \circ (id_{M_2(K_0)} \otimes \sigma) \circ (\varphi \oplus \psi)],$$

where

$$\sigma(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and where η_0 is an isomorphism of $M_4(K_0)$ with $M_2(K_0)$. If we take

$$\eta_0(x) = M_2(\eta)(uxu^*),$$

where u is as in Definition 2.2 and $\eta : M_2(K_0) \longrightarrow K_0$ is the isomorphism used in the definition of addition in $\bar{P}_0(A)$, then the formula for Ψ defines a homomorphism even before taking homotopy classes. In the case in which A is not unital, one should also check that the identity and inversion are in fact given by the formulas in Proposition 1.14. This is easy and is omitted. The proof is now complete.

THEOREM 2.12. *Let A be a unital σ-C^*-algebra. Then there is a natural isomorphism of abelian groups $[P, A]_1 \cong RK_0(A)$.*

Proof. We have natural isomorphisms of abelian groups

$$[P, A]_1 \cong [W_\infty(q\mathbf{C})^+, A]_1 \cong [q\mathbf{C}, K_0 \tilde{\otimes} A] \cong \bar{P}_0(A) \cong \bar{P}(A) \cong RK_0(A),$$

using, in order, Definition 2.1, Proposition 1.14, Lemma 2.10, Lemma 2.6, and Proposition 2.8.

3. A classifying algebra for RK_1. In this section we prove the analog for RK_1 of the theorem of the last section. We will not use any of the results of the last section, and we will use only a small part of the material in Section 1. (In Section 4, we will come back and explain the connection between our classifying algebra for RK_1 and the results of Section 1.)

Definition 3.1. Let $U_{nc}(n)$ be the "noncommutative unitary group" defined at the end of [**7**]. That is, $U_{nc}(n)$ is the universal unital C^*-algebra on the generators $x_{n,i,j}$ for $1 \leqq i,j \leqq n$, subject to the relations that the $n \times n$ matrix

$$x_n = \begin{pmatrix} x_{n,1,1} & \cdots & x_{n,1,n} \\ \vdots & \ddots & \vdots \\ x_{n,n,1} & \cdots & x_{n,n,n} \end{pmatrix}$$

be a unitary element of $M_n(U_{nc}(n))$. (The subscript nc stands for "noncommutative".) Define a unital homomorphism $\varphi_n : U_{nc}(n+1) \longrightarrow U_{nc}(n)$ by

$$\varphi_n(x_{n+1,i,j}) = \begin{cases} x_{n,i,j} & 1 \leqq i,j \leqq n \\ 1 & i=j=n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then define $U_{nc} = \varprojlim U_{nc}(n)$ with respect to the homomorphisms φ_n. We write $x_{\infty,i,j}$ for the element of U_{nc} determined by the coherent sequence whose value is $x_{n,i,j}$ for $n \geqq \max(i,j)$ and whose value is δ_{ij} for $n < \max(i,j)$. Define unital homormorphisms $\chi : U_{nc} \longrightarrow \mathbf{C}$, $\mu : U_{nc} \longrightarrow U_{nc} *_{\mathbf{C}} U_{nc}$, and $\iota : U_{nc} \longrightarrow U_{nc}$ as follows:

$$\chi(x_{\infty,i,j}) = \delta_{ij}$$

$$\mu(x_{\infty,i,j}) = \sum_{k=1}^{\infty} x^{(1)}_{\infty,i,k} x^{(2)}_{\infty,k,j}$$

$$\iota(x_{\infty,i,j}) = x^*_{\infty,j,i}.$$

In the formula for μ, we have written $x^{(l)}_{\infty,i,j}$ for the generator of the l-th copy of U_{nc} in the free product corresponding to the generator $x_{\infty,i,j}$ of U_{nc}.

PROPOSITION 3.2. *$(U_{nc}, \chi, \mu, \iota)$ is a dual group in the sense of Voiculescu* [**29**].

Proof. One easily checks that the homomorphisms φ_n preserve the standard dual group structure on $U_{nc}(n)$ (see [**29**], 5.6), and that the formulas for χ, μ, and ι define the inverse limits over n of the corresponding homomorphisms for $U_{nc}(n)$.

The goal of this section is to prove that $RK_1(A) \cong [U_{nc}, A]_1$ for unital σ-C^*-algebras A. As in the previous section, the proof will proceed via an intermediate group which is of interest in its own right, given as follows.

Definition 3.3. If A is a unital σ-C^*-algebra, we define

$$\bar{U}(A) = (U/U_0)((K \otimes A)^+).$$

As in the introduction, U stands for the unitary group, U_0 for its path component of the identity, and U/U_0 for the quotient. We will also use this notation for such algebras as $(K_0 \tilde{\otimes} A)^+$, with $K_0 \tilde{\otimes} A$ as in Definition 1.8. By analogy with the definition of homotopy in $K_0 \tilde{\otimes} A$ (see Definition 1.8), we will take $U_0((K_0 \tilde{\otimes} A)^+)$ to be the set of unitaries in $(K_0 \tilde{\otimes} A)^+$ which can be connected to the identity via a unitary path of the form $t \longmapsto a_t + \lambda_t \cdot 1$, where $\lambda_t \in \mathbf{C}$ and $t \longmapsto a_t$ defines an element of $K_0 \tilde{\otimes} (A \otimes C([0, 1]))$. This is, at least apparently, a stronger condition than the existence of some continuous path of unitaries in $(K_0 \tilde{\otimes} A)^+$.

We will also adopt throughout this section the convention that whenever we write $A = \varprojlim A_n$, then the A_n are C^*-algebras, the maps $\kappa_n : A \longrightarrow A_n$ are surjective, the map from A_{n+1} to A_n is called π_n, and maps on multiplier algebras, tensor products, etc. determined by κ_n and π_n are also called κ_n and π_n.

LEMMA 3.4. *Let A be a unital σ-C^*-algebra. Then the obvious homomorphism from $\bar{U}_0(A) = (U/U_0)((K_0 \tilde{\otimes} A)^+)$ to $\bar{U}(A)$ is an isomorphism.*

This lemma asserts that every unitary in $(K \otimes A)^+$ is homotopic to one in $(K_0 \tilde{\otimes} A)^+$, and that two unitaries in $(K_0 \tilde{\otimes} A)^+$ which are homotopic in $(K \otimes A)^+$ are in fact homotopic in $(K_0 \tilde{\otimes} A)^+$. The proof is similar in concept to the proofs of Lemmas 2.5 and 2.6, but technically simpler. We therefore omit most of the details, noting only that extensive use is made of the function $a \longmapsto a(a^*a)^{-\frac{1}{2}}$ from invertible elements to unitaries, and that appropriate paths of unitaries can be lifted directly using the surjectivity of $U_0(B) \longrightarrow U_0(C)$ when $B \longrightarrow C$ is a surjective map of unital σ-C^*-algebras ([**20**], Lemma 1.11).

PROPOSITION 3.5. *There is a natural isomorphism of groups $[U_{nc}, A]_1 \cong \bar{U}(A)$ for unital σ-C^*-algebras A.*

Proof. We will actually establish a natural isomorphism of $\mathrm{Hom}_1(U_{nc}, A)$ with the group

$$G(A) = \{u \in U((K_0 \tilde{\otimes} A)^+) : 1 - u \in K_0 \tilde{\otimes} A\}.$$

Here, the group structure on $\mathrm{Hom}_1(U_{nc}, A)$ comes from the dual group structure on U_{nc}, as in [**29**], 2.3. Applying this result with $A \otimes C([0, 1])$ in place of A will then give a natural isomorphism

$$[U_{nc}, A]_1 \cong G(A)/G_0(A),$$

where $G_0(A)$ is the path component of the identity in $G(A)$. Since every element of $U((K_0 \tilde{\otimes} A)^+)$ can be written uniquely as ζu, with $\zeta \in S^1$ and $u \in G(A)$, it follows easily that

$$G(A)/G_0(A) \cong (U/U_0)((K \tilde{\otimes} A)^+).$$

By the previous lemma, the right hand side is $\bar{U}(A)$.

Given a unitary $u \in G(A)$, with $A = \varprojlim A_n$, we can write

$$\kappa_n(u) = 1 - e_n + u_n,$$

where $u_n \in U(M_{k(n)} \otimes A_n), e_n$ is the identity of $M_{k(n)} \otimes A_n$, and $n \longmapsto k(n)$ is an increasing sequence of integers. We can then define $\varphi_n : U_{nc} \longrightarrow A_n$ to be the unital homomorphism such that $\varphi_n(x_{\infty,i,j}) = (u_n)_{i,j}$ for $i, j \leqq k(n)$ and $\varphi_n(x_{\infty,i,j}) = \delta_{ij}$ otherwise. Note that φ_n is a continuous homormorphism because it factors through $U_{nc}(k(n))$. Also notice that it does not depend on the choice of $k(n)$, as long as $\kappa_n(u) \in (M_{k(n)} \otimes A_n)^+$ and satisfies $1 - \kappa_n(u) \in M_{k(n)} \otimes A_n$. Clearly the φ_n define a coherent sequence of unital homomorphisms from U_{nc} to A_n, and therefore a unital homomorphism $\varphi : U_{nc} \longrightarrow A$. It is also clear that the assignment $u \longmapsto \varphi$ is bijective. (A homomorphism φ can only correspond to an element of $G(A)$ because $\kappa_n \circ \varphi$ must factor through some $U_{nc}(k(n))$, so that the corresponding unitary u satisfies $1 - \kappa_n(u) \in M_{k(n)} \otimes A_n$.) It is easy to prove that the assignment $u \longmapsto \varphi$ is a group homomorphism.

We now want to relate $\bar{U}(A)$ to $RK_1(A)$. The first step is to show that $\bar{U}$ is part of a generalized cohomology theory on σ-C^*-algebras.

Lemma 3.6. *Extend the definition of $\bar{U}$ to general σ-C^*-algebras by defining $\bar{U}(A)$ to be the kernel of the obvious map $\bar{U}(A^+) \longrightarrow \bar{U}(\mathbf{C})$ for a not necessarily unital σ-C^*-algebra A. Then $\bar{U}$ becomes a well defined homotopy invariant functor from σ-C^*-algebras to abelian groups such that if*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is an exact sequence of σ-C^*-algebras, then there is a functorial long exact sequence

$$\cdots \longrightarrow \bar{U}(SI) \longrightarrow \bar{U}(SA) \longrightarrow \bar{U}(SB) \longrightarrow \bar{U}(I) \longrightarrow \bar{U}(A) \longrightarrow \bar{U}(B),$$

where SA is the suspension $C_0(\mathbf{R}) \otimes A$ for any A.

The proof is similar to, but simpler then, the proofs of the corresponding statements for RK_0 in Propositions 2.2 and 2.4 and Corollary 2.5 of [**20**]. We omit the details.

Lemma 3.7. *If A is a nonunital σ-C^*-algebra, then the obvious map*

$$(U/U_0)((K \otimes A)^+) \longrightarrow \bar{U}(A)$$

is an isomorphism.

Proof. This proof uses standard K-theory techniques. For surjectivity, one needs to use the fact that if $A \longrightarrow B$ is surjective and A and B are unital, then

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

$U_0(A) \longrightarrow U_0(B)$ is surjective ([**20**], Lemma 1.11), and for injectivity one needs the fact that $U(K^+)$ is path connected. We omit further details.

COROLLARY 3.8. *For any σ-C^*-algebra A there is a natural isomorphism $\bar{U}(K \otimes A) \cong \bar{U}(A)$.*

Proof. There is an isomorphism $K \otimes K \cong K$, which is unique up to homotopy.

LEMMA 3.9. *For any unital C^*-algebra B and any unital σ-C^*-algebra A, the group $U(B \otimes_{\min} M(K \otimes A))$ is path-connected.*

Proof. The proof is the same as the proof of Lemma 1.9 of [**20**], replacing everywhere $M(K \otimes A)$ by $B \otimes M(K \otimes A)$ and $M(K \otimes A_n)$ by $B \otimes M(K \otimes A_n)$. The references to Theorem 2.5 of [**15**] and its corollary remain valid in this more general situation.

