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CIRCLE MEANS OF GREEN'S FUNCTIONS

To Professor Yukinari Tôki on the occasion of his 60th birthday

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Consider the polar coordinate differentials $(dr,\ d\theta)$ on a hyperbolic Riemann surface R with center $z_0 \in R$ which are given by

$$(1)\qquad \begin{aligned} dr(z)/r(z) &= -dG_R(z, z_0)\,; \\ d\theta(z) &= -{}^*dG_R(z, z_0)\ , \end{aligned}$$

where $G_R(z, \zeta)$ is the Green's function on R with pole $\zeta \in R$. Under the initial condition $r(z_0) = 0$, dr is given as the differential of the global function on R:

$$(2)\qquad r(z) = e^{-G_R(z, z_0)}\ .$$

The range of $r(z)$ is $[0, 1)$ and we denote the level line of $r(z)$ by

$$(3)\qquad C_\rho = \{z \in R\,; r(z) = \rho\}\ (\rho \in (0, 1))\ .$$

The purpose of this paper is to establish the following circle mean formula which has applications in the Green potential theory on hyperbolic Riemann surfaces:

THEOREM. *The mean of the Green's function $G_R(z, \zeta)$ along the level line $C_\rho (\rho \in (0, 1))$ is given by*

$$(4)\qquad \frac{1}{2\pi}\int_{C_\rho} G_R(z, \zeta) d\theta(z) = -\max(\log \rho,\ \log r(\zeta))\ .$$

If R reduces to the unit disk $\boldsymbol{D} = \{|z| < 1\}$ and if we take $z_0 = 0$, then,

$$G_D(z, \zeta) = \log\left|\frac{1 - \bar{\zeta}z}{z - \zeta}\right|,\quad r(z) = |z|$$

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and (4) takes on the form

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{1-\bar{\zeta}\rho e^{i\theta}}{\rho e^{i\theta}-\zeta}\right|d\theta = -\max(\log\rho,\ \log|\zeta|)\ ,$$

which is useful in classical potential theory. This is just an easy consequence of the Gauss mean value theorem applied to $|z| < \rho$ or $|z| > \rho$ which is easily derived from the Green formula. The general case (4) can also be proven along the same line of idea but one difficulty arises from the fact that the "circle" C_ρ may not be compact whereas $C_\rho = \{|z| = \rho\}$ is always compact for the disk $\boldsymbol{D}$. The proof of our theorem will be given in nos. 4–6 after preparations in nos. 1–3.

1. We denote by Ω_ρ the set $\{z \in R;\ r(z) < \rho\}$ for each $\rho \in (0, 1)$. Since $\Omega_\rho = \{z \in R;\ G(z, z_0) > \log \rho^{-1}\}$, where we simply denote by $G(z, \zeta)$ the Green's function $G_R(z, \zeta)$ on R, by the maximum principle, Ω_ρ is a subregion of R, and clearly $\partial\Omega_\rho = C_\rho$. If ρ is sufficiently small, then Ω_ρ is simply connected and relatively compact. But in general $\bar{\Omega}_\rho$ may not be compact. However we have the following Kuramochi theorem [2] (and an alternative proof of it was given by Kusunoki-Mori [3]):

LEMMA. *The double $\hat{\Omega}_\rho$ of Ω_ρ along $C_\rho = \partial\Omega_\rho$ is parabolic.*

2. Let $\{R_n\}_0^\infty$ be a regular exhaustion of R such that $\bar{R}_0 \subset \Omega_\rho$. Consider the harmonic function w_n on $\Omega_\rho \cap R_n - \bar{R}_0$ with the continuous boundary values 0 on ∂R_0, 1 on $\partial R_n \cap \Omega_\rho$, and $^*dw_n = 0$ on $C_\rho \cap R_n$ except for an isolated set of points where $dr = d\theta = 0$. We also set $w_n = 0$ on $\bar{R}_0$ and $w_n = 1$ on $\Omega_\rho - R_n$. Then we have the

LEMMA. *The sequence $\{w_n\}_1^\infty$ satisfies*

$$(5)\qquad \lim_{n\to\infty} w_n = 0\ , \qquad \lim_{n\to\infty} D_{\Omega_\rho}(w_n) = 0\ .$$

This is a direct consequence of Lemma 1, and is actually equivalent to Lemma 1. In fact, let S_n be the double (not necessarily connected) of $\Omega_\rho \cap R_n$ along $C_\rho \cap R_n$. Then $\{S_n\}_0^\infty$ be an exhaustion of $\hat{\Omega}_\rho$. Let the harmonic measure of ∂S_n with respect to $S_n - \bar{S}_0$ be ω_n. Then $\omega_n | \Omega_\rho \cap R_n = w_n$ and the parabolicity of $\hat{\Omega}_\rho$, i.e. $\lim_{n\to\infty}\omega_n = 0$, is equivalent to $\lim_{n\to\infty} w_n = 0$. In this case Dirichlet integrals satisfy $2D_{\Omega_\rho \cap R_n}(w_n) = D_{S_n}(\omega_n)$ and, by the Fatou lemma, these converge to zero. For a direct proof of Lemma 2, we refer to Nakai [4].

