Canad. Math. Bull. Vol. **61** (1), 2018 pp. 55–69
http://dx.doi.org/10.4153/CMB-2017-050-9

Enumerating Unlabelled Embeddings of Digraphs

Yichao Chen, Xiaojian Gao, and Yuanqiu Huang

Abstract. A 2-cell embedding of an Eulerian digraph D into a closed surface is said to be *directed* if the boundary of each face is a directed closed walk in D. In this paper, a method is developed with the purpose of enumerating unlabelled embeddings for an Eulerian digraph. As an application, we obtain explicit formulas for the number of unlabelled embeddings of directed bouquets of cycles B_n, directed dipoles OD_{2n} and for a class of regular tournaments T_{2n+1}.

1 Introduction

1.1 Directed Embedding

A *directed graph* or *digraph* D consists of a finite nonempty set $V(D)$ of vertices together with a set $A(D)$ of ordered pairs of vertices called *arcs* or *directed edges*. The *outdegree* $\text{out}(v)$ of a vertex v of a digraph D is the number of out-arcs at v. The *indegree* $\text{in}(v)$ of v is the number of in-arcs at v. A digraph D is said to be *connected* if its underlying graph G is connected. A digraph D is called an *Eulerian digraph* if $\text{in}(v) = \text{out}(v)$ for each vertex v of D. In this paper all digraphs considered are both Eulerian and connected. An *orientation* of a graph is obtained by assigning a direction to each edge. Any digraph constructed this way is called an *oriented graph*. A *surface* is a compact 2-manifold without boundary.

A *directed embedding* of an Eulerian directed graph D into an orientable surface S_g is a homeomorphism $i: D \to S_g$ of D into S_g such that every face is bounded by a directed closed walk in D. An embedding here is taken to be *cellular*. Given a digraph embedding of D, each arc of D is on the boundary of exactly two faces: one we call a *face* (each arc is traversed in the forward direction), and the other we call an *antiface* (each arc is traversed against its given orientation).

Directed embeddings (Tutte called them plane alternating dimaps) were studied by Tutte [14] in 1948. Tutte's original purpose was to study the dissections of equilateral triangles into equilateral triangles. He then generalized the concept of dual planar maps to a trinity of directed plane embeddings. In [15], Tutte also noted the possibility of extending his theory to other surfaces. Bonnington, Conder, Morton,

Received by the editors October 30, 2016; revised June 23, 2017.
Published electronically September 29, 2017.
Author Y. C. is supported by the NNSFC under Grant No. 11471106. Author Y. H. is supported by NNSFC under Grant No. 11371133.
AMS subject classification: **05C10**.
Keywords: Eulerian digraph, directed embedding, unlabelled embedding.

and McKenna [2] made a systematic study of directed embeddings of an Eulerian digraph into surfaces. We refer the reader to [1, 4, 9] for more connections with directed embeddings and other areas of mathematics.

In [13], Mull, Rieper, and White enumerated unlabelled graph embeddings (or congruent embeddings). Their method was generalized to any graph with loops or multiple edges by Feng, Kwak and Zhou [5]. One can refer to [5–8, 12, 13] for more enumeration work on the unlabelled graph embeddings case. In [3], the authors enumerated labelled digraph embeddings. Here, we direct our attention to unlabelled digraph embeddings. Two 2-cell embeddings $i: D \to S$ and $j: D \to S$ of an Eulerian digraph D into an orientable surface S_g are *equivalent* if there is a surface homeomorphism h on S_g and a digraph automorphism α of D such that $hi = j\alpha$; *i.e.,* we regard two digraph embeddings as equivalent if they look alike when the labels of vertices and arcs are removed. Though the counting theorem of Mull–Rieper–White on graph embeddings is generalized to digraph embeddings, we will see that there are different things that need to be dealt with. For example, counting unlabelled graph embeddings of the complete graph K_n is an easy task [13]; however, counting unlabelled digraph embeddings of any regular tournament does not appear to be easy.

1.2 Combinatorial Representation

In this paper, we assume that the graph underlying the embedded digraph is simple. A directed embedding has a standard combinatorial representation which is called a 3-*constellation*; see [4, Proposition 1] for details. Here we use another combinatorial representation of a directed embedding known as an alternating rotation system. An *alternating rotation at a vertex* v of a Eulerian digraph D is a cyclic ordering of the vertices connected to v via in-arcs and of the vertices connected to v via out-arcs such that the in-arcs and out-arcs at v alternate. An *alternating rotation system* ρ of a graph D is an assignment of an alternating rotation at every vertex of D. We denote the set of alternating rotation systems of D by $R(D)$. It is easy to see that

$$|R(D)| = \prod_{v \in V(D)} \Big(\frac{d(v)}{2} - 1 \Big)! \frac{d(v)}{2}!,$$

where $d(v)$ is the degree of the vertex v.

1.3 Directed Map Automorphism

A *directed map* is a pair (D, ρ), where D is a connected Eulerian digraph and ρ is an alternating rotation system for D. An *automorphism* α of a digraph D is a permutation α of the vertex set V, such that the pair of vertices $\overrightarrow{uv}$ form an arc if and only if the pair $\overrightarrow{\alpha(u)\alpha(v)}$ also form an arc. The *automorphism group* of D is denoted by $\operatorname{Aut} D$. Let $\alpha \in \operatorname{Aut} D$, and $\rho \in R(D)$. We define $\alpha(\rho) \in R(D)$ by

$$\alpha(\rho)_{\alpha(v)} = \alpha \rho_v \alpha^{-1}$$

for all $v \in V(D)$; *i.e.,* if ρ_v takes x to y, then $\alpha(\rho)_{\alpha(v)}$ takes $\alpha(x)$ to $\alpha(y)$.