COROLLARY 3.10. *For any C^*-algebra B and any unital σ-C^*-algebra A, we have*

$$\bar{U}(B \otimes_{\min} M(K \otimes A)) = 0.$$

Proof. Consider the split exact sequence

$$0 \longrightarrow K \otimes B \otimes_{\min} M(K \otimes A) \longrightarrow (K \otimes B)^+ \otimes_{\min} M(K \otimes A) \underset{\pi}{\overset{\iota}{\rightleftarrows}} M(K \otimes A) \longrightarrow 0.$$

If $u \in U((K \otimes B \otimes_{\min} M(K \otimes A))^+)$, then by the lemma there is a path $t \longmapsto u_t$ in $U((K \otimes B)^+ \otimes_{\min} M(K \otimes A))$ which connects u to 1. Then

$$t \longmapsto v_t = u_t(\iota \circ \pi)(u_t^*)$$

is a continuous path in $U([K \otimes B \otimes_{\min} M(K \otimes A)]^+)$ connecting $v_0 = u(\iota \circ \pi)(u^*)$ to 1. Since

$$u \in (K \otimes B \otimes_{\min} M(K \otimes A))^+,$$

the element $(\iota \circ \pi)(u^*)$ is a scalar multiple of 1, and thus can be trivially connected to 1. So

$$u \in U_0([K \otimes B \otimes_{\min} M(K \otimes A)]^+).$$

In view of Lemma 3.7, we have shown that

$$\bar{U}(K \otimes B \otimes_{\min} M(K \otimes A)) = 0,$$

as desired.

COROLLARY 3.11. *For any unital σ-C^*-algebra A there is a natural isomorphism $\bar{U}(SQ(A)) \cong \bar{U}(A)$, where $Q(A) = M(K \otimes A)/(K \otimes A)$.*

Proof. Apply Lemma 3.6 to the natural short exact sequence

$$0 \longrightarrow K \otimes A \longrightarrow M(K \otimes A) \longrightarrow Q(A) \longrightarrow 0$$

to obtain the natural exact sequence

$$\bar{U}(SM(K \otimes A)) \longrightarrow \bar{U}(SQ(A)) \longrightarrow \bar{U}(K \otimes A) \longrightarrow \bar{U}(M(K \otimes A)).$$

According to the previous corollary, the two end terms are zero, and according to Lemma 3.8, we have $\bar{U}(K \otimes A) \cong \bar{U}(A)$ naturally.

To complete the proof that $[U_{nc}, A]_1 \cong RK_1(A)$, we now have to show that $\bar{U}(SQ(A)) \cong RK_1(A)$, that is, that $(U/U_0)([K \otimes SQ(A)]^+)$ is isomorphic to the kernel of the map

$$(U/U_0)(Q((SA)^+)) \longrightarrow (U/U_0)(Q(\mathbf{C})).$$

This requires three more lemmas. The first generalizes Lemma 1.12 of [**20**] and its corollary, in the same direction in which Lemma 3.9 generalizes Lemma 1.9 of [**20**]. If $\varphi : A \longrightarrow B$ is a unital homomorphism of σ-C^*-algebra, then

$$Q\varphi : Q(A) \longrightarrow Q(B)$$

will denote the homomorphism provided by Lemma 1.6 of [**20**], and, for any σ-C^*-algebra A,

$$ev_x : C(X, A) \longrightarrow A$$

will be evaluation at $x \in X$.

LEMMA 3.12. *Let B be a unital nuclear C^*-algebra, and let A be a unital σ-C^*-algebra. If $u \in U(B \otimes QC([0,1], A))$, then*

$$(id_B \otimes Qev_0)(u) \cdot [(id_B \otimes Qev_1)(u)]^* \in U_0(B \otimes Q(A)).$$

Proof. We can clearly assume that $(id_B \otimes Qev_1)(u) = 1$. Then the proof is obtained from the proof of Lemma 1.12 of [**20**] by replacing $Q(\cdot)$ by $B \otimes Q(\cdot)$ and $M(\cdot)$ by $B \otimes M(\cdot)$ throughout. Of course, we use our Lemma 3.9 in place of Lemma 1.9 of [**20**]. The only other point that requires comment is in the second paragraph of the proof, where we now need to know that

$$(*) \qquad (U/U_0)(B \otimes C([0,1], Q(A_n))) \longrightarrow (U/U_0)(B \otimes QC([0,1], A_n))$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

is an isomorphism. The lemma following Proposition 1.13 of [**15**] no longer applies directly, but does give an isomorphism

$$(**)\qquad (U/U_0)([R \otimes_{\min} M(K \otimes S)]/[R \otimes_{\min} (K \otimes S)]) \to (U/U_0)(Q(R \otimes_{\min} S)),$$

for any unital C^*-algebras R and S. If R is nuclear, then the algebra on the left hand side is $R \otimes Q(S)$. Using $(**)$ with $R = B \otimes C([0, 1])$ and $S = A_n$, and again with $R = B$ and $S = C([0, 1], A_n)$, we obtain $(*)$.

LEMMA 3.13. *Let B be a unital nuclear C^*-algebra, and let A be a unital σ-C^*-algebra. Then there is a natural isomorphism*

$$(U/U_0)(B \otimes Q(A)) \cong (U/U_0)([K_0 \tilde{\otimes} (B \otimes Q(A))]^+).$$

Proof. We take $K = K(H)$, where H is a separable infinite dimensional Hilbert space. Fix a countable family $u_1, u_2, \ldots$ of isometries in $L(H)$ with pairwise orthogonal ranges which span H. We now define homomorphisms

$$\Phi : (U/U_0)(B \otimes Q(A)) \longrightarrow (U/U_0)([K_0 \tilde{\otimes} (B \otimes Q(A))]^+)$$

and

$$\Psi : (U/U_0)([K_0 \tilde{\otimes} (B \otimes Q(A))]^+) \longrightarrow (U/U_0)(B \otimes Q(A)).$$

Let (e_{ij}) be a standard set of matrix units in K_0, so that K_0 is their linear span. Then every element x of $K_0 \tilde{\otimes} (B \otimes Q(A))$ can be written as

$$x = \sum_{i,j=1}^{\infty} e_{ij} \otimes x_{ij},$$

where $x_{ij} \in B \otimes Q(A)$ and this sum is finite in each continuous C^*-seminorm on $B \otimes Q(A)$. Now define

$$\Phi([v]) = [e_{11} \otimes v + (1 - e_{11}) \otimes 1]$$

and, for $w = x + \lambda \cdot 1 \in U([K_0 \tilde{\otimes} (B \otimes Q(A))]^+)$ with $x \in K_0 \tilde{\otimes} (B \otimes Q(A))$, define

$$\Psi([w]) = \left[\lambda + \sum_{i,j=1}^{\infty} u_i x_{ij} u_j^*\right].$$

The sum in the definition of Ψ is finite in each continuous C^*-seminorm on $B \otimes Q(A)$, and is easily seen to define a unitary in $B \otimes Q(A)$. The elements u_i are regarded as being in $B \otimes Q(A)$ via the composite

$$L(H) \longrightarrow B \otimes L(H) \otimes A \longrightarrow B \otimes M(K \otimes A) \longrightarrow B \otimes Q(A).$$

For each n choose a $*$-strong operator continuous path $t \longmapsto u_n^{(t)}$ of isometries in $L(H)$ such that $u_n^{(0)} = 1$ and $u_n^{(1)} = u_n$. This path defines an element in $M(K \otimes C([0,1]))$ and hence an element $\bar{u}_n \in B \otimes QC([0,1], A)$. For $v \in U(B \otimes Q(A))$, we now observe that

$$\Psi \circ \Phi([v]) = [u_1 v u_1^* + 1 - u_1 u_1^*].$$

Then

$$\bar{v} = \bar{u}_1 (v \otimes 1_{C([0,1])}) \bar{u}_1^* + 1 - \bar{u}_1 \bar{u}_1^*$$

is an element of $U(B \otimes QC([0,1], A))$ such that

$$(id_B \otimes Qev_0)(\bar{v}) = v \quad \text{and} \quad (id_B \otimes Qev_1)(\bar{v}) = u_1 v u_1^* + 1 - u_1 u_1^*.$$

It follows from the previous lemma that $\Psi \circ \Phi([v]) = [v]$.

We now consider the composite in the other order. Let

$$w \in U([K_0 \tilde{\otimes} (B \otimes Q(A))]^+).$$

Writing elements of $[K_0 \tilde{\otimes} (B \otimes Q(A))]^+$ as matrices with entries in $B \otimes Q(A)$, we have

$$w = \begin{pmatrix} w_{11} & w_{12} & \cdots \\ w_{21} & w_{22} & \\ \vdots & & \ddots \end{pmatrix},$$

and $\Phi \circ \Psi([w])$ is the class of

$$x = \begin{pmatrix} \sum_{i,j=1}^{\infty} u_i w_{ij} u_j^* & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now define x_{s+n}, for $n \in \mathbf{N} \cup \{0\}$ and $s \in (0,1]$, inductively by $x_0 = x$ and

$$\begin{aligned} x_{s+n} = [1 \otimes p_{n+1} + (I_n \oplus c_s \oplus 1) \otimes (1 - p_{n+1})] \\ \cdot x_n [1 \otimes p_{n+1} + (I_n \oplus c_s \oplus 1) \otimes (1 - p_{n+1})]^*, \end{aligned}$$

where $p_{n+1} = \sum_{i=1}^{n+1} u_i u_i^*$, where $s \longmapsto c_s$ is a continuous path of unitaries in M_2 with

$$c_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and where I_n is an $n \times n$ identity matrix. Then one checks that

$$x_n =$$

$$\begin{pmatrix} u_1w_{11}u_1^* + 1 - u_1u_1^* & u_1w_{12}u_2^* & \dots & \sum_{j=n+1}^{\infty} u_1w_{1j}u_j^* & 0 & \dots \\ u_2w_{21}u_1^* & u_2w_{22}u_2^* + 1 - u_2u_2^* & & \vdots & & \\ \vdots & & \ddots & \vdots & & \\ \sum_{i=n+1}^{\infty} u_iw_{i1}u_1^* & \dots & \dots & p_n + \sum_{i,j=n+1}^{\infty} u_iw_{ij}u_j^* & 0 & \dots \\ \vdots & & & \vdots & & \ddots \end{pmatrix}$$

and that the upper left $(n-1) \times (n-1)$ block of x_t agrees with that of x_n for $t \geqq n$. If now p is any continuous C^*-seminorm on A, then there is k such that $p(w_{ij}) = 0$ unless $i, j \leqq k$. With $A_p = A/\mathrm{Ker}(p)$, the image of $t \longmapsto x_t$ in $[K_0 \tilde{\otimes} (B \otimes Q(A_p))]^+$ actually lies in $(M_k \otimes B \otimes Q(A_p))^+$, and furthermore has a limit as $t \longmapsto \infty$. Therefore $t \longmapsto x_t$ defines a homotopy in $U([K_0 \tilde{\otimes} (B \otimes Q(A))]^+)$ from x to

$$x_\infty =$$

$$\begin{pmatrix} u_1w_{11}u_1^* + 1 - u_1u_1^* & u_1w_{12}u_2^* & u_1w_{13}u_3^* & \dots \\ u_2w_{21}u_1^* & u_2w_{22}u_2^* + 1 - u_2u_2^* & u_2w_{23}u_3^* & \\ u_3w_{31}u_1^* & u_3w_{32}u_2^* & u_3w_{33}u_3^* + 1 - u_3u_3^* & \\ \vdots & & & \ddots \end{pmatrix}.$$

So $\Phi \circ \Psi([w]) = [x_\infty]$.