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3. We denote by $H^C B(\Omega_\rho)$ the class of bounded harmonic functions on Ω_ρ which have continuous extensions to $\Omega_\rho \cup C_\rho$. Then we have the following Gauss mean value theorem (cf. Brelot-Choquet [1], Nakai [5]):

LEMMA. *Let u belong to the class $H^C B(\Omega_\rho)$. Then*

$$u(z_0) = \frac{1}{2\pi}\int_{C_\rho} u(z) d\theta(z) . \tag{6}$$

For a proof we refer to Sario-Nakai [6; p. 200]. In particular the total mass of the positive measure $d\theta$ on C_ρ is 2π:

$$\int_{C_\rho} d\theta(z) = 2\pi , \tag{7}$$

and therefore the left hand side of (4) may be called as a mean on C_ρ with respect to $d\theta$.

4. We are ready to prove (4). First assume that $\zeta \notin \Omega_\rho$, i.e. $r(\zeta) \geq \rho$. If $r(\zeta) > \rho$, then $G(\cdot, \zeta)$ belongs to $H^C B(\Omega_\rho)$ and an application of Lemma 3 with (2) yields

$$\begin{aligned} \frac{1}{2\pi}\int_{C_\rho} G(z, \zeta) d\theta(z) &= G(z_0, \zeta) = G(\zeta, z_0) \\ &= -\log r(\zeta) \\ &= -\max(\log \rho, \ \log r(\zeta)) . \end{aligned}$$

5. Next suppose that $r(\zeta) = \rho$, i.e. $\zeta \in C_\rho$. Let V be a parametric disk $|z - \zeta| < 1$ such that $\overline{V} = \{|z - \zeta| \leq 1\}$ and that $C_\rho \cap V$ consists of radii of V, and that there is no point in V with $dr = d\theta = 0$ except possibly for ζ. Explicitly, we take

$$(z - \zeta)^m = c((G(z, z_0) + iH(z, z_0)) - (G(\zeta, z_0) + iH(\zeta, z_0))) \tag{$*$}$$

as the local parameter at ζ, where H is a branch of the conjugate of G, m the order of ζ for $G + iH$, and c a suitable positive constant. Take a smaller disk $V_\eta = \{|z - \zeta| < \eta\}$ $(\eta < 1/2)$. Let $c(\eta)$ be the Harnack constant for the set $\overline{V}_\eta$ with respect to $V_{2\eta}$, i.e. $c(\eta)$ is the smallest number such that

$$u(w') \leq c(\eta) u(w) \tag{8}$$

for every pair of points w, w' in $\overline{V}_\eta$ and for every positive harmonic function u on $V_{2\eta}$. Set $k(\eta) = \max(c(\eta) - 1, \ 1 - c(\eta)^{-1})$. The Harnack theorem implies that

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$$\lim_{\eta\to 0} c(\eta) = 1\,, \qquad \lim_{\eta\to 0} k(\eta) = 0\,. \tag{9}$$

Since $w \to G(z, w)$ is positive and harmonic on $V_{2\eta}$ for any fixed $z \in C_\rho - V_{2\eta}$, (8) implies that

$$|G(z, \zeta) - G(z, w)| \leq k(\eta) G(z, w) \tag{10}$$

for any $w \in \overline{V}_\eta$ and $z \in C_\rho - V_{2\eta}$.

On the other hand, if $\eta < 1/4$ is sufficiently small, then

$$G(z, w) \leq -2 \log |z - w| \tag{11}$$

for all points z and w in $V_{2\eta}$. This follows from the fact that $u(z, w) = G(z, w) - \log(|1 - \overline{w}z|/|z - w|)$ is positive and harmonic in either one of each variables z and w in V if the other is arbitrarily fixed in V. Then, by the Harnack theorem, $u(z, w)$ is continuous on $V \times V$. Therefore $u(z, w)$ is bounded on $V_{2\eta} \times V_{2\eta}$ and (11) follows.