Two alternating rotation systems $\rho, \sigma \in R(D)$ are said to be *equivalent* if there is an automorphism $\alpha \in \operatorname{Aut} D$ so that the action of α on $R(D)$ is such that $\sigma = \alpha(\rho)$, *i.e.*, $\alpha\rho_v\alpha^{-1} = \sigma_{\alpha(v)}$ for all $v \in V(D)$. We define the set $C(D)$ as the number of inequivalent embeddings (inequivalent class) of D. Our task is to count $C(D)$ for D.

A *directed map automorphism* α for $\overrightarrow{M} = (D, \rho)$ is a digraph automorphism such that $\alpha(\rho) = \rho$. For every ρ in $R(D)$, we define the *directed map automorphism group* of ρ as the set of all elements in $\operatorname{Aut} D$ that fix ρ:

$$\operatorname{Aut}\overrightarrow{M} = \{\alpha \in \operatorname{Aut} D \text{ acting on } R(D) \mid \alpha(\rho) = \rho\}.$$

A direct consequence of the directed map automorphism and the alternating rotation system is that a permutation $\alpha \in \operatorname{Aut}\overrightarrow{M}$ if only if $(v_1, v_2, \ldots, v_k)$ being a face (antiface) of $\overrightarrow{M}$ implies that $(\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_k))$ is a face (antiface) of $\overrightarrow{M}$. The following result is the orbit-stabilizer theorem.

Theorem 1.1 *The number of alternating rotation systems of an Eulerian digraph D equivalent to a given alternating rotation system ρ is the index $|\operatorname{Aut} D\colon \operatorname{Aut}\overrightarrow{M}|$, where $\overrightarrow{M}$ is the directed map (D, ρ).*

We have the following version of Burnside's lemma for enumerating unlabelled directed embeddings.

Theorem 1.2 *The number of inequivalent unlabelled embeddings of the digraph D is*

$$|C(D)| = \frac{1}{|\operatorname{Aut} D|} \sum_{\alpha \in \operatorname{Aut} D} |F(\alpha)|,$$

where $F(\alpha) = \{\rho \in R(D) \mid \alpha(\rho) = \rho\}$ is the fixed point set of α.

The following two theorems appear in [13]. They are also valid for digraph embeddings.

Theorem 1.3 *If $\alpha \in \operatorname{Aut} D$ fixes two adjacent vertices, then either α is the identity permutation or $|F(\alpha)| = 0$.*

For $\alpha \in \operatorname{Aut} D$ and $v \in V(D)$, we define the *fixed set at v*, denoted by $F_v(\alpha)$, to be the set of alternating rotations at v fixed under conjugation by α. Given a disjoint cycle decomposition of α, let $l(v)$ be the *length* of the cycle containing v.

Theorem 1.4 *If $\alpha \in \operatorname{Aut} D$, then*

$$|F(\alpha)| = \prod_{v \in S} |F_v(\alpha^{l(v)})|,$$

where the product extends over a complete set S of orbit representatives for $\langle\alpha\rangle$ acting on $V(D)$.

Let ϕ be the *Euler totient function* and let $N(v)$ be the set of neighbors of v. Let $\alpha^{l(v)}|_{N(v)}$ be the restriction of $\alpha^{l(v)}$ to the set of neighbors of v, and we also assume that $|N(v)| = 2n$. The *cycle type* of a permutation of $2n$ elements is a $2n$-tuple whose k-th entry is the number of k-cycles present in the disjoint cycle representation of the permutation. If γ is the permutation, we write $j(\gamma)$ for the $2n$-tuple and j_k for

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press

the k-th entry. A *d-uniform permutation* is a permutation in which every cycle in the disjoint cycle decomposition has the same length d. The following theorem is slightly different from the result of Mull, Rieper, and White. The proof is similar to [13], but, for completeness, we give a detailed proof.

Theorem 1.5 *If the vertices connected to v via in-arcs and the vertices connected to v via out-arcs are in different cycles of α, then*

$$|F_v(\alpha^{l(v)})| = \begin{cases} \phi(d)(\frac{n}{d})!(\frac{n}{d}-1)!d^{\frac{2n}{d}-1} & \text{if } \alpha^{l(v)}|_{N(v)} \text{ is } d\text{-uniform,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let the permutation γ of $N(v)$ (the restriction of $\alpha^{l(v)}$ to $N(v)$) be a $2n$-cycle with the property that the vertices connected to v via in-arcs and the vertices connected to v via out-arcs belong to different cycles of γ. Let $\rho = (x_1y_1x_2y_2\dots x_ny_n)$ be an alternating rotation at a vertex v such that x_i and y_i, for $1 \le i \le n$, are vertices connected to an in-arc and an out-arc of v, respectively. Suppose that γ satisfies $\gamma\rho\gamma^{-1} = \rho$. If x_{k+1} is the image of x_1 under γ, then $\gamma(x_{i+1}) = \gamma\rho^{2i}(x_1) = \rho^{2i}\gamma(x_1) = \rho^{2i}(x_{k+1}) = \rho^{2i}\rho^{2k}(x_1) = \rho^{2k}(x_{i+1})$; similarly, if y_{k+1} is the image of y_1 under γ, we have $\gamma(y_{i+1}) = \rho^{2k}(y_{i+1})$, for $i = 0, 1, \dots, n-1$. Thus, $\gamma = \rho^{2k}$. Furthermore γ is d-uniform, where $d = 2n/(\gcd(2k, 2n)) = n/(\gcd(k, n))$, else it fixes no $2n$-cycle under conjugation.