Now define $\bar{x} \in U([K \otimes B \otimes QC([0,1], A)]^+)$ by replacing in the formula for x_∞ each u_n by $\bar{u}_n$ and each w_{ij} by $w_{ij} \otimes 1_{C([0,1])}$. Then

$$(id_{(K\otimes B)^+} \otimes Qev_0)(\bar{x}) = w \quad \text{and} \quad (id_{(K\otimes B)^+} \otimes Qev_1)(\bar{x}) = x_\infty.$$

By the previous lemma, w and x_∞ therefore define the same class in $(U/U_0)((K \otimes B)^+ \otimes Q(A))$. To see that they define the same class in $(U/U_0)([K \otimes B \otimes Q(A)]^+)$, let $t \longmapsto z_t$ be a homotopy from w to x_∞. Let

$$\iota : (K \otimes B)^+ \otimes Q(A) \longrightarrow (K \otimes B)^+ \otimes Q(A)$$

be the homomorphism coming from the map $b + \lambda \cdot 1 \longmapsto \lambda \cdot 1$ from $(K \otimes B)^+$ to $(K \otimes B)^+$. Then $t \longmapsto z_t\iota(z_t^*)$ is a homotopy in $U([K \otimes B \otimes Q(A)]^+)$ from a scalar multiple of w to a scalar multiple of x_∞. So $[w] = [x_\infty]$ in $(U/U_0)([K \otimes B \otimes Q(A)]^+)$, and so also in $(U/U_0)([K_0 \tilde{\otimes} (B \otimes Q(A))]^+)$ by Lemma 3.4. Thus $\Phi \circ \Psi([w]) = [w]$.

LEMMA 3.14. *Let B be a unital nuclear C^*-algebra and let A be a unital σ-C^*-algebra. Then there is a natural isomorphism*

$$\Phi : (U/U_0)(B \otimes Q(A)) \longrightarrow (U/U_0)(Q(B \otimes A)).$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

This lemma generalizes the lemma following Proposition 1.13 in [**15**], except for the nuclearity hypothesis (which is probably unnecessary). Of course, the map comes from the map $B\otimes Q(A) \longrightarrow Q(B\otimes A)$ that one gets from $B\otimes M(K\otimes A) \longrightarrow M(K\otimes B\otimes A)$.

Proof of Lemma 3.14. Consider the following commutative diagram with exact rows:

$$(*)\qquad \begin{array}{ccccccccc} 0 \longrightarrow & K\otimes B\otimes A & \longrightarrow & B\otimes M(K\otimes A) & \longrightarrow & B\otimes Q(A) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & K\otimes B\otimes A & \longrightarrow & M(K\otimes B\otimes A) & \xrightarrow{\pi} & Q(B\otimes A) & \longrightarrow 0 \end{array}$$

(Note that the top row is exact if A is a C^*-algebra, because B is nuclear. To show that it is exact in the case at hand, we note that Proposition 5.3(2) of [**19**] can be applied, because of the surjectivity statements in Lemmas 1.5 and 1.6 of [**20**].) The left hand vertical map in $(*)$ is the identity, and the middle vertical map is injective. Therefore the right hand vertical map is injective. Consequently we can identify $B\otimes Q(A)$ with a subalgebra of $Q(B\otimes A)$.

We now prove that Φ is injective. This is the same as showing that

$$U(B\otimes Q(A))\cap U_0Q(B\otimes A)\subset U_0(B\otimes Q(A)).$$

Let u be an element of the left hand side. By [**20**], Lemma 1.11, there is $x\in UM(K\otimes B\otimes A)$ such that $\pi(x)=u$. But then we must actually have

$$x\in U(B\otimes M(K\otimes A)),$$

since $u\in B\otimes Q(A)$. Now $U(B\otimes M(K\otimes A))$ is path connected by Lemma 3.9, so it follows that

$$u=\pi(x)\in U_0(B\otimes Q(A)),$$

as desired.

Before proving that Φ is surjective, we establish the following claim: If B and A are unital C^*-algebras, with B nuclear, and if

$$D=\{a:[0,1]\longrightarrow Q(B\otimes A) : a \text{ is continuous},$$
$$a(0)\in \mathbf{C}\cdot 1, \quad \text{and} \quad a(1)\in B\otimes Q(A)\},$$

then $U(D)$ is connected. To prove it, let $u\in U(D)$. Without loss of generality, we can assume $u(0)=1$. We want to find a continuous function

$$v:[0,1]^2\longrightarrow UQ(B\otimes A)$$

such that $v(t,0) = u(t), v(t,1) = 1, v(0,s) \in \mathbf{C} \cdot 1$, and $v(1,s) \in B \otimes Q(A)$, for all $t, s \in [0,1]$. Begin by choosing a continuous function

$$z : [0,1]^2 \longrightarrow UQ(B \otimes A)$$

such that $z(t,0) = u(t)$ and $z(t,1) = 1$ for all t. Replacing $z(t,s)$ by $z(0,s)^* z(t,s)$, we may also assume that $z(0,s) = 1$. Using the fact that

$$(U/U_0)(B \otimes Q(A)) \longrightarrow (U/U_0)(Q(B \otimes A))$$

is an isomorphism (by the lemma in **[15]** we are generalizing), we can write

$$z(1,s) = x(s)y(s)$$

with

$$x \in U_0 C([0,1], Q(B \otimes A)) \quad \text{and} \quad y \in UC([0,1], B \otimes Q(A)).$$

Furthermore, we will have $x(0) = u(1)y(0)^*$ and $x(1) = y(1)^*$ in the same component of $U(B \otimes Q(A))$. Replacing $x(s)$ by $x(s)c(s)$ and $y(s)$ by $c(s)^* y(s)$ for an appropriate $c \in UC([0,1], B \otimes Q(A))$, we may also assume that $x(0) = x(1) = 1$.

As in the proof of Lemma 3.12, the lemma we are generalizing also implies that

$$(U/U_0)(C(S^1) \otimes B \otimes Q(A)) \longrightarrow (U/U_0)(C(S^1) \otimes Q(B \otimes A))$$

is an isomorphism. Therefore there is a factorization $x(s) = w(1,s)d(s)$, where

$$w : [0,1]^2 \longrightarrow UQ(B \otimes A)$$

satisfies $w(t,0) = w(t,1)$ and $w(0,s) = 0$ (so $w(1,\cdot) \in U_0(C(S^1) \otimes Q(B \otimes A))$), and where $d(s) \in U(B \otimes Q(A))$ for all s. Replacing $w(t,s)$ by $w(t,s)w(t,0)^*$ and $d(s)$ by $w(1,0)d(s)$, we may assume in addition that $w(t,0) = w(t,1) = 1$ for all t. (Note that $w(1,0) = x(1)d(1)^* \in B \otimes Q(A)$.) Now define

$$v(t,s) = w(t,s)^* z(t,s).$$

The only property required of v that we must check is that $v(1,s) \in B \otimes Q(A)$. However, we have

$$v(1,s) = (d(s)x(s)^*)(x(s)y(s)),$$

and $d(s), y(s) \in B \otimes Q(A)$. This proves the claim

Now we prove that Φ is surjective. We return to the case in which A is a σ-C^*-algebra $\varprojlim A_n$. Let $u \in UQ(B \otimes A)$. We construct by induction continuous paths $t \longmapsto u_n^{(t)}$ in $UQ(B \otimes A_n)$, defined for $t \in [0, \infty]$ and satisfying

$$u_n^{(t)} = u_n^{(n)} \in U(B \otimes Q(A_n)) \quad \text{for } t \geqq n \text{ and } \pi_n(u_{n+1}^{(t)}) = u_n^{(t)} \quad \text{for all } t.$$

To start, use the isomorphism

$$(U/U_0)(B \otimes Q(A_1)) \cong (U/U_0)(Q(B \otimes A_n))$$

from the lemma following Proposition 1.13 in **[15]** to choose $u_1^{(t)}$ in $UQ(B \otimes A_1)$ such that

$$u_1^{(0)} = \kappa_1(u) \quad \text{and} \quad u_1^{(1)} \in B \otimes Q(A_1).$$

Then set $u_1^{(t)} = u_1^{(1)}$ for $t \geqq 1$. Given $u_n^{(t)}$, we use the surjectivity of

$$Q(B \otimes A_{n+1}) \longrightarrow Q(B \otimes A_n)$$

(**[20]**, Lemma 1.6) and the corresponding surjectivity of the components of the identity of the unitary groups to lift the path

$$t \longmapsto (u_n^{(0)})^* u_n^{(t)}$$

to a path $t \longmapsto c_t$ in $UQ(B \otimes A_{n+1})$, and we define

$$u_{n+1}^{(t)} = \kappa_{n+1}(u) c_0^* c_t \quad \text{for } t \in [0, n].$$

Next, use the argument above to choose a path $t \longmapsto v_t$ from $v_0 = u_{n+1}^{(n)}$ to some $v_1 \in U(B \otimes Q(A_{n+1}))$. Then

$$t \longmapsto \pi_n(v_t)(u_n^{(n)})^*$$

is a path in $UQ(B \otimes A_n)$ which is 1 at $t = 0$ and is in $U(B \otimes Q(A_n))$ at $t = 1$. Let D_l, for $l = n, n+1$, be the result of putting A_l for A in the definition of the algebra used in the claim above. Then $D_{n+1} \longrightarrow D_n$ is surjective. By the claim, there is therefore a unitary path $t \longmapsto w_t$ defining an element of D_{n+1} whose image in D_n is

$$t \longmapsto \pi_n(v_t)(u_n^{(n)})^*.$$

Now set

$$u_{n+1}^{(n+t)} = w_0 w_t^* v_t \quad \text{for } t \in [0, 1].$$

This gives the correct value for $u_{n+1}^{(n)}$, satisfies

$$\pi_n(u_{n+1}^{(n+t)}) = 1 \cdot (\pi_n(v_t)(u_n^{(n)})^*)^*\pi_n(v_t) = u_n^{(n)}$$

as required, and gives

$$u_{n+1}^{(n+1)} = w_0 w_1^* v_1 \in U(B \otimes Q(A_{n+1}))$$

by the definition of D_{n+1}. So we can set

$$u_{n+1}^{(t)} = u_{n+1}^{(n+1)} \quad \text{for } t \geqq n+1,$$

thus completing the induction.

We now set $u^{(t)}$ equal to the element of $UQ(B \otimes A)$ determined by the coherent sequence $(u_n^{(t)})$. Then

$$u = u(u^{(\infty)})^* \cdot u^{(\infty)},$$

where $u(u^{(\infty)})^* \in U_0 Q(B \otimes A)$ and $u^{(\infty)} \in U(B \otimes Q(A))$. So Φ is surjective. This completes the proof.

THEOREM 3.15. *There is a natural isomorphism*

$$(U/U_0)((K \otimes A)^+) \cong RK_1(A)$$

for any σ-C^*-algebra A.

Proof. We first consider the case in which A is unital. Then we have the following chain of natural isomorphisms, each of which follows easily from previous results as listed below:

$$\begin{aligned} RK_1(A) = RK_0(SA) &\cong \operatorname{Ker}[RK_0(C(S^1) \otimes A) \to RK_0(A)] \\ &= \operatorname{Ker}[(U/U_0)Q(C(S^1) \otimes A) \to (U/U_0)Q(A)] \\ &\cong \operatorname{Ker}[(U/U_0)(C(S^1) \otimes Q(A)) \to (U/U_0)Q(A)] \\ &\cong \operatorname{Ker}[(U/U_0)([K_0 \tilde{\otimes} (C(S^1) \otimes Q(A))]^+) \\ &\to (U/U_0)([K_0 \tilde{\otimes} Q(A)]^+)] \\ &\cong \operatorname{Ker}[\bar{U}(C(S^1) \otimes Q(A)) \to \bar{U}(Q(A))] \\ &\cong \bar{U}(SQ(A)) \cong \bar{U}(A) = (U/U_0)((K \otimes A)^+). \end{aligned}$$

On the first line, the equality is the definition of RK_1 and the isomorphism is obtained by applying the long exact sequence for RK_* to the split exact sequence

$$(*) \qquad 0 \to SA \to C(S^1) \otimes A \to A \to 0.$$

The equality which comes next is the definition of RK_0. The three isomorphisms connecting the second line to the second line from the end are, in order, Lemmas 3.14, 3.13, and 3.4. The isomorphism between the second last line and the last line is obtained by applying Lemma 3.6 to the sequence $(*)$, the isomorphism $\bar{U}(SQ(A)) \cong \bar{U}(A)$ is Corollary 3.11, and the equality at the end is the definition of $\bar{U}(A)$.

