Can. J. Math., Vol. XXIX, No. 6, 1977, pp. 1312–1339

COVERING LINKAGE INVARIANTS

RICHARD HARTLEY AND KUNIO MURASUGI

Let K be a knot in a manifold M. Corresponding to a representation of $\pi_1(M - K)$ into a transitive group of permutations there is a branched covering space $\tilde{M}$ of M. K is covered by $\tilde{K}$ which may be a link of several components. The set of linking numbers between the various components of $\tilde{K}$ has long been recognised as a useful knot invariant. Bankwitz and Schumann used this invariant in considering dihedral coverings of Viergeflechte. In this case $\tilde{M}$ is simply connected and the linking numbers can be computed without great difficulty [**1**]. More recently, Perko used this linking invariant in completing the list of amphicheiral knots in Reidemeister's table [**13**]. He used a geometrical method which is generally applicable, but requires considerable geometric intuition [**12**, p. 141]. There was an obvious need for a purely algebraic method of computing this "covering linkage" invariant, although Perko refers to the "apparent intractability" of the algebraic problem. The need was further highlighted by Riley's complaint that he did not know how to compute linking numbers [**16**, p. 613]. Furthermore, the work of Cappell and Shaneson indicates that these invariants may perhaps be applied to obtain a negative resolution of the Poincaré conjecture [**4**].

In this paper, a completely general and purely algebraic method is given for computing the "covering linkage" invariants corresponding to a given branched covering. The method is relatively simple and eminently suitable for computer calculations. Important in this method is the Reidemeister-Schreier algorithm for finding a presentation of a subgroup. Section 2 of this paper gives a new formalisation of this algorithm designed for easy application in the calculation of covering linkage invariants. For a different formulation of the Reidemeister-Schreier algorithm, the reader is referred to [**10**, § 2.3]. For theoretical study of covering linkage invariants the closely related concept of a linking function is introduced in § 4 and its use is demonstrated by the basic Theorem 4.1.

Sections 5 and 7 complete the formalism necessary to calculate covering linkage invariants, Section 5 proving the existence of linking functions, and Section 7 showing how the method is applied by performing an explicit calculation.

The value of covering linkage invariants has been demonstrated previously by their power in distinguishing different knot-types. As a result of our basic method, however, we demonstrate that they bear a close relationship to many invariants previously considered in the literature of knot theory, and hence they are of considerable theoretical importance also. In