Let C be the set of $n!(n-1)!$ alternating cyclic permutation of $N(v)$. Since the vertices connected to v via in-arcs and the vertices connected to v via out-arcs are in different cycles of γ, the number of such γ whose cycle type is d-uniform equals $(n!/(d^{\frac{n}{d}}(\frac{n}{d})!))^2$. Let J be the set of such $(n!/(d^{\frac{n}{d}}(\frac{n}{d})!))^2$ permutations. For each $\gamma \in J$, let T_γ denote those members of C fixed by γ under conjugation and each element of J is the same as any other, so that each T_γ has the same cardinality. Denote this common value by t. Note that each $2n$-cycle of C is fixed under conjugation by precisely $\phi(d)$ $(d \le n)$ members of J, and hence

$$\sum_{\gamma \in J} |T_\gamma| = t|J| = \phi(d) \cdot |C|. \tag{1.1}$$

By equation (1.1), we have

$$t = \frac{\phi(d) \cdot |C|}{|J|} = \frac{\phi(d)n!(n-1)!}{(n!/(d^{\frac{n}{d}}(\frac{n}{d})!))^2}.$$

By a simple calculation, we have the desired result. ∎

Example 1.6 Let $\alpha = (xy)(x_1y_1x_2y_2x_3y_3x_4y_4)(x_5y_5x_6y_6x_7y_7x_8y_8)$; then α is an automorphism of Aut OD_8 (the directed dipole graph OD_{2n} will be defined in the next section). We have $\alpha^2 = (x)(y)(x_1x_2x_3x_4)(x_5x_6x_7x_8)(y_1y_2y_3y_4)(y_5y_6y_7y_8)$. This means that $\alpha^2|_{N(x)}$ is 4-uniform. By Theorem 1.5,

$$|F_x(\alpha^2)| = \phi(4)\Big(\frac{8}{4}\Big)!\Big(\frac{8}{4}-1\Big)!4^{\frac{2\cdot 4}{4}-1} = 2 \cdot 2!1! \cdot 4 = 16.$$

2 Unlabelled Embeddings for Directed Dipoles

A *dipole graph* D_n is a multigraph consisting of two vertices connected with n parallel edges. Given a dipole graph D_{2n}, there is a unique Eulerian orientation of D_{2n}, denoted by OD_{2n}. We call it *directed dipole graph*. In [5], Feng, Kwak, and Zhou calculated unlabelled embeddings of D_n, here we give our attention to unlabelled directed embeddings of OD_{2n}. A *subdivision* of a digraph D is a digraph resulting from the subdivision of arcs in D. Given a directed embedding of a non-simple digraph, and then the embedding can be subdivided. Recall that the embedding of the non-simple digraph and its subdivision are homeomorphic to each other; this means that we only need to consider its subdivision.

Let the vertex set of OD_{2n} be $\{x, y\}$ and the arcs of OD_{2n} be $\overrightarrow{e_i} = \overrightarrow{xy}$ and $\overleftarrow{f_i} = \overleftarrow{xy}$, for $i = 1, 2, \ldots, n$. Now we subdivide the arcs $\overrightarrow{e_i} = \overrightarrow{xy}$ and $\overleftarrow{f_i} = \overleftarrow{xy}$ to form two new arcs $\overrightarrow{xx_i}$, $\overrightarrow{x_iy}$ and $\overleftarrow{yy_i}$, $\overleftarrow{y_ix}$, respectively, as shown in Figure 1. In the following discussion, the automorphism group of the subdivided graph OD_{2n} is denoted by Aut OD_{2n}. We have the following theorem for the automorphism group of the subdivided graph OD_{2n}.

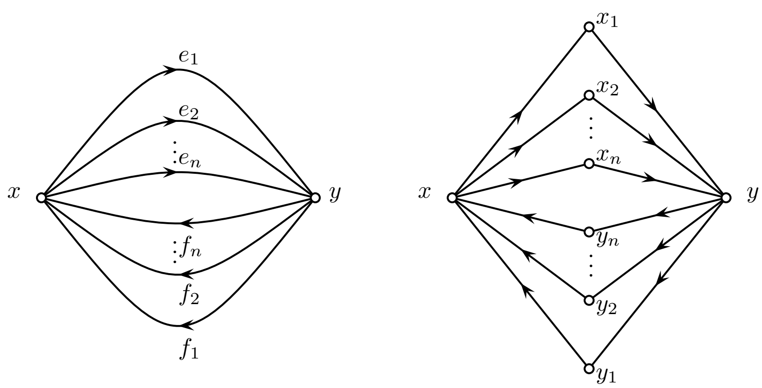

Figure 1: The directed dipole graph OD_{2n} and its subdivision

Theorem 2.1 *Let $V_1 = \{x_1, x_2, \ldots, x_n\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$. Suppose that $n \geq 2$; then every $\alpha \in$ Aut OD_{2n} can be expressed in one of the following forms:*

(i) *$\alpha = (x)(y)\sigma\tau$, where σ is a permutation on V_1 and τ is a permutation on V_2;*

(ii) *$\alpha = (xy)\alpha_1$, where α_1 is a permutation of the form*

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ x_{j_1} & x_{j_2} & \cdots & x_{j_n} \end{pmatrix}.$$

Proof For any $\alpha \in$ Aut OD_{2n}, α must fix x and y, or send x to y and y to x. In the former case, $\alpha = (x)(y)\sigma\tau$, where σ is a permutation on V_1 and τ is a permutation on V_2. In the latter case, $\alpha = (xy)\alpha_1$, where α_1 is a permutation of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ x_{j_1} & x_{j_2} & \cdots & x_{j_n} \end{pmatrix}.$$

The result follows. ■

By Theorem 2.1, $|\operatorname{Aut} OD_{2n}| = 2(n!)^2$.