Now we consider the case in which A is not unital. Naturality in the unital case, and the results $RK_1(A^+) \cong RK_1(A)$ and $\bar{U}(A^+) \cong \bar{U}(A)$, give $RK_1(A) \cong \bar{U}(A)$, and $\bar{U}(A) \cong (U/U_0)((K \otimes A)^+)$ is Lemma 3.7.

THEOREM 3.16. *Let A be a unital σ-C^*-algebra. Then there is a natural isomorphism of abelian groups $[U_{nc}, A]_1 \cong RK_1(A)$.*

Proof. Combine the previous theorem with Proposition 3.5.

4. The K-theory of the classifying algebras. In this section we compute $RK_*(P)$ and $RK_*(U_{nc})$, where P and U_{nc} are the classifying algebras of the previous two sections. In order to be able to handle them both in the same way, we prove, as a preliminary result, that U_{nc} is isomorphic to $W_\infty(S)^+$ for an appropriate algebra S. Our K-theory results will enable us to show that the algebras of continuous functions on the traditional classifying spaces for K-theory do not work in the noncommutative setting, even for the K-theory of C^*-algebras.

LEMMA 4.1. *Let $S = \{f \in C(S^1) : f(1) = 0\}$, where S^1 is identified with $\{\zeta \in \mathbf{C} : |\zeta| = 1\}$. Then there are isomorphisms $U_{nc}(n) \cong W_n(S)^+$ which respect the maps of the corresponding inverse systems.*

Proof. First observe that S is the universal C^*-algebra on the single generator s subject to the relations $ss^* = s^*s = -s - s^*$. (The element $s + 1$ is unitary in S^+; conversely, if u is a unitary element of any unital C^*-algebra, then $s = u - 1$ satisfies the given relations.) Therefore $W_n(S)^+$ is the universal C^*-algebra on the generators $x_n(s, i, j)$ and 1, subject to the relations that 1 be an identity and that the matrices

$$x_n(s) = (x_n(s, i, j))_{i,j=1}^n$$

satisfy

$$x_n(s)x_n(s)^* = x_n(s)^*x_n(s) = -x_n(s) - x_n(s)^*.$$

The equations $x_{n,i,j} = x_n(s, i, j) + \delta_{ij}1$ relate these generators and relations to the standard generators and relations for $U_{nc}(n)$ in Definition 3.1. Naturality with respect to the maps of the inverse systems is immediate.

PROPOSITION 4.2. *Let $(W_\infty(S)^+, \chi, \mu, \iota)$ be the homotopy dual group obtained via Definition* 1.10 *from the algebra S of the previous lemma, with $\iota_0 : S \longrightarrow$*

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

S given by $\iota_0(f)(\zeta) = f(\bar{\zeta})$. Let $(U_{nc}, \chi', \mu', \iota')$ be the dual group of Definition 3.1. Then there exists an isomorphism $\varphi : W_\infty(S)^+ \longrightarrow U_{nc}$ such that $\chi' \circ \varphi = \chi$, $\mu' \circ \varphi \simeq (\varphi * \varphi) \circ \mu$, and $\iota' \circ \varphi = \varphi \circ \iota$.

This proposition implies that $W_\infty(S)^+$ and U_{nc} are homotopy equivalent as homotopy dual groups. It follows that, for unital σ-C^*-algebras A, we have a natural isomorphism

$$RK_1(A) \cong [W_\infty(S)^+, A]_1,$$

so that we have unified the construction of the classifying algebras for RK_0 and RK_1. Note in particular that this proposition implies that U_{nc} is homotopy abelian.

Proof of Proposition 4.2. We first check that ι_0 satisfies the conditions of Definition 1.10. We certainly have $\iota_0 \circ \iota_0 = id_S$. To construct a homotopy from

$$a \longmapsto \begin{pmatrix} \iota_0(a) & 0 \\ 0 & a \end{pmatrix}$$

to the zero homomorphism, let $s(\zeta) = \zeta - 1$ be the generator of S and choose a continuous path $t \longmapsto u_t$ of scalar unitary matrices such that

$$u_0 = 1 \quad \text{and} \quad u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then let $\psi_t : S \longrightarrow M_2(S)$ be the homomorphism sending s to

$$\begin{pmatrix} \iota_0(s) + 1 & 0 \\ 0 & 1 \end{pmatrix} u_t \begin{pmatrix} 1 & 0 \\ 0 & s+1 \end{pmatrix} u_t^* - 1.$$

We now let $\varphi : W_\infty(S)^+ \longrightarrow U_{nc}$ be the inverse limit of the isomorphisms of the previous lemma, taking the generator s of S as above. Obviously φ is an isomorphism satisfying $\chi' \circ \varphi = \chi$. To check that $\iota' \circ \varphi = \varphi \circ \iota$, we let $\bar{\varphi} : S \longrightarrow K_0 \tilde{\otimes} U_{nc}$ be the homomorphism corresponding to φ under Proposition 1.9. Then the homomorphisms corresponding to $\iota' \circ \varphi$ and $\varphi \circ \iota$ are $(id_{K_0} \otimes \iota') \circ \bar{\varphi}$ and $\bar{\varphi} \circ \iota_0$ respectively. These are equal because both send s to $\bar{\varphi}(s^*)$.

Finally, we prove that $\mu' \circ \varphi \simeq (\varphi * \varphi) \circ \mu$. This is the same as proving that

$$\mu' \simeq (\varphi * \varphi) \circ \mu \circ \varphi^{-1}.$$

If x_∞ is the matrix over U_{nc} whose ij entry is $x_{\infty,i,j}$, and if $x_\infty^{(1)}$ and $x_\infty^{(2)}$ are the analogous matrices over $U_{nc} *_{\mathbf{C}} U_{nc}$ using the generators of the first and second free factors respectively, then $\mu'(x_{\infty,i,j})$ is the ij entry of the matrix product

$x_\infty^{(1)}x_\infty^{(2)}$. Also it is easily shown that $(\varphi * \varphi) \circ \mu \circ \varphi^{-1}(x_{\infty,i,j})$ is the ij entry of the matrix product $y^{(1)}y^{(2)}$, where

$$y^{(1)} = \left(\begin{array}{cc|cc|c} x^{(1)}_{\infty,1,1} & 0 & x^{(1)}_{\infty,1,2} & 0 & \\ 0 & 1 & 0 & 0 & \cdots \\ \hline x^{(1)}_{\infty,2,1} & 0 & x^{(1)}_{\infty,2,2} & 0 & \\ 0 & 0 & 0 & 1 & \cdots \\ \hline & \vdots & & \vdots & \ddots \end{array}\right)$$

and

$$y^{(2)} = \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & \\ 0 & x^{(2)}_{\infty,1,1} & 0 & x^{(2)}_{\infty,1,2} & \cdots \\ \hline 0 & 0 & 1 & 0 & \\ 0 & x^{(2)}_{\infty,2,1} & 0 & x^{(2)}_{\infty,2,2} & \cdots \\ \hline & \vdots & & \vdots & \ddots \end{array}\right).$$

Thus, if $t \longmapsto \sigma_t^{(l)}$ is a homotopy connecting $id_{U_{nc}}$ to a homomorphism $\sigma_1^{(l)}$ sending $x_{\infty,i,j}$ to the ij entry of $y^{(l)}$, for $l = 1, 2$, then

$$t \longmapsto (\sigma_t^{(1)} * \sigma_t^{(2)}) \circ \mu'$$

is a homotopy connecting μ' to $(\varphi * \varphi) \circ \mu \circ \varphi^{-1}$.

To produce $\sigma_t^{(l)}$, we identify U_{nc} with $W_\infty(S)^+$ via φ, and then use Proposition 1.9. Let

$$m_S : S \longrightarrow K_0 \tilde{\otimes} W_\infty(S)$$

be the homomorphism corresponding to $id_{W_\infty(S)}$. Then the homomorphisms

$$S \longrightarrow K_0 \tilde{\otimes} W_\infty(S)$$

corresponding to $\sigma_1^{(l)}$ have the form $(\eta^{(l)} \otimes id_{W_\infty(S)}) \circ m_S$, for appropriate homomorphisms $\eta^{(l)} : K_0 \longrightarrow K_0$ sending rank one projections to rank one projections. (On matrix units e_{ij}, we have $\eta^{(1)}(e_{ij}) = e_{2i-1,2j-1}$ and $\eta^{(2)}(e_{ij}) = e_{2i,2j}$.) The existence of the desired homotopies then follows from Lemma 1.12 and Proposition 1.9. This completes the proof.

We now turn to the computation of the K-theory of U_{nc} and P.

LEMMA 4.3. *Let A be a C^*-algebra. Then the homomorphisms*

$$\pi_n : W_{n+1}(A) \longrightarrow W_n(A)$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

of Definition 1.7 *are all isomorphisms on K-theory.*

Proof. Define $\varphi_n : W_n(A) \longrightarrow A$ by

$$\varphi_n(x_n(a,i,j)) = \delta_{i1}\delta_{j1}a.$$

Since $\varphi_n \circ \pi_n = \varphi_{n+1}$, it suffices to show that each φ_n is an isomorphism on K-theory. Let $m_A : A \longrightarrow M_n(W_n(A))$ be given by

$$m_A(a) = (x_n(a,i,j))_{i,j=1}^n.$$

Then $M_n(\varphi) \circ m_A$ sends $a \in A$ to the $n \times n$ matrix whose (1, 1) entry is a and whose other entries are zero. Identifying $K_*(M_n(A))$ with $K_*(A)$ in the standard way, we see that $M_n(\varphi)_* \circ (m_A)_*$ is the identity on $K_*(A)$.

We now consider the composite in the other order, namely

$$m_A \circ \varphi : W_n(A) \longrightarrow M_n(W_n(A)).$$

Under the identification of Proposition 1.5, this map corresponds to the homomorphism $\eta : A \longrightarrow M_{n^2}(W_n(A))$ given by

$$\eta(a) = \begin{pmatrix} (x_n(a,i,j))_{i,j=1}^n & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}.$$

In this expression each 0 is an $n \times n$ zero matrix. The homomorphism η is easily seen to be homotopic to the homomorphism λ defined by

$$\lambda(a) = \left(\begin{pmatrix} x_n(a,i,j) & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right)_{i,j=1}^n.$$

In this expression each matrix is $n \times n$. It follows that $m_A \circ \varphi$ is homotopic to the homomorphism $\psi : W_n(A) \longrightarrow M_n(W_n(A))$ corresponding to λ. Since ψ is just the standard embedding

$$x_n(a,i,j) \longmapsto \begin{pmatrix} x_n(a,i,j) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

of $W_n(A)$ in the upper left corner of $M_n(W_n(A))$, it follows that $\varphi_* \circ (m_A)_*$ is the identity on $K_*(W_n(A))$.

Note that, in cases for which it makes sense, we have actually proved that $W_n(A)$ and A are KK-equivalent.

COROLLARY 4.4. *For any C^*-algebra A there is a natural isomorphism*

$$RK_*(W_\infty(A)) \cong K_*(A).$$

Proof. Use the previous lemma and the Milnor $\varprojlim^1$-sequence, [**20**], Theorem 3.2.