We consider those $\eta < 1/4$ for which (11) is valid. Let w be an arbitrary point in $V_\eta - \overline{\Omega}_\rho$. By (10) and (11) we have

$$\begin{aligned} &\left| \int_{C_\rho} G(z, w) d\theta(z) - \int_{C_\rho} G(z, \zeta) d\theta(z) \right| \\ &\leq \int_{C_\rho - V_{2\eta}} |G(z, w) - G(z, \zeta)|\, d\theta(z) \\ &+ \int_{C_\rho \cap V_{2\eta}} (G(z, w) + G(z, \zeta)) d\theta(z) \\ &\leq k(\eta) \int_{C_\rho - V_{2\eta}} G(z, w) d\theta(z) \\ &- 2 \int_{C_\rho \cap V_{2\eta}} (\log |z - w| + \log |z - \zeta|) d\theta(z)\,. \end{aligned} \tag{12}$$

Since $w \notin \overline{\Omega}_\rho$, i.e. $r(w) > \rho$, by no. 4, we deduce

$$\int_{C_\rho - V_{2\eta}} G(z,\, w) d\theta(z) < \int_{C_\rho} G(z, w) d\theta(z) = -\log r(w)\,. \tag{13}$$

Suppose that $C_\rho \cap V$ consists of $2m$ radii $r_1, \cdots, r_{2m}$. Clearly (cf. (*) in no. 5)

$$-\int_{C_\rho \cap V_{2\eta}} \log |z - \zeta|\, d\theta(z) = \mathcal{O}\left(-\int_0^{2\eta} \log t\, dt\right) = \mathcal{O}(\eta(1 - \log 2\eta))\,. \tag{14}$$

Let Γ_j be the diameter of V containing r_j. Denote by w_j the projection of w on Γ_j. Since $-\log |z - w| \leq -\log |z - w_j|$,

$$-\int_{r_j}\log|z-w|\,d\theta(z)\le-\int_{r_j}\log|z-w_j|\,d\theta(z)$$
$$=\mathcal{O}\left(-\int_{\pm|w_j-\zeta|}^{2\eta}\log|t|dt\right)=\mathcal{O}\left(-\int_0^{2\eta}\log t\,dt\right)$$

and therefore

$$-\int_{C_\rho\cap V_{2\eta}}\log|z-w|\,d\theta(z)=\mathcal{O}(\eta(1-\log 2\eta))\,. \tag{15}$$

From (12)–(15) it follows that

$$\left|\int_{C_\rho}G(z,\,w)d\theta(z)-\int_{C_\rho}G(z,\,\zeta)d\theta(z)\right|\le-k(\eta)\log r(w)+\mathcal{O}(\eta(1-\log 2\eta))\,.$$

By (9), this shows that the function $w\to\int_{C_\rho}G(z,w)d\theta(z)$ is continuous. By no. 4,

$$\frac{1}{2\pi}\int_{C_\rho}G(z,w)d\theta(z)=-\max(\log\rho,\ \log r(w))$$

for $w\notin\bar{\Omega}_\rho$, and on letting $w\to\zeta$, we conclude that (4) is true for any $\zeta\in C_\rho$. Thus far we have proven (4) for $\zeta\notin\Omega_\rho$.

6. The essential part of the proof is for the case $\zeta\in\Omega_\rho$, i.e. $r(\zeta)<\rho$. Take a regular exhaustion $\{R_n\}_0^\infty$ such that $z_0,\,\zeta\in R_0\subset\bar{R}_0\subset\Omega_\rho$. We denote by $g_n(z)$ the Green's function on $\Omega_\rho\cap R_n$ with pole z_0 and set $g_n=0$ on $\Omega_\rho-R_n$. Set $G_n(z)=g_n(z)-\log\rho$. Let $a(\varepsilon)$ and $b(\varepsilon)$ be the small disks with centers z_0 and ζ and with radii ε, respectively, such that $\overline{a(\varepsilon)\cup b(\varepsilon)}\subset R_0$ and that $\overline{a(\varepsilon)}\cap\overline{b(\varepsilon)}=\phi$ if $z_0\ne\zeta$ and $a(\varepsilon)=b(\varepsilon)$ if $z_0=\zeta$. The Green formula yields

$$\int_{\partial(\Omega_\rho\cap R_n)-\partial(a(\varepsilon)\cup b(\varepsilon))}(G(\cdot,\zeta)^*dg_n-g_n{}^*dG(\cdot,\zeta))$$
$$=\int_{\Omega_\rho\cap R_n-a(\varepsilon)\cup b(\varepsilon)}(G(z,\zeta)\Delta g_n(z)-g_n(z)\Delta G(z,\zeta))dxdy=0$$

and therefore

$$\begin{aligned}&\int_{\partial(\Omega_\rho\cap R_n)-\partial(a(\varepsilon)\cup b(\varepsilon))}G(\cdot,\zeta)^*dG_n\\&+\int_{\partial(a(\varepsilon)\cup b(\varepsilon))}G_n{}^*dG(\cdot,\zeta)\\&+\log\rho\cdot\int_{\partial(\Omega_\rho\cap R_n)}{}^*dG(\cdot,\zeta)=0\,.\end{aligned}\tag{16}$$