Lemma 2.2 *Let $V_1 = \{x_1, x_2, \ldots, x_n\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$. Let $\alpha = (x)(y)\sigma\tau$, where σ is a permutation on V_1 and τ is a permutation on V_2. Suppose every cycle in $\sigma\tau$ is of length d; then the number of members in the conjugacy class of $\sigma\tau$ equals*

$$\Big(\frac{n!}{d^{\frac{n}{d}}(\frac{n}{d})!}\Big)^2.$$

Proof From Cauchy's formula, the result follows. ■

Lemma 2.3 *Let $V_1 = \{x_1, x_2, \ldots, x_n\}$ and let $V_2 = \{y_1, y_2, \ldots, y_n\}$. Suppose α_1 is a permutation of the form*

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ x_{j_1} & x_{j_2} & \cdots & x_{j_n} \end{pmatrix}.$$

Let $\alpha = (xy)\alpha_1$, where α_1^2 is a permutation of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{t_1} & x_{t_2} & \cdots & x_{t_n} \end{pmatrix}\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_{t_1} & y_{t_2} & \cdots & y_{t_n} \end{pmatrix}.$$

Suppose every cycle in α_1^2 of length d; then the number of members in the conjugacy class of α_1 equals

$$\frac{(n!)^2}{d^{\frac{n}{d}}(\frac{n}{d})!}.$$

Proof Since α_1^2 is d-uniform, we set $\alpha_1^2 = (x_{i_1}x_{i_2}\cdots x_{i_d})(y_{j_1}y_{j_2}\cdots y_{j_d})\cdots$. Then α_1 is $2d$-uniform and has the form $(x_{i_1}y_{j_1}x_{i_2}y_{j_2}\cdots x_{i_d}y_{j_d})\cdots$. To count the contribution from such an α_1, we think of α_1 as being constructed from pairs xy, where $x \in V_1$ and $y \in V_2$. This constructs a bijection $h\colon V_1 \to V_2$, and there are $n!$ such bijections. From Cauchy's formula, there are $n!/(d^{\frac{n}{d}}(\frac{n}{d})!)$ such types (we consider the pair $x_{i_k}y_{j_k}$ as an element in α_1). Thus, the number of members in the conjugacy class of α_1 equals

$$n!\frac{n!}{d^{\frac{n}{d}}(\frac{n}{d})!}.$$

The result follows. ■

Theorem 2.4 *The number of inequivalent unlabelled embeddings of OD_{2n} equals*

$$|C(OD_{2n})| = \frac{1}{2}\bigg(\sum_{d|n}\Big(\phi(d)\Big(\frac{n}{d}-1\Big)!d^{\frac{n}{d}-1}\Big)^2 + \sum_{d|n}\phi(d)\Big(\frac{n}{d}-1\Big)!d^{\frac{n}{d}-1}\bigg).$$

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press

Proof The proof has two cases.

Case 1. When $\alpha = (x)(y)\sigma\tau$, $|F(\alpha)| \neq 0$ if and only if $|F_v(\alpha^{l(v)})| \neq 0$, for all $v \in V(OD_{2n})$ by Theorem 1.4. By Theorem 1.5, $|F_x(\alpha)| \neq 0$ if and only if $\sigma\tau|_{N(x)}$ is d-uniform, where d depends on x. Therefore, we have

$$|F_x(\alpha)| = \phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}.$$

Similarly, we have

$$|F_y(\alpha)| = \phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}.$$

Now, we suppose that a vertex v other than x and y, is in an orbit whose cycle length is $l(v) = d$. Since the identity permutation fixes all of the $\text{out}(v)!(\text{out}(v)-1)!$ rotations, thus $|F_v(\alpha^d)| = |F_v(e)| = \text{out}(v)!(\text{in}(v)-1)! = 1!0! = 1$. By Theorem 1.4,

$$|F(\alpha)| = \Big(\phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}\Big)^2.$$

Case 2. When $\alpha = (xy)\alpha_1$, where α_1 is a permutation of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ x_{j1} & x_{j2} & \dots & x_{jn} \end{pmatrix}.$$

By Theorem 1.4, $|F(\alpha)| \neq 0$ if and only if $|F_v(\alpha^{l(v)})| \neq 0$, for all $v \in V(OD_{2n})$. Recall that the vertices x, y are in the same cycle (xy), we choose x as orbit representative. By Theorem 1.5, $|F_x(\alpha^2)| \neq 0$ if and only if $\alpha_1^2|_{N(x)}$ is d-uniform, where d depends on x; therefore,

$$|F_x(\alpha)| = \phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}.$$

Now we suppose the vertex v other than x and y, is in an orbit whose cycle length is $l(x) = 2d$, so $|F_v(\alpha^{2d})| = |F_v(e)| = 1$. By Theorem 1.4,

$$|F(\alpha)| = \phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}.$$

In all, by Theorem 1.2 and Lemmas 2.2 and 2.3,

$$\begin{aligned}|C(OD_{2n})| = &\frac{1}{2(n!)^2}\sum_{d|n}\Big(\phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}\frac{n!}{d^{\frac{n}{d}}(\frac{n}{d})!}\Big)^2\\ &+\frac{1}{2(n!)^2}\sum_{d|n}\phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}\frac{(n!)^2}{d^{\frac{n}{d}}(\frac{n}{d})!}.\end{aligned}$$

The result follows. ∎

Table 1 shows a picture for the values of $|C(OD_{2n})|$ when $n \leq 9$.