PROPOSITION 4.5. (1) *With P as in definition* 2.1, *we have $RK_0(P) \cong \mathbf{Z}^2$ and $RK_1(P) = 0$.*

(2) *With U_{nc} as in Definition* 3.1, *we have $RK_0(U_{nc}) \cong \mathbf{Z}$ and $RK_1(U_{nc}) \cong \mathbf{Z}$.*

Proof. (1) Proposition 1.16 yields an obvious exact sequence

$$0 \longrightarrow C_0((0,1)) \otimes M_2 \longrightarrow q\mathbf{C} \longrightarrow \mathbf{C} \oplus \mathbf{C} \longrightarrow 0.$$

Examining the corresponding long exact sequence in K-theory, it is easily shown that $K_0(q\mathbf{C}) \cong \mathbf{Z}$, generated by the class of the quasihomomorphism

$$\mathbf{C} \rightrightarrows \mathbf{C} * \mathbf{C} \triangleright q\mathbf{C},$$

and that $K_1(q\mathbf{C}) = 0$. The result now follows from Corollary 4.4.

(2) By Proposition 4.2, we have $U_{nc} \cong W_\infty(S)^+$, while clearly $S \cong C_0(\mathbf{R})$. The result now follows from Corollary 4.4.

We should point out that Lemma 4.3 and Corollary 4.4 are also valid for σ-C^*-algebras, since

$$W_n(\varprojlim_k A_k) \cong \varprojlim_k W_n(A_k).$$

This enables one to compute the K-theory of some of the dual groups defined in [**29**]. For example, $GL_{nc}(n)$ is isomorphic to $W_n(A)^+$, where A is the universal pro-C^*-algebra on the generators y_0 and z_0, subject to the relations

$$y_0 z_0 = z_0 y_0 = -y_0 - z_0.$$

Equivalently, A^+ is the universal unital pro-C^*-algebra on generators y and z (namely, $y_0 + 1$ and $z_0 + 1$) subject to the relations

$$yz = zy = 1.$$

One can then show that A^+ is homotopy equivalent to $C(S^1)$ by using an appropriate variation on the retraction $(t, x) \longmapsto x(x^*x)^{-t/2}$, for $t \in [0,1]$ and x

invertible, from the group of invertible elements of a C^*-algebra to its unitary group. It follows that

$$RK_0(GL_{nc}(n)) \cong RK_1(GL_{nc}(n)) \cong \mathbf{Z}.$$

This can also be obtained more directly by using the retraction above to show that $GL_{nc}(n)$ is homotopy equivalent to $U_{nc}(n)$.

We also note that Proposition 4.5 shows that the K-theory of our classifying algebras is about as simple as it could be. In fact, we will prove in the next section that there are natural isomorphisms

$$[\mathrm{Ker}(\chi : P \longrightarrow \mathbf{C}), A] \cong RK_0(A) \quad \text{and}$$

$$[\mathrm{Ker}(\chi : U_{nc} \longrightarrow \mathbf{C}), A] \cong RK_1(A)$$

for arbitrary σ-C^*-algebras A, with the restriction in the RK_0 case that A have a countable approximate identity. For the first of these kernels, we have $RK_0 \cong \mathbf{Z}$ and $RK_1 = 0$, while for the second we have $RK_0 = 0$ and $RK_1 \cong \mathbf{Z}$. In either case, the element of $RK_*(A)$ corresponding to a homomorphism $\varphi : \mathrm{Ker}(\chi) \longrightarrow A$ is $\varphi_*(\eta)$ for an appropriate generator η of $RK_*(\mathrm{Ker}(\chi))$.

This situation is rather different from what happens in the case of spaces. We will take as our standard models for the classifying spaces the infinite unitary group $U = \varinjlim U(n)$ for RK^1 and the space $\mathbf{Z} \times BU$ for RK^0. (See for instance Example 2.2 of [**1**].) The representable K-theory of these spaces is known, and will be described in the proof of the next proposition. In both cases, it is quite large. In particular, we have

$$RK_*(P) \ncong RK_*(C(\mathbf{Z} \times BU)) \quad \text{and} \quad RK_*(U_{nc}) \ncong RK_*(C(U)).$$

It follows that the σ-C^*-algebras $C(\mathbf{Z} \times BU)$ and $C(U)$, of all continuous complex valued functions on $\mathbf{Z} \times BU$ and U, cannot be used as classifying algebras for RK_* on the category of σ-C^*-algebras. Indeed, we would otherwise have homotopy equivalences

$$P \simeq C(\mathbf{Z} \times BU) \quad \text{and} \quad U_{nc} \simeq C(U),$$

which contradicts the K-theory results above. (This is a standard result in category theory; see the Yoneda Lemma and its corollary on page 61 of [**14**].)

In fact, our results are strong enough to show that $C(\mathbf{Z} \times BU)$ and $C(U)$ do not even classify K-theory for C^*-algebras. This justifies the construction of our noncommutative analogs of these algebras. We need a lemma, which generalizes the remark on operations at the end of [**24**].

LEMMA 4.6. *Let $A \longmapsto \psi_A$ be a natural transformation from K_i to itself* ($i = 0$ *or* 1)*, where K_i is regarded as a functor from unital C^*-algebras to sets. Then ψ_A has the form*

$$\psi_A(\eta) = m\eta + n[1] \quad \textit{for some } m, n \in \mathbf{Z}.$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

(*Of course, the second term is zero if* $i = 1$.)

Proof. We cannot directly apply the remark of [**24**] referred to above, since we assume neither that ψ_A is a group homomorphism nor that it is natural with respect to nonunital homomorphisms. However, using [**8**], we see that every element η of $K_0(A)$ has the form $\gamma_* \circ \beta_*^{-1} \circ (\alpha^+)_*(\eta_0)$, where $\alpha : q\mathbf{C} \longrightarrow K \otimes A$ is some homomorphism, $\beta : A^+ \longrightarrow (K \otimes A)^+$ is

$$\beta(a + \lambda \cdot 1) = a \otimes p + \lambda \cdot 1$$

for some fixed rank one projection $p \in K, \gamma : A^+ \longrightarrow A$ is

$$\gamma(a + \lambda \cdot 1_{A^+}) = a + \lambda \cdot 1_A,$$

and $\eta_0 \in K_0(q\mathbf{C}^+)$ is the class determined by the homomorphism $q\mathbf{C} \longrightarrow q\mathbf{C}^+$. Now

$$\psi_{q\mathbf{C}^+}(\eta_0) = m\eta_0 + n[1], \quad \text{for some } m, n \in \mathbf{Z},$$

by the proof of the previous proposition, and the argument of [**24**] implies that $\psi_A(\eta) = m\eta + n[1]$. This does the case $i = 0$. The case $i = 1$ is done similarly, using $C_0(\mathbf{R})$ and geometric realization as in Lemma 3.1 and Remark 3.2 of [**23**], in place of $q\mathbf{C}$ and quasihomomorphisms.

PROPOSITION 4.7. (1) *There is no natural isomorphism of sets*

$$K_0(A) \cong [C(\mathbf{Z} \times BU), A]_1$$

for unital C^*-algebras A.

(2) *There is no natural isomorphism of sets*

$$K_1(A) \cong [C(U), A]_1$$

for unital C^*-algebras A.

Note that, in this proposition, we do not assume any structure of homotopy dual group on $C(Z \times BU)$ or $C(U)$.

Proof of Proposition 4.7. Let $i = 0$ or 1, let $Z = \mathbf{Z} \times BU$ or U as appropriate, and let $E = \varprojlim E_n$ be $W_\infty(B)^+ = \varprojlim W_n(B)^+$, with $B = q\mathbf{C}$ or $C_0(\mathbf{R})$ as appropriate. Further let Z_0 be a compact subset of Z such that $j^* : K^*(Z) \longrightarrow K^*(Z_0)$ is surjective; Z_0 will be chosen at the end of the proof in such a way as to obtain a contradiction.

The fact that Z is a classifying space for RK^i on countable direct limits of compact spaces implies that there is a canonical element $\eta_0 \in RK^i(Z)$ such that the isomorphism $[X, Z] \cong RK^i(X)$ is determined by $[f] \longmapsto f^*(\eta_0)$. (The existence of η_0 follows abstractly from the Yoneda Lemma, on page 61 of [**14**].) Similarly, there is a canonical element $\xi_0 \in RK_i(E)$ such that the canonical isomorphism

$$\Psi_A : [E, A]_1 \longrightarrow RK_i(A)$$

is given by $\Psi_A([\varphi]) = \varphi_*(\xi_0)$. (Note that ξ_0 is a generator of $RK_i(\mathrm{Ker}(E \longrightarrow \mathbf{C}))$.) Clearly the abelianization $E/[E,E]$ of E, where $[E,E]$ is the closed ideal in E generated by all commutators $xy - yx$, is a classifying algebra for RK_i of commutative unital σ-C^*-algebras. The usual Yoneda Lemma argument, together with Proposition 5.7 of [**19**], therefore yields a unital homotopy equivalence $E/[E,E] \simeq C(Z)$. Composing it with the quotient map produces a unital homomorphism $\pi : E \longrightarrow C(Z)$ such that $\pi_*(\xi_0) = \eta_0$. Let

$$\pi_A^* : [C(Z), A]_1 \longrightarrow [E, A]_1$$

denote the function induced by π.

Now suppose that we are given some natural isomorphism

$$\Phi_A : [C(Z), A]_1 \longrightarrow K_i(A)$$

for unital C^*-algebras A. Then

$$A \longmapsto \Psi_A \circ \pi_A^* \circ \Phi_A^{-1} = \Lambda_A$$

is a natural transformation of sets from $K_i(A)$ to $K_i(A)$. Therefore there are, by the previous lemma, $m, n \in \mathbf{Z}$ such that

$$\Lambda_A(\eta) = m\eta + n[1] \quad \text{for all } A \text{ and all } \eta \in K_i(A).$$

Using the definition of π_A^* and Ψ_A, and $\pi_*(\xi_0) = \eta_0$, we obtain

$$\varphi_*(\eta_0) - n \cdot [1_A] = m\Phi_A([\varphi])$$

for every unital $\varphi : C(Z) \longrightarrow A$. If A is restricted to be of the form $C(X)$ for X compact, then $[\varphi] \longmapsto \varphi_*(\eta_0)$ is bijective, whence $m = \pm 1$. Therefore $[\varphi] \longmapsto \varphi_*(\eta_0)$ is bijective from $[C(Z), A]_1$ to $K_i(A)$ for all A, commutative or not.

Let $\rho : C(Z) \longrightarrow C(Z_0)$ be the restriction map. Then $\rho \circ \pi$ has a factorization as $\lambda \circ \kappa_n$ for some n, with $\kappa_n : E \longrightarrow E_n$ being the canonical map, and for some $\lambda : E_n \longrightarrow C(Z_0)$. Choose, by the previous paragraph, a unital homomorphism $\sigma : C(Z) \longrightarrow E_n$ such that $\sigma_*(\eta_0) = (\kappa_n)_*(\xi_0)$. Then

$$(\lambda \circ \sigma)_*(\eta_0) = (\rho \circ \pi)_*(\xi_0) = \rho_*(\eta_0),$$

so that $\lambda \circ \sigma$ and ρ must be homotopic. Since $\rho_* = j^*$ is surjective, λ_* must also be surjective.

It now remains only to choose Z_0 such that $j^* : K^i(Z) \longrightarrow K^i(Z_0)$ is surjective, but such that there is no surjective map from $K_i(E_n)$ to $K^i(Z_0)$ for any n. For $i = 0$ and $Z = \mathbf{Z} \times BU$, we note that

$$BU = \varinjlim BU(n) \quad \text{and} \quad BU(n) = \varinjlim_N G_n(\mathbf{C}^N),$$

where $G_n(\mathbf{C}^N)$ is the Grassmannian of n-dimensional subspaces of $\mathbf{C}^N$. The representable K-theory of these spaces is known: it is in every case concentrated in degree 0, and we have

$$RK^0(BU) \cong \mathbf{Z}[[c_1, c_2, \ldots]], \quad RK^0(BU(n)) \cong \mathbf{Z}[[c_1, c_2, \ldots, c_n]],$$

and

$$RK^0(G_n(\mathbf{C}^N)) \cong \mathbf{Z}[[c_1, c_2, \ldots, c_n]]/I_{n,N}$$

for some suitable ideal $I_{n,N}$. The canonical maps between these spaces are all the obvious ones, and are surjective. (See [**27**], 16.32 and 16.33, and [**11**], IV.3.19, IV.3.22, and its proof.) Take $Z_0 = G_1(\mathbf{C}^3) = \mathbf{C}P^2$, embedded in BU as above and thence in $\mathbf{Z} \times BU$ by identifying BU with $\{0\} \times BU$. Then $K^0(Z_0) \cong \mathbf{Z}^3$, while $K_0(E_n) \cong \mathbf{Z}^2$ for all n, by Lemma 4.3 and Proposition 4.5(1). Since there is no surjective homomorphism from $\mathbf{Z}^2$ to $\mathbf{Z}^3$, we have the desired contradiction.