Since

$$\lim_{\varepsilon\to 0}\int_{-\partial(a(\varepsilon)\cup b(\varepsilon))} G(\cdot,\zeta)^*dG_n = 2\pi G(z_0,\zeta)$$

and

$$\lim_{\varepsilon\to 0}\int_{\partial(a(\varepsilon)\cup b(\varepsilon))} G_n{}^*dG(\cdot,\zeta) = -2\pi G_n(\zeta)\ ,$$

on letting $\varepsilon\to 0$ in (16) we therefore obtain

$$\frac{1}{2\pi}\int_{\partial(\Omega_\rho\cap R_n)} G(\cdot,\zeta)^*dG_n = \log\rho - (G(\zeta,a_0) - G_n(\zeta))\ , \tag{17}$$

where $G(\zeta,z_0) - G_n(\zeta)$ is understood to be zero if $\zeta = z_0$. Observe that

$$\int_{\partial(\Omega_\rho\cap R_n)} G(\cdot,\ \zeta)\,^*dG_n = \int_{C_\rho\cap R_n} G(\cdot,\ \zeta)\,^*dG_n + \int_{\Omega_\rho\cap\partial R_n} G(\cdot,\zeta)\,^*dG_n \tag{18}$$

and

$$G(z,z_0) = \lim_{n\to\infty} G_n(z) \tag{19}$$

increasingly on $\bar{\Omega}_\rho$. Let w_n be as in no. 2. By the Green formula

$$\left|\int_{\Omega_\rho\cap\partial R_n} {}^*dG_n + \int_{C_\rho\cap R_n} w_n{}^*dG_n\right|^2 = \left|\int_{\partial(\Omega_\rho\cap R_n - \bar{R}_0)} w_n{}^*dG_n\right|^2$$
$$\leq D_{\Omega_\rho-\bar{R}_0}(G_n)D_{\Omega_\rho}(w_n) \leq D_{\Omega_\rho-\bar{R}_0}(G(\cdot,z_0))D_{\Omega_\rho}(w_n) \to 0$$

as $n\to\infty$. On the other hand $0 \geq w_n{}^*dG_n \geq {}^*dG(\cdot,z_0)$ on $C_\rho\cap R_n$, $w_n{}^*dG_n \to 0$ on C_ρ, and by (7), $-{}^*dG(\cdot,z_0) = d\theta$ has the finite total mass 2π on C_ρ. Thus by the Lebesgue convergence theorem

$$\lim_{n\to\infty}\int_{C_\rho\cap R_n} w_n{}^*dG_n = 0$$

and therefore

$$\lim_{n\to\infty}\int_{\Omega_\rho\cap\partial R_n} {}^*dG_n = 0\ .$$

Since $G(\cdot,\zeta)$ is uniformly bounded on ∂R_n, we obtain

$$\lim_{n\to\infty}\int_{\Omega_\rho\cap\partial R_n} G(\cdot,\zeta)^*dG_n = 0\ . \tag{20}$$

Let $K = \sup_{C_\rho} G(\cdot,\zeta) < \infty$. Again $0 \geq G(\cdot,\zeta)^*dG_n \geq K^*dG(\cdot,z_0) = -Kd\theta$

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on $C_\rho \cap R_n$ and in view of (19) $G(\cdot,\zeta)^*dG_n$ converges to $G(\cdot,\zeta)^*dG(\cdot,z_0)$ $= -G(\cdot,\zeta)d\theta$. The Lebesgue convergence theorem yields

$$\lim_{n\to\infty}\int_{C_\rho\cap R_n} G(\cdot,\zeta)^*dG_n = -\int_{C_\rho} G(\cdot,\zeta)d\theta\ . \tag{21}$$

By (18)–(21), on letting $n\to\infty$ in (17), we deduce

$$\frac{1}{2\pi}\int_{C_\rho} G(z,\zeta)d\theta(z) = -\log\rho = -\max(\log\rho,\ \log r(\zeta))$$

and the proof of (4) is herewith complete.

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