Table 1:

n	1	2	3	4	5	6	7	8	9
$\lvert C(OD_{2n})\rvert$	1	2	6	27	310	7320	259581	12704542	812872047

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press

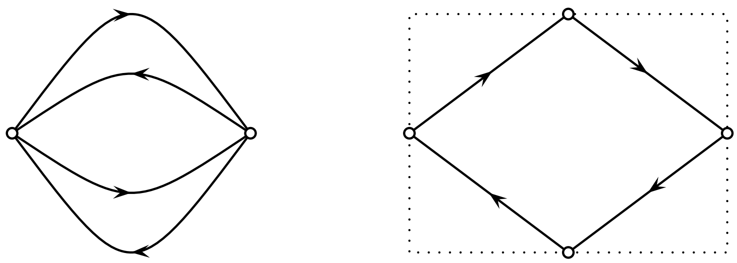

Figure 2: Two unlabelled embeddings of OD_4

Table 2:

Class	Alternating rotation systems
1	$x:(x_1,y_1,x_2,y_2,x_3,y_3),\ \ y:(y_1,x_1,y_2,x_2,y_3,x_3),$ $x_i:(xy),\ \ y_i:(xy),\ i=1,2,3$
2	$x:(x_1,y_1,x_2,y_2,x_3,y_3),\ \ y:(y_1,x_1,y_2,x_3,y_3,x_2),$ $x_i:(xy),\ \ y_i:(xy),\ i=1,2,3$
3	$x:(x_1,y_1,x_2,y_2,x_3,y_3),\ \ y:(y_1,x_1,y_3,x_3,y_2,x_2),$ $x_i:(xy),\ \ y_i:(xy),\ i=1,2,3$
4	$x:(x_1,y_1,x_2,y_2,x_3,y_3),\ \ y:(y_1,x_2,y_3,x_1,y_2,x_3),$ $x_i:(xy),\ \ y_i:(xy),\ i=1,2,3$
5	$x:(x_1,y_1,x_2,y_2,x_3,y_3),\ \ y:(y_1,x_2,y_2,x_3,y_3,x_1),$ $x_i:(xy),\ \ y_i:(xy),\ i=1,2,3$
6	$x:(x_1,y_1,x_2,y_2,x_3,y_3),\ \ y:(y_1,x_3,y_2,x_1,y_3,x_2),$ $x_i:(xy),\ \ y_i:(xy),\ i=1,2,3$

Figure 2 and Table 2 show two unlabelled embeddings of OD_4 and six unlabelled embeddings of OD_6, respectively.

2.1 Asymptotic Behavior

Theorem 2.5 $|C(OD_{2n})| \sim \dfrac{|R(OD_{2n})|}{|\mathrm{Aut}(OD_{2n})|}.$

Proof Define

$$g=\lim_{n\to\infty}\frac{\sum_{d|n}(\phi(d)(\frac{n}{d}-1)!d^{\frac{n}{d}-1})^2}{(n-1)!^2},$$
$$h=\lim_{n\to\infty}\frac{\sum_{d|n}\phi(d)(\frac{n}{d}-1)!d^{\frac{n}{d}-1}}{(n-1)!^2}.$$

Recall that $\frac{|R(OD_{2n})|}{|\operatorname{Aut} OD_{2n}|} = \frac{n!^2(n-1)!^2}{2n!^2}$. By Theorem 2.4,

$$\begin{aligned}\lim_{n\to\infty} \frac{|C(OD_{2n})|}{\frac{|R(OD_{2n})|}{|\operatorname{Aut} OD_{2n}|}} &= \lim_{n\to\infty} \frac{|C(OD_{2n})|}{n!^2(n-1)!^2/2n!^2}\\ &= \lim_{n\to\infty} \frac{(\sum_{d|n}(\phi(d)(\frac{n}{d}-1)!d^{\frac{n}{d}-1})^2 + \sum_{d|n}\phi(d)(\frac{n}{d}-1)!d^{\frac{n}{d}-1})}{(n-1)!^2}\\ &= g+h.\end{aligned}$$

We have

$$\begin{aligned} g+h &\geq \lim_{n\to\infty} \frac{\phi(1)(n-1)!^2 + \phi(1)(n-1)!}{(n-1)!^2} = 1,\\ g &\leq \lim_{n\to\infty} \frac{\phi(1)(n-1)!^2 + (n-1)^2(\frac{n}{2}-1)!^2(2^{\frac{n}{2}-1})^2}{(n-1)!^2}\\ &= \lim_{n\to\infty} \frac{(n-1)!^2}{(n-1)!^2} + \lim_{n\to\infty} \frac{(\frac{n}{2}-1)!^2(2^{\frac{n}{2}-1})^2}{(n-2)!^2}\\ &= 1+0 = 1\end{aligned}$$

and

$$\begin{aligned} h &\leq \lim_{n\to\infty} \frac{\phi(1)(n-1)! + (n-1)(\frac{n}{2}-1)!2^{\frac{n}{2}-1}}{(n-1)!^2}\\ &= \lim_{n\to\infty} \frac{(n-1)!}{(n-1)!^2} + \lim_{n\to\infty} \frac{(n-1)(\frac{n}{2}-1)!2^{\frac{n}{2}-1}}{(n-1)!^2}\\ &= 0+0 = 0.\end{aligned}$$

Thus,

$$g+h = \lim_{n\to\infty} \frac{|C(OD_{2n})|}{\frac{|R(OD_{2n})|}{|\operatorname{Aut} OD_{2n}|}} = 1.$$

The result follows. ■

3 Unlabelled Embeddings for a Bouquet of Directed Circles

A *bouquet of directed circles* is a digraph obtained by gluing together a collection of directed loops at a single point. A bouquet of directed circles digraph containing n loops $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}\}$ is denoted by B_n. We subdivide each arc $\overrightarrow{a_i}$ of B_n with two new vertices x_i and y_i, for $i = 1, 2, \ldots, n$, the resulted digraph is simple, as shown in Figure 3. Recall that Feng, Kwak, and Zhou [5] counted unlabelled embeddings for bouquets of circles. We shall see that the method here is different from that of [5], and we calculate the automorphism group of the subdivision graph of B_n. We will denote $\operatorname{Aut} B_n$ as the automorphism group of the subdivided graph B_n.