For $i = 1$ and $Z = U = \varinjlim U(n)$, we observe that it is known that $K^*(U(n))$ is an exterior algebra over $\mathbf{Z}$ on odd degree generators $\beta_1, \ldots, \beta_n$, with the homomorphism from $K^*(U(n+1))$ to $K^*(U(n))$ given by killing β_{n+1}. (I am grateful to Jonathan Rosenberg for pointing this out to me.) However, we do not need the full strength of this statement, but only the fact that the rational K-theory $K^*(U(n)) \otimes \mathbf{Q}$ is an exterior algebra over $\mathbf{Q}$, with generators and homomorphisms as above. This version is much more easily derived from results in the literature: for example, use the computation of $H^*(U(n), \mathbf{Z})$ in Theorem VII.4.1 of [**31**], and the fact that the Chern character is a rational isomorphism. The reasoning of the main part of the proof applies equally well to rational K-theory, and the necessary contradiction is supplied by observing that there is no surjective linear map from $K_1(U_{nc}(n)) \otimes \mathbf{Q} \cong \mathbf{Q}$ to $K^1(U(2)) \cong \mathbf{Q}^2$, for any n.

As a corollary of the proof, we obtain:

Example 4.8. There exists a unital C^*-algebra A and a class $\eta \in K_1(A)$ such that η is not represented by any unitary matrix over A whose entries and their adjoints commute with each other. Indeed, it is easily derived from the proof above that

$$A = U_{nc}(2) \quad \text{and} \quad \eta = \left[\begin{pmatrix} x_{2,1,1} & x_{2,1,2} \\ x_{2,2,1} & x_{2,2,2} \end{pmatrix} \right]$$

is such an example. (Note that $\eta = (\kappa_2)_*(\xi_0)$.)

It seems very difficult to produce such an example directly. Obviously we cannot take A to be commutative, and it is only slightly less obvious that we cannot take A to have the form $M_n \otimes C(X)$. On the other hand, it seems quite possible that no such example exists with A simple. Indeed, in all known examples with A simple and unital, every class in $K_1(A)$ is already represented by a unitary in A. (See Section 7 of [**6**] for a more detailed discussion of the evidence.)

We should also point out that our computations show that the "noncommutative" unitary groups $U_{nc}(n)$ are topologically quite different from the ordinary unitary groups $U(n)$. Similar considerations hold for the "noncommutative" finite approximations $W_n(q\mathbf{C})$ to $\mathbf{Z} \times BU$ and their abelianizations, which are presumably homotopy equivalent to the algebras of continuous functions on certain finite disjoint unions of Grassmannians.

5. Other classifying algebras. In this section we define and discuss several other pro-C^*-algebras which classify RK_0 and RK_1 for σ-C^*-algebras. One of our constructions can be generalized to produce classifying algebras for functors of the form $KK^0(D, -)$. These constructions are in certain ways more natural than the definitions of P and U_{nc}, but they do not yield σ-C^*-algebras. The resulting algebras are therefore much harder to handle. In particular, we do not know whether or not the various classifying algebras we construct for RK_0 and RK_1 are homotopy equivalent to each other.

Our constructions come from two sources. One source is a noncommutative analog of the loop space functor, originally defined in Section 2.6 of [**21**]. This functor sends a classifying algebra for RK_i to one for RK_{1-i}. The other source is an adjoint W to the functor $K \otimes -$, which can be used in place of W_∞. We consider the loop algebra construction first.

We will find it convenient to work with pointed (pro-)C^*-algebras and their reduced K-theory. The category of pointed (pro-)C^*-algebras is a noncommutative analog of the category of pointed spaces. Thus, the version of our earlier results in terms of pointed algebras, given in Theorem 5.4 below, is actually the most direct analog of the standard topological approach, which uses pointed spaces and pointed maps.

Definition 5.1 ([**21**], Definition 2.5.1). A *pointed (pro-)C^*-algebra* is a pair (A, α) consisting of a unital (pro-)C^*-algebra A and a unital homomorphism $\alpha : A \longrightarrow \mathbf{C}$. A *pointed morphism* $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ is a unital homomorphism (continuous and adjoint preserving, as always) $\varphi : A \longrightarrow B$ such that $\beta \circ \varphi = \alpha$.

We will generally omit the homomorphism α from the notation. If A is any (pro-)C^*-algebra, then A^+, equipped with the homomorphism $a + \lambda \cdot 1 \longmapsto \lambda$, is a pointed (pro-)$C^*$-algebra. In fact, $A \longmapsto A^+$ and $\varphi \longrightarrow \varphi^+$ is easily seen to be a category equivalence from (pro-)C^*-algebras and arbitrary homomorphisms to pointed (pro-)C^*-algebras and pointed morphisms. The inverse is

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

$(A, \alpha) \longmapsto \mathrm{Ker}(\alpha)$. One should also note that $(X, x) \longmapsto (C(X), ev_x)$, where ev_x is evaluation at x, defines a category equivalence from the category of pointed compact Hausdorff spaces (respectively, pointed completely Hausdorff quasitopological spaces) and basepoint preserving continuous maps to the category of pointed commutative C^*-algebras (respectively, pointed commutative pro-C^*-algebras) and pointed morphisms. By abuse of language, we will therefore refer to α in the pair (A, α) as the basepoint of the algebra A.

The pointed category has noncommutative analogs of wedge and smash products, given by

$$A^+ \vee B^+ = (A \oplus B)^+ \quad \text{and} \quad A^+ \wedge B^+ = (A \otimes B)^+.$$

Of course the usual ambiguity concerning tensor products of C^*-algebras extends to the smash product. As a special case, suspension in the pointed category is $A \longmapsto \Sigma A = C(S^1) \wedge A$, where the basepoint of $C(S^1)$ is taken to be ev_1. There is also a pointed free product, namely

$$(A^+, B^+) \longmapsto A^+ *_{\mathbf{C}} B^+ = (A * B)^+.$$

Extending our earlier notation, the set of pointed morphisms between pointed (pro-)C^*-algebras A and B is denoted by $\mathrm{Hom}_+(A, B)$, and the set of homotopy classes in $\mathrm{Hom}_+(A, B)$ is denoted by $[A, B]_+$. Note that pointed homotopy has an obvious formulation in terms of pointed morphisms from A to $B \wedge C([0, 1])^+$ which is analogous to the usual formulation of homotopy in terms of homomorphisms from A to $B \otimes C([0, 1])$.

Definition 5.2. If (A, α) is a pointed σ-C^*-algebra, then the *reduced representable K-theory* of A is

$$\widetilde{RK}_*(A) = \mathrm{Ker}[\alpha_* : RK_*(A) \longmapsto RK_*(\mathbf{C})].$$

Of course, we have $\widetilde{RK}_*(A^+) \cong RK_*(A)$. In particular,

$$RK_0(A^+) \cong \widetilde{RK}_0(A^+) \oplus \mathbf{Z} \quad \text{and} \quad RK_1(A^+) \cong \widetilde{RK}_1(A^+).$$

Note that $\widetilde{RK}_*$ is functorial for pointed morphisms.

If (A, χ, μ, ι) is a homotopy dual group, then (A, χ) is a pointed (pro-)C^*-algebra. We did not require the maps and homotopies in the definition of a homotopy dual group to be pointed. However, in the definition of the homotopy dual groups U_{nc} and $W_\infty(A)^+$, the maps are all pointed, and in the proof of Theorem 1.11, all homotopies are in fact pointed. Thus, we have:

PROPOSITION 5.3. *The homotopy dual groups $W_\infty(A)^+$ of Definition* 1.10 *and U_{nc} of Definition* 3.1 *are abelian homotopy dual groups in the category of pointed pro-C^*-algebras.*

For abelianness of U_{nc}, see Proposition 4.2. It follows from this proposition that $[W_\infty(A)^+, B]_+$ and $[U_{nc}, B]_+$ are abelian groups in a natural way.

The pointed version of our earlier theorems is then:

THEOREM 5.4. (1) *If A is a σ-C^*-algebra with a countable approximate identity, then there is a natural isomorphism of abelian groups $\widetilde{RK}_0(A^+) \cong [P, A^+]_+$.*

(2) *If A is any σ-C^*-algebra, then there is a natural isomorphism of abelian groups $\widetilde{RK}_1(A^+) \cong [U_{nc}, A^+]_+$.*

Presumably we do not actually need the assumption in (1) that A have a countable approximate identity. In our proof, it enters through Lemma 2.10, where we used the stabilization theorem for Hilbert modules and the contractibility of the unitary group of $M(K \otimes A_p)$ for a C^*-algebra quotient A_p of A.

Proof of Theorem 5.4. (1) We have natural isomorphisms of sets

$$\begin{aligned}\widetilde{RK}_0(A^+) \cong RK_0(A) \cong \bar{P}(A) &\cong \bar{P}_0(A)\\ &\cong [q\mathbf{C}, K_0 \tilde{\otimes} A] \cong [W_\infty(q\mathbf{C}), A]\\ &\cong [W_\infty(q\mathbf{C})^+, A^+]_+ = [P, A^+]_+.\end{aligned}$$

These follow from, in order, the definition of $\widetilde{RK}_0$, Proposition 2.8, Lemma 2.6, Lemma 2.10, Proposition 1.9, the definition of a pointed morphism, and the definition of P. Comparing the composite of these isomorphisms with the proof of Theorem 2.12 now yields a commutative diagram with bijective horizontal arrows:

$$\begin{array}{ccc}\widetilde{RK}_0(A^+) & \xrightarrow{\cong} & [P, A^+]_+\\ \downarrow & & \downarrow\\ RK_0(A^+) & \xrightarrow{\cong} & [P, A^+]_1.\end{array}$$

Since the left vertical arrow is injective, so is the right one. That the top horizontal arrow is a group homomorphism now follows from the fact that the other three arrows are group homomorphisms.

(2). Let α be the basepoint of A^+, let

$$\varphi = (id_{K_0} \otimes \alpha)^+ : (K_0 \tilde{\otimes} A^+)^+ \longrightarrow K_0^+,$$

and let

$$\psi : (K_0 \tilde{\otimes} A^+)^+ \longrightarrow \mathbf{C}$$

be the obvious map. Set

$$G(A) = \{u \in U((K_0 \tilde{\otimes} A^+)^+) : \psi(u) = 1\}$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

and

$$U^+(A) = \{u \in U((K_0 \tilde{\otimes} A^+)^+) : \varphi(u) = 1\},$$

and let $G_0(A)$ and $U_0^+(A)$ be the path components of the identity in $G(A)$ and $U^+(A)$. In the proof of Proposition 3.5, it was shown that $[U_{nc}, A^+]_1 \cong G(A)/G_0(A)$, and essentially the same proof yields $[U_{nc}, A^+]_+ \cong U^+(A)/U_0^+(A)$. An easy argument, using the path connectedness of

$$G(\mathbf{C}) = \{u \in U(K_0^+) : u - 1 \in K_0\},$$

shows that

$$U^+(A)/U_0^+(A) \longrightarrow G(A)/G_0(A)$$

is an isomorphism. (Note that $G(\mathbf{C})/G_0(\mathbf{C}) \cong RK_1(\mathbf{C}) = 0$.) Therefore we have isomorphisms

$$\begin{aligned} \widetilde{RK}_1(A^+) &\cong RK_1(A^+) \cong [U_{nc}, A^+]_1 \\ &\cong G(A)/G_0(A) \cong U^+(A)/U_0^+(A) \cong [U_{nc}, A^+]_+, \end{aligned}$$

as desired. (The second isomorphism is Theorem 3.16.)