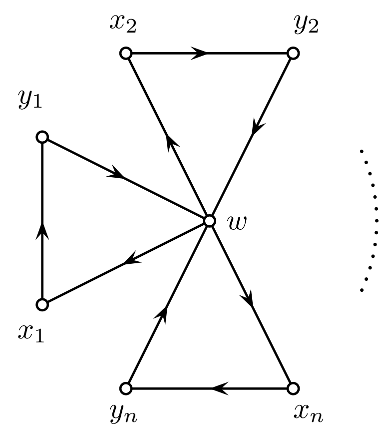

Figure 3: The subdivision of B_n

Theorem 3.1 *Suppose that $n > 1$. Let $V_1 = \{x_1, x_2, \ldots, x_n\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$. Then every $\alpha \in \operatorname{Aut} B_n$ can be expressed as $\alpha = (w)\alpha_1$, where α_1 is a permutation of the form*

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}.$$

Proof In the subdivided graph B_n, the degree of w is $2n$ and $d(x_i) = d(y_i) = 2$, for $i = 1, 2, \ldots, n$. Let $\alpha \in \operatorname{Aut} B_n$; then α must fix w. In order to preserve the adjacency of two vertices x_i and y_i, for $i = 1, 2, \ldots, n$, α must send y_i to y_j if α send x_i to x_j, where $1 \leq i \neq j \leq n$. Thus, $\alpha = (w)\alpha_1$, where α_1 is a permutation of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}. \qquad \blacksquare$$

By Theorem 3.1, we have $|\operatorname{Aut} B_n| = n!$. The following lemma follows directly from Cauchy's formula.

Lemma 3.2 *Let $\alpha = (w)\alpha_1$, where α_1 is a permutation of the form*

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_{i_1} & y_{i_2} & \cdots & y_{i_n} \end{pmatrix}.$$

Suppose every cycle in α_1 of length d; then the number of member in the conjugacy class of α_1 is

$$\frac{n!}{d^{\frac{n}{d}}(\frac{n}{d})!}.$$

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press

Theorem 3.3 *The number of inequivalent unlabelled embeddings of B_n equals*

$$|C(B_n)| = \sum_{d|n} \phi(d)\Big(\frac{n}{d}-1\Big)!d^{\frac{n}{d}-1}.$$

Proof Let $\alpha = (w)\alpha_1$. By Theorem 1.4, $|F(\alpha)| \neq 0$ if and only if $|F_v(\alpha^{l(v)})| \neq 0$, for all $v \in V(B_{2n})$. Let v be the central vertex w. By Theorem 1.5, $|F_w(\alpha)| \neq 0$ if and only if $\alpha_1|_{N(w)}$ is d-uniform. By Theorem 1.5,

$$|F_w(\alpha)| = \phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}.$$

If the vertex v is any other vertex, then $l(v) = d$, so $|F_v(\alpha^d)| = |F_v(e)|$. Since the identity permutation fixes all of the $\text{out}(v)!(\text{out}(v)-1)!$ rotations, we have $|F_v(\alpha^d)| = 1$, by Theorem 1.4,

$$|F(\alpha)| = \phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}.$$

By Theorem 1.2 and Lemma 3.2,

$$|C(B_n)| = \frac{1}{n!}\sum_{d|n}\phi(d)\Big(\frac{n}{d}\Big)!\Big(\frac{n}{d}-1\Big)!d^{\frac{2n}{d}-1}\frac{n!}{d^{\frac{n}{d}}(\frac{n}{d})!},$$

which simplifies to the desired result. ■

We list some values of $|C(B_n)|$ for $n = 1, 2, \ldots, 10$.

Table 3:

n	1	2	3	4	5	6	7	8	9	10
$\lvert C(B_n)\rvert$	1	2	4	10	28	136	726	5100	40362	363288

Figure 4 shows four unlabelled embeddings of B_3. One can see the first and the third embeddings of Figure 4 are different, since the third embedding cannot be obtained by any $\alpha \in \operatorname{Aut} B_3$ acting on the first embedding.

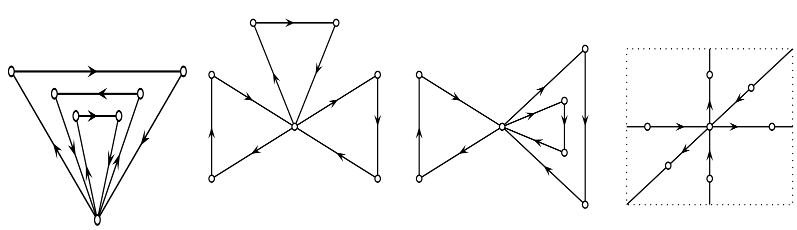

Figure 4: Four unlabelled embeddings of B_3

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press

3.1 Asymptotic Behavior

Theorem 3.4 $|C(B_n)| \sim \dfrac{|R(B_n)|}{|\operatorname{Aut}(B_n)|}$.

Proof By Theorem 3.3, we have

$$f = \lim_{n\to\infty} \frac{|C(B_n)|}{\frac{|R(B_n)|}{|\operatorname{Aut}(B_n)|}} = \lim_{n\to\infty} \frac{|C(B_n)|}{(n-1)!} \geq \lim_{n\to\infty} \frac{\phi(1)(n-1)!}{(n-1)!} = 1,$$

and

$$f \leq \lim_{n\to\infty} \frac{\phi(1)(n-1)! + (n-1)(\frac{n}{2}-1)!2^{\frac{n}{2}-1}}{(n-1)!}$$
$$= \lim_{n\to\infty} \frac{(n-1)!}{(n-1)!} + \lim_{n\to\infty} \frac{(\frac{n}{2}-1)!2^{\frac{n}{2}-1}}{(n-2)!} = 1 + 0 = 1.$$

Combining this with the discussion above, the result follows. ■

4 Unlabelled Embeddings for a Class of Regular Tournaments

A tournament is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. A tournament of odd order $2n+1$ is regular if the out-degree of each vertex is n. We observe that they are many ways to assign directions to the edges of a complete graph to get a tournament. Although the number of regular tournaments on 5 vertices is one, there is more than one regular tournament with n vertices for $n = 7, 9, 11, \ldots$. Let T_{2n+1} be the regular tournament on vertices labeled $x_1, x_2, \ldots, x_{2n+1}$, with arcs $\overrightarrow{x_i x_{i+1}}, \overrightarrow{x_i x_{i+2}}, \ldots, \overrightarrow{x_i x_{i+n}}$, for all $i = 1, 2, \ldots, 2n+1$, with subtraction modulo $2n+1$. A drawing of T_5 is shown in Figure 5.

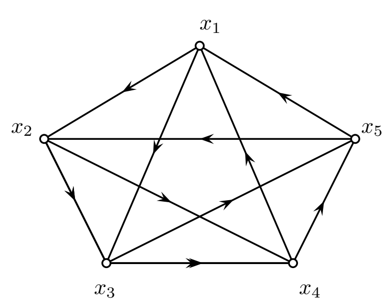

Figure 5: The regular tournament T_5

Theorem 4.1 *Every* $\alpha \in \operatorname{Aut} T_{2n+1}$ *can be expressed as* $\alpha = (x_1, x_2, \ldots, x_{2n+1})^k$, *where* $1 \leq k \leq 2n+1$.

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press

Proof From the definition of T_{2n+1}, it is routine to check that Aut T_{2n+1} is isomorphic to a cyclic group of order $2n+1$. The result follows. ■

Theorem 4.2 *The number of inequivalent unlabelled embeddings of T_{2n+1} equals*

$$|C(T_{2n+1})| = \frac{1}{2n+1} \sum_{d|(2n+1)} \phi(d)\big(n!(n-1)!\big)^{\frac{2n+1}{d}}.$$

Proof Since $\alpha = (x_1, x_2, \dots, x_{2n+1})^k$, where $1 \le k \le 2n+1$, it follows that α is a uniform permutation. Suppose α is d-uniform; then $|F_v(\alpha^d)| \ne 0$, for any $v \in V(T_{2n+1})$. Since the identity permutation fixes all of the $n!(n-1)!$ alternating rotation system; thus, $|F_v(\alpha^{l(v)})| = |F_v(\alpha^d)| = |F_v(e)| = n!(n-1)!$. Note that the number of cycles in α is $(2n+1)/d$, so by Theorem 1.4, we have $|F(\alpha)| = (n!(n-1)!)^{\frac{2n+1}{d}}$. There are $\phi(d)$ such α. By Theorem 1.2, we have

$$|C(T_{2n+1})| = \frac{1}{2n+1} \sum_{d|(2n+1)} \phi(d)\big(n!(n-1)!\big)^{\frac{2n+1}{d}}.$$ ■

Table 4 lists some values of $|C(T_n)|$ for $n = 3, 5, 7, 9$. Representatives of the eight classes of $C(T_5)$ are detailed in Table 5.

Table 4:

n	3	5	7	9
$\lvert C(T_n)\rvert$	1	8	5118840	295810000

Table 5:

Class	Alternating rotation systems
1	$x_1 : (x_2, x_4, x_3, x_5)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_1, x_5, x_2)$ $x_4 : (x_5, x_2, x_1, x_3)$, $x_5 : (x_1, x_3, x_2, x_4)$
2	$x_1 : (x_2, x_4, x_3, x_5)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_1, x_5, x_2)$ $x_4 : (x_5, x_3, x_1, x_2)$, $x_5 : (x_1, x_3, x_2, x_4)$
3	$x_1 : (x_2, x_5, x_3, x_4)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_1, x_5, x_2)$ $x_4 : (x_5, x_3, x_1, x_2)$, $x_5 : (x_1, x_3, x_2, x_4)$
4	$x_1 : (x_2, x_4, x_3, x_5)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_2, x_5, x_1)$ $x_4 : (x_5, x_2, x_1, x_3)$, $x_5 : (x_1, x_3, x_2, x_4)$
5	$x_1 : (x_2, x_4, x_3, x_5)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_2, x_5, x_1)$ $x_4 : (x_5, x_3, x_1, x_2)$, $x_5 : (x_1, x_3, x_2, x_4)$
6	$x_1 : (x_2, x_5, x_3, x_4)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_2, x_5, x_1)$ $x_4 : (x_5, x_3, x_1, x_2)$, $x_5 : (x_1, x_3, x_2, x_4)$
7	$x_1 : (x_2, x_4, x_3, x_5)$, $x_2 : (x_3, x_5, x_4, x_1)$, $x_3 : (x_4, x_1, x_5, x_2)$ $x_4 : (x_5, x_2, x_1, x_3)$, $x_5 : (x_1, x_3, x_2, x_4)$
8	$x_1 : (x_2, x_5, x_3, x_4)$, $x_2 : (x_3, x_1, x_4, x_5)$, $x_3 : (x_4, x_2, x_5, x_1)$ $x_4 : (x_5, x_3, x_1, x_2)$, $x_5 : (x_1, x_4, x_2, x_3)$

4.1 Asymptotic Behavior

Theorem 4.3 $|C(T_{2n+1})| \sim \dfrac{|R(T_{2n+1})|}{|\operatorname{Aut} T_{2n+1}|}$.