This theorem can be restated without using pointed algebras, as follows.

COROLLARY 5.5. (1) *If A is a σ-C^*-algebra with a countable approximate identity, then there is a natural isomorphism*

$$RK_0(A) \cong [W_\infty(q\mathbf{C}), A].$$

(2) *If A is any σ-C^*-algebra, then there is a natural isomorphism*

$$RK_1(A) \cong [\mathrm{Ker}(U_{nc} \longrightarrow \mathbf{C}), A],$$

where $U_{nc} \longrightarrow \mathbf{C}$ is given by $x_{\infty,i,j} \longmapsto \delta_{ij}$.

This version is not as nice as the previous version because $W_\infty(q\mathbf{C})$ and $\mathrm{Ker}(U_{nc} \longrightarrow \mathbf{C})$ are not homotopy dual groups. The reason for introducing pointed pro-C^*-algebras is to be able to state this result in the form of Theorem 5.4.

We now construct the noncommutative analog of the loop space. (Compare Section 2.6 of [**21**]). We include here an easy direct proof of its existence, not relying on the generator and relation material in Section 1.3 of [**21**].

THEOREM 5.6. *There is a functor $A \longmapsto \Omega A$ from pointed pro-C^*-algebras to pointed pro-C^*-algebras such that there are natural isomorphisms of sets*

$$\mathrm{Hom}_+(\Omega A, B) \cong \mathrm{Hom}_+(A, \Sigma B) \quad \text{and} \quad [\Omega A, B]_+ \cong [A, \Sigma B]_+$$

for any pointed pro-C^*-algebra B.

Proof. Let A have the basepoint $\alpha : A \longrightarrow \mathbf{C}$. Let D be the set of isomorphism classes of pairs $(Z, (\pi_\zeta)_{\zeta \in S^1})$ in which Z is a pointed pro-C^*-algebra and each π_ζ is a pointed morphism from A to Z, satisfying the following conditions:

(1) $(a, \zeta) \longmapsto \pi_\zeta(a)$ is (jointly) continuous.

(2) $\pi_1 = \varepsilon_Z \circ \alpha$. (Recall that $\varepsilon_Z : \mathbf{C} \longrightarrow Z$ is given by $\varepsilon_Z(1) = 1$.)

(3) Z is the C^*-algebra generated by $\bigcup_{\zeta \in S^1} \pi_\zeta(A)$.

For $d \in D$ choose a representative $(Z_d, (\pi_\zeta^d)_{\zeta \in S^1})$. Define $d \geqq e$ if there exists a pointed morphism (necessarily unique) $\varphi_{de} : Z_d \longrightarrow Z_e$ such that

$$\varphi_{de} \circ \pi_\zeta^d = \pi_\zeta^e \quad \text{for all } \zeta \in S^1.$$

This makes D a partially ordered set, and it is in fact directed, as can be seen by looking at the C^*-subalgebra of $Z_d \oplus Z_e$ generated by all $(\pi_\zeta^d(a), \pi_\zeta^e(a))$. Then set

$$\Omega A = \varprojlim_{d \in D} Z_d,$$

taken with respect to the maps φ_{de}, and let $\pi_\zeta : A \longrightarrow \Omega A$ be the obvious pointed morphisms.

Define a map $\mathrm{Hom}_+(\Omega A, B) \longrightarrow \mathrm{Hom}_+(A, \Sigma B)$ by sending ψ to the pointed morphism given by

$$\eta(a)(\zeta) = \psi \circ \pi_\zeta(a) \quad \text{for } a \in A \text{ and } \zeta \in S^1.$$

It suffices to prove that this map is defined and bijective when B is a pointed C^*-algebra, by the definition of an inverse limit. Continuity of η follows from the joint continuity of $(a, \zeta) \longmapsto \pi_\zeta(a)$ and the fact that S^1 is compact. Injectivity of the assignment $\psi \longmapsto \eta$ is obvious. For surjectivity, let $\eta : A \longrightarrow \Sigma B$ be a pointed morphism, and define $\sigma_\zeta : A \longrightarrow B$ by $\sigma_\zeta = ev_\zeta \circ \eta$. Let Z be the pointed C^*-subalgebra of B generated by $\bigcup_{\zeta \in S^1} \sigma_\zeta(A)$. One readily checks that $(Z, (\sigma_\zeta)_{\zeta \in S^1})$ defines a class in D. (Joint continuity of $(a, \zeta) \longmapsto \sigma_\zeta(a)$ follows from the definition of the norm on ΣB.) Then ψ is the composite $\Omega A \longrightarrow Z \longrightarrow B$. This shows that

$$\mathrm{Hom}_+(\Omega A, B) \cong \mathrm{Hom}_+(A, \Sigma B).$$

Naturality is obvious.

To prove that $[\Omega A, B]_+ \cong [A, \Sigma B]_+$, use the previous result with $B \wedge C([0, 1])^+$ in place of B.

The loop algebra construction gives as a new set of classifying algebras by using the Bott periodicity isomorphisms $\widetilde{RK}_i(\Sigma A) \cong \widetilde{RK}_{1-i}(A)$, as follows.

Corollary 5.7. (1) *If A is any σ-C^*-algebra, then there is a natural isomorphism*

$$\widetilde{RK}_0(A^+) \cong [\Omega U_{nc}, A^+]_+.$$

(2) *If A is a σ-C^*-algebra with a countable approximate identity, then there is a natural isomorphism*

$$\widetilde{RK}_1(A^+) \cong [\Omega P, A^+]_+.$$

The usual considerations concerning H-space structures from topology (see Sections 1.5 and 1.6 of [**26**], especially the example following Theorem 1.5.7 and the remark following Corollary 1.6.11) carry over to homotopy dual group structures on loop algebras; we will not worry about the details here.

Unfortunately, this corollary does not imply the analog of the "original" Bott periodicity theorem (see for example Theorem 11.60 of [**27**]), which would be the existence of homotopy equivalences $\Omega U_{nc} \simeq P$ and $\Omega P \simeq U_{nc}$, because ΩU_{nc} and ΩP are not σ-C^*-algebras. We do not know whether or not these homotopy equivalences hold. It does, however, seem to be possible to construct a certain canonical quotient $\Omega_0 A$ of ΩA which for $A = U_{nc}$ or $A = P$ would be a pointed σ-C^*-algebra still satisfying $[\Omega_0 A, B] \cong [A, \Sigma B]$ for σ-C^*-algebras B. Then one could conclude that $\Omega_0 U_{nc} \simeq P$ and $\Omega_0 P \simeq U_{nc}$. (This would, incidentally, enable one to remove the countable approximate identity hypotheses in Theorem 5.4(1) and Corollary 5.7(2).) We hope to investigate this possibility in a future paper.

We now consider a more straightforward method of obtaining classifying algebras from $C(S^1)$ and $q\mathbf{C}$, which, however, also has the disadvantage of giving uncountable inverse limits of C^*-algebras.

Proposition 5.8. *There exists a functor $A \longmapsto W(A)$ from pro-C^*-algebras to pro-C^*-algebras such that there are natural isomorphisms of sets* $\mathrm{Hom}(W(A), B) \simeq \mathrm{Hom}(A, K \otimes B)$ *and* $[W(A), B] \cong [A, K \otimes B]$ *for any pro-C^*-algebra B.*

Proof. Let D be the set of all isomorphism classes of pairs $(Z, (x(\cdot, i, j))_{i,j=1}^{\infty})$ such that Z is a C^*- algebra and the $x(\cdot, i, j)$ are functions from A to Z satisfying:

(1) The infinite matrix $x(a) = (x(a, i, j))_{i,j=1}^{\infty}$ is in $K \otimes Z$.

(2) $a \longmapsto x(a)$ is a homomorphism from A to $K \otimes Z$.

(3) $\bigcup_{i,j=1}^{\infty} x(A, i, j)$ generates Z as a C^*- algebra.

Note that D is not empty: take $Z = 0$. Pick representatives $(Z_d, (x^d(\cdot, i, j)))$, define a partial order on D, and define homomorphisms $\varphi_{de} : Z_d \longrightarrow Z_e$ for $d \geqq e$, as in the proof of Theorem 5.6.

To show that D is directed, let $d, e \in D$, define $x(\cdot, i, j) : A \longrightarrow Z_d \oplus Z_e$ by

$$x(a, i, j) = (x^d(a, i, j), x^e(a, i, j)),$$

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

and let Z be the C^*-subalgebra of $Z_1 \oplus Z_2$ generated by $\bigcup_{i,j} x(A,i,j)$. To verify that (1) holds for $(Z, (x(\cdot,i,j)))$, let p_n be the sum of the first n standard rank one projections in K, and regard p_n as a multiplier of $K \otimes (Z_1 \oplus Z_2)$ in the obvious way. Then $p_n x(a) p_n \longrightarrow x(a)$ in $K \otimes (Z_1 \oplus Z_2)$, and $p_n x(a) p_n$ is clearly in $K \otimes Z$. Therefore $x(a)$ is in fact in $K \otimes Z$. Condition (2) is now immediate, and (3) holds by definition. So D is directed.

We can now set

$$W(A) = \varprojlim_{d \in D} Z_d.$$

One shows that

$$\mathrm{Hom}(W(A), B) \cong \mathrm{Hom}(A, K \otimes B) \quad \text{and} \quad [W(A), B] \cong [A, K \otimes B]$$

by the same reasoning as was used to get the analogous statements in the proof of Theorem 5.6, except that the continuity arguments are easier here.

Unfortunately, there is no reason to think that $W(A)$ is a σ-C^*-algebra.

Note that $W(A)$ is generated by elements $x(a,i,j)$ whose images in Z_d are $x^d(a,i,j)$ and that the obvious map $x : A \longrightarrow K \otimes W(A)$ is the homomorphism corresponding to the identity map of $W(A)$. The assignment $x(a,i,j) \longmapsto x_\infty(a,i,j)$, where $x_\infty(a,i,j)$ is as in Definition 1.10, extends to a homomorphism from $W(A)$ to $W_\infty(A)$ which has dense range.

PROPOSITION 5.9. *Let A and $\iota_0 : A \longrightarrow A$ be as in Definition* 1.10. *Then $W(A)^+$ has a structure of pointed abelian homotopy dual group, with $\chi : W(A)^+ \longrightarrow \mathbf{C}$ being the obvious map, $\iota = W(\iota_0)$, and with*

$$\mu : W(A)^+ \longrightarrow W(A)^+ *_{\mathbf{C}} W(A)^+$$

given as follows. Define $\eta : A \longrightarrow K \otimes (W(A) * W(A))$ by

$$\eta(a) = \left(\begin{array}{cc|cc|c} x^{(1)}(a,1,1) & 0 & x^{(1)}(a,1,2) & 0 & \cdots \\ 0 & x^{(2)}(a,1,1) & 0 & x^{(2)}(a,1,2) & \cdots \\ \hline x^{(1)}(a,2,1) & 0 & x^{(1)}(a,2,2) & 0 & \cdots \\ 0 & x^{(2)}(a,2,1) & 0 & x^{(2)}(a,2,2) & \cdots \\ \hline & \vdots & & \vdots & \ddots \end{array}\right),$$

where $x^{(l)}(a,i,j)$ is the element of the l-th copy of $W(A)$ in the free product corresponding to $x(a,i,j) \in W(A)$. Then μ is the unitization of the homomorphism $W(A) \longrightarrow W(A) * W(A)$ corresponding to η under the previous proposition.

https://doi.org/10.4153/CJM-1989-046-2 Published online by Cambridge University Press

Proof. The proof is the same as the proof of Theorem 1.11, using Proposition 5.8 in place of Proposition 1.9 and using the (well known) analog of Lemma 1.12 with K replacing K_0.

It follows that $W(qA)^+$ is a pointed abelian homotopy dual group for any C^*-algebra A. (See Proposition 1.15.) Furthermore, with

$$S = \{f \in C(S^1) : f(1) = 0\}$$

as in Lemma 4.1, $W(S)^+$ is a pointed abelian homotopy dual group. (In fact, without allowing homotopy, $W(S)^+$ is a nonabelian dual group with μ and ι given by formulas analogous to those in the definition of U_{nc}. The dual group $W(S)^+$ is the "noncommutative" analog, in the sense of Voiculescu [**29**], of the group $\{u \in U(K^+) : u - 1 \in K\}$.)

PROPOSITION 5.10. (1) *If A is a σ-C^*-algebra with a countable approximate identity, then there is a natural isomorphism of abelian groups*

$$\widetilde{RK}_0(A^+) \cong [W(q\mathbf{C})^+, A^+]_+.$$

(2) *If A is any σ-C^*-algebra, then there is a natural isomorphism of abelian groups*

$$\widetilde{RK}_1(A^+) \cong [W(S)^+, A^+]_+.$$

Note that the canonical homomorphisms $W(q\mathbf{C})^+ \longrightarrow P$ and $W(S)^+ \longrightarrow U_{nc}$ thus induce isomorphisms on the sets of pointed homotopy classes of maps to A^+ for any σ-C^*-algebra A (with a countable approximate identity in the case of $W(q\mathbf{C})^+ \longrightarrow P$). In the commutative situation, it is known, for example, that

$$U \longrightarrow \{u \in U(K^+) : u - 1 \in K\}$$

is in fact a homotopy equivalence. Unfortunately, it is far from clear whether or not, say, $W(S)^+ \longrightarrow U_{nc}$ is a homotopy equivalence.

Proof of Proposition 5.10. (1) Replacing $K_0 \tilde{\otimes} A$ everywhere in the proof of Lemma 2.10 by $K \otimes A$ yields a proof that $[q\mathbf{C}, K \otimes A] \cong \bar{P}(A)$. Now use the natural isomorphisms

$$\begin{aligned}\widetilde{RK}_0(A^+) &\cong RK_0(A) \cong \bar{P}(A)\\ &\cong [q\mathbf{C}, K \otimes A] \cong [W(q\mathbf{C}), A] \cong [W(q\mathbf{C})^+, A^+]_+.\end{aligned}$$

where the second step is Proposition 2.8, the second last step is the definition of W, and the last step is the definition of pointed homotopy.

(2) We combine modifications to the proof of Proposition 3.5 analogous to the modification used above on Lemma 2.10 and analogous to the modifications

used in the proof of Theorem 5.4(2). This yields a proof that $[W(S)^+, A^+]_+$ is naturally isomorphic to $(U/U_0)((K \otimes A^+)^+)$. Using Theorem 3.15, we have isomorphisms

$$(U/U_0)((K \otimes A^+)^+) \cong RK_1(A^+) \cong \widetilde{RK}_1(A^+).$$

This completes the proof.

Just as before, we now also get natural isomorphisms

$$\widetilde{RK}_0(A^+) \cong [\Omega(W(S)^+), A^+]_+$$

if A is a σ-C^*-algebra and

$$\widetilde{RK}_1(A^+) \cong [\Omega(W(q\mathbf{C})^+), A^+]_+$$

if in addition A has a countable approximate identity. One can now ask whether, for example, there is a homotopy equivalence $\Omega(W(S)^+) \simeq W(q\mathbf{C})^+$. This might be true even if $\Omega U_{nc} \simeq P$ fails.

The functor W also enables us to construct classifying algebras for KK-theory, at least for C^*-algebras. The following result is immediate from the definition.

Proposition 5.11. *If A and B are C^*-algebras, then there is a natural isomorphism of abelian groups of $KK(A, B)$ as in Definition* 1.5 *of* [**8**] *with $[W(qA), B]$, equipped with the group structure it gets by being identified with $[W(qA)^+, B^+]_+$.*

If A and B are separable, then the group $KK(A, B)$ as defined in [**8**] is the same as the usual $KK^0(A, B)$ as defined in [**12**] or in [**5**]. See Theorem 3.6.5 of [**18**] for a detailed proof.

For KK^1, there are two alternatives. One is to use $W(\varepsilon A)$, where εA is as defined in [**32**]. The other is to use periodicity in KK-theory and the loop algebra, which gives

$$KK^1(A, B) \cong [\Omega(W(qA)^+), B^+]_+$$

if, say, A and B are separable. These results suggest that one might hope for a close relationship between $\Omega(W(qA)^+)$ and $W(\varepsilon A)^+$.

Proposition 5.11 suggests two other questions. First, when can one use $W_\infty(qA)$ instead of $W(qA)$? This is an advantage because $W_\infty(qA)$ is a σ-C^*-algebra. Unfortunately, the answer seems to be "almost never". One would need to be able to deform, for any (separable) B, any homomorphism from qA to $K \otimes B$ so as to have its range contained in $K_0 \tilde{\otimes} B$. Thus, qA would presumably have to be semiprojective in the sense of Effros and Kaminker [**10**]. We have seen that this happens with $q\mathbf{C}$ and with $C_0(\mathbf{R})$, but in general it is very rare, even rarer than A itself being semiprojective. Note, for example, that Loring has shown in [**13**] that $C(S^1 \times S^1)$ is not semiprojective.

The second question is to what extent Proposition 5.11 holds for the KK-theory defined by Weidner in [**30**] when A and B are no longer C^*-algebras. Weidner's theory uses an analog of a Kasparov bimodule which consists of a Hilbert B-module E, a representation of A in $L(E)$, and, rather than a single operator in $L(E)$, a whole family of operators on the Hilbert B_p-modules E_p obtained from the continuous C^*-seminorms p on B. If A and B are σ-C^*-algebras, then it should be possible to use an induction argument to show that one needs only a single operator on E. One should then be able to generalize Proposition 5.11 to this case. However, we have no idea of what happens in the general case.

Returning to representable K-theory, we see that we have accumulated quite a number of definitions of groups isomorphic to $RK_0(A)$. If, say, A is a pointed σ-C^*-algebra such that the kernel of its basepoint has a countable approximate identity, then the following eight groups are all isomorphic:

$$[W_\infty(q\mathbf{C})^+, A]_+, \quad [\Omega U_{nc}, A]_+,$$
$$[W(q\mathbf{C})^+, A]_+, \quad [\Omega(W(S)^+), A]_+,$$
$$\operatorname{Ker}[\bar{P}(A) \longrightarrow \bar{P}(\mathbf{C})], \quad \operatorname{Ker}[(U/U_0)Q(A) \longrightarrow (U/U_0)Q(\mathbf{C})], \quad \bar{U}(\Sigma A),$$
$$\operatorname{Ker}[KK(\mathbf{C}, A) \longrightarrow KK(\mathbf{C}, \mathbf{C})] \quad \text{as defined in [30]}.$$

Question 5.12. If A is allowed to be an arbitrary pointed pro-C^*-algebra, which of these groups are still isomorphic? Which ones gives the "right" definition of the representable K-theory of A?

Note that we don't know that $M(K \otimes A)/(K \otimes A)$ is complete, so there are two possible choices for $Q(A)$, making nine groups in all. We can, of course, also ask the analogous question for RK_1.

References

1. J. F. Adams, *Stable homotopy and generalized homology*, part III, Chicago Lectures in Mathematics (University of Chicago Press, 1974).

2. J. Anderson, B. Blackadar and U. Haagerup, *Minimal projections in the reduced group C^*-algebra of $\mathbf{Z}_n * \mathbf{Z}_m$*, J. Operator Theory, to appear.

3. B. Blackadar, *A simple C^*-algebra with no nontrivial projections*, Proc. Amer. Math. Soc. *78* (1980), 504–508.

4. ——— *Shape theory for C^*-algebras*, Math. Scand. *56* (1985), 249–275.

5. ——— *K-theory for operator algebras*, MSRI publications no. 5 (Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1986).

6. ——— *Comparison theory in simple C^*-algebras*, pages 21–54 in: *Operator algebras and applications Volume 1: Structure theory; K-theory, geometry and topology*, London Mathematical Society Lecture Note Series no. 135 (Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sydney, 1988).

7. L. G. Brown, *Ext of certain free product C^*- algebras*, J. Operator Theory *6* (1981), 135–141.

8. J. Cuntz, *A new look at KK-theory*, K-Theory *1* (1987), 31–51.

9. J. Cuntz and N. Higson, *Kuiper's theorem for Hilbert modules*, pages 429–435 in: *Operator algebras and mathematical physics*, Contemporary Math. *62* (Amer. Math. Soc., 1987).

10. E. G. Effros and J. Kaminker, *Homotopy continuity and shape theory for C^*-algebras*, pages 152–180 in: *Geometric methods in operator algebras*, Pitman Research Notes in Math. *123* (Longman Scientific and Technical, 1986).

11. M. Karoubi, *K-theory: an introduction*, Grundlehren der Mathematischen Wissenschaften *226* (Springer-Verlag, Berlin, Heidelberg, New York, 1978).

12. G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR, Ser. Mat. *44* (1980), 571–636 (in Russian); English translation in Math. USSR Izvestija *16* (1981), 513–572.

13. T. A. Loring, *The torus and noncommutative topology*, Ph.D. Thesis, Berkeley (1986).

14. S. MacLane, *Categories for the working mathematician*, Graduate Texts in Math *5* (Springer-Verlag, Berlin, Heidelberg, New York, 1971).

15. J. A. Mingo, *K-theory and multipliers of stable C^*-algebras*, Trans. Amer. Math. Soc. *299* (1987), 397–411.

16. J. A. Mingo and W. J. Phillips, *Equivariant triviality theorems for Hilbert C^*-modules*, Proc. Amer. Math. Soc. *91* (1984), 225–230.

17. G. K. Pedersen, *SAW^*-algebras and corona C^*-algebras, contributions to noncommutative topology*, J. Operator Theory *15* (1986), 15–32.

18. N. C. Phillips, *Equivariant K-theory and freeness of group actions on C^*-algebras*, Lecture Notes in Math. *1274* (Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987).

19. ——— *Inverse limits of C^*-algebras*, J. Operator Theory *19* (1988), 159–195.

20. ——— *Representable K-theory for σ-C^*-algebras*, *K-Theory*, to appear.

21. ——— *Inverse limits of C^*-algebras and applications*, pages 127–185 in: *Operator algebras and applications Volume 1: Structure theory; K-theory, geometry and topology*, London Mathematical Society Lecture Note Series *135* (Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sydney, 1988).

22. J. Rosenberg, *The role of K-theory in noncommutative algebraic topology*, pages 155–182 in: *Operator algebras and K-theory*, Contemporary Math. *10* (Amer. Math. Soc., 1982).

23. C. Schochet, *Topological methods for C^*-algebras II: geometric resolutions and the Künneth formula*, Pacific J. Math. *98* (1982), 443–458.

24. ——— *Topological methods for C^*-algebras IV: mod p homology*, Pacific J. Math. *114* (1984), 447–468.

25. E. Spanier, *Quasitopologies*, Duke Math. J. *30* (1963), 1–14.

26. ——— *Algebraic topology* (McGraw-Hill, New York, San Francisco, St. Louis, Toronto, London, Sydney, 1966).

27. R. M. Switzer, *Algebraic topology – homotopy and homology*, Grundlehren der Mathematischen Wissenschaften *212* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

28. M. Takesaki, *Theory of operator algebras I* (Springer-Verlag, New York, Heidelberg, Berlin, 1979).

29. D. Voiculescu, *Dual algebraic structures on operator algebras related to free products*, J. Operator Theory *17* (1987), 85–98.

30. J. Weidner, *Topological invariants for generalized operator algebras*, Ph.D. Thesis, Heidelberg (1987).

31. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math. *61* (Springer-Verlag, New York, Heidelberg, Berlin, 1978).

32. R. Zekri, *A new description of Kasparov's theory of C^*-algebra extensions*, Ph.D. Thesis, Université d'Aix-Marseille II, Faculté des Sciences de Luminy (1986).

University of Georgia,
Athens, Georgia