Proof We have

$$f = \lim_{n\to\infty} \frac{|C(T_{2n+1})|}{\frac{|R(T_{2n+1})|}{|\operatorname{Aut} T_{2n+1}|}} = \lim_{n\to\infty} \frac{|C(T_{2n+1})|}{(n!(n-1)!)^{2n+1}/(2n+1)}$$
$$= \lim_{n\to\infty} \frac{\frac{1}{2n+1}\left(\sum_{d|n} \phi(d)(n!(n-1)!)^{\frac{2n+1}{d}}\right)}{(n!(n-1)!)^{2n+1}/2n+1} = \lim_{n\to\infty} \frac{\sum_{d|n} \phi(d)(n!(n-1)!)^{\frac{2n+1}{d}}}{(n!(n-1)!)^{2n+1}}$$
$$\geq \lim_{n\to\infty} \frac{\phi(1)(n!(n-1)!)^{2n+1}}{(n!(n-1)!)^{2n+1}} = 1$$

and

$$f = \lim_{n\to\infty} \frac{\sum_{d|n} \phi(d)(n!(n-1)!)^{\frac{2n+1}{d}}}{(n!(n-1)!)^{2n+1}}$$
$$\leq \lim_{n\to\infty} \frac{\phi(1)(n!(n-1)!)^{2n+1} + 2n\cdot(n!(n-1)!)^{\frac{2n+1}{2}}}{(n!(n-1)!)^{2n+1}} = 1 + 0 = 1,$$

so $f = 1$. The result follows. ∎

Note that there are many ways to assign directions to the edges of a complete graph K_{2n+1} ($n \geq 3$), so as to obtain an Eulerian digraph. For example, McKay [10] gives asymptotic numbers of regular tournaments. It seems that the classification for the automorphism group of all regular tournaments is not a easy task [11]. Let T be any regular tournament with $2n+1$ vertices, we pose the following problem.

Problem 4.4 Calculate the number of unlabelled embeddings for any regular tournament T. Does it hold that

$$\lim_{n\to\infty} \frac{|C(T)|}{(n!(n-1)!)^{2n+1}/|\operatorname{Aut}(T)|} = 1?$$

Acknowledgment We are grateful to the two anonymous referees for some very helpful comments.

References

[1] K. Bonetta-Martin and T. A. McCourt, *On which groups can arise as the canonical group of a spherical latin bitrade.* Ars Math. Contemp. **13**(2017), 167–185.

[2] C. P. Bonnington, M. Conder, M. Morton, and P. McKenna, *Embedding digraphs on orientable surfaces.* J. Combin. Theory Ser. B **85**(2002), 1–20. http://dx.doi.org/10.1006/jctb.2001.2085

[3] Y. Chen, J. L. Gross, and X. Hu, *Enumeration of digraph embeddings.* European J. Combin. **36**(2014), 660–678. http://dx.doi.org/10.1016/j.ejc.2013.10.003

[4] G. E. Farr, *Minors for alternating dimaps.* arxiv:1311.2783

[5] Y.-Q. Feng, J.-H. Kwak, and J. Zhou, *Congruence classes of orientable 2-cell embeddings of bouquets of circles and dipoles.* Electron. J. Combin. **17**(2010), Research Paper 41, 15.

[6] J. H. Kwak and J. Lee, *Enumeration of graph embeddings.* Discrete Math. **135**(1994), 129–151. http://dx.doi.org/10.1016/0012-365X(93)E0075-F

[7] J. H. Kim and Y. K. Park, *Incongruent embeddings of a bouquet into surfaces.* Bull. Austral. Math. Soc. **61**(2000), 89–96. http://dx.doi.org/10.1017/S0004972700022048
[8] J. H. Kim, H. K. Kim, and D. K. Lim, *Enumerating embeddings of a dartboard graph into surfaces.* Comm. Korean Math. Soc. **11**(1996), 1095–1104.
[9] T. A. McCourt, *Growth rates of groups associated with face 2-coloured triangulations and directed eulerian digraphs on the sphere.* Electron. J. Combin. **23**(2016), no. 1, Paper 1.51, 23.
[10] B. D. McKay, *The asymptotic numbers of regular tournaments, Eulerian digraphs and Eulerian oriented graphs.* Combinatorica **10**(1990), 367–377. http://dx.doi.org/10.1007/BF02128671
[11] J. W. Moon, *Tournaments with a given automorphism group.* Canad. J. Math. **16**(1964), 485–489. http://dx.doi.org/10.4153/CJM-1964-050-9
[12] B. M. Mull, *Enumerating the orientable 2-cell imbeddings of complete bipartite graphs.* J. Graph Theory **103**(1999), 77–90. http://dx.doi.org/10.1002/(SICI)1097-0118(199902)30:2<77::AID-JGT2>3.0.CO;2-W
[13] B. M. Mull, R. G. Rieper, and A. T. White, *Enumerating 2-cell imbeddings of connected graphs.* Proc. Amer. Math. Soc. **103**(1988), 321–330. http://dx.doi.org/10.2307/2047573
[14] W. T. Tutte, *The dissection of equilateral triangles into equilateral triangles.* Proc. Cambridge Phil. Soc. **44**(1948), 463–482. http://dx.doi.org/10.1017/S030500410002449X
[15] ———, *Duality and trinity. Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his 60th birthday), Vol. III.* Colloq. Math. Soc. János Bolyai, 10, North-Holland, Amsterdam, 1975, pp. 1459–1472.

Department of Mathematics, Hunan University, 410082 Changsha, China
e-mail: ycchen@hnu.edu.cn

Department of Mathematics, Hunan University, 410082 Changsha, China
e-mail: s12061017@hnu.edu.cn

Department of Mathematics, Normal University of Hunan, 410081 Changsha, China
e-mail: hyqq@hunnu.edu.cn

https://doi.org/10.4153/CMB-2017-050-9 Published online by Cambridge University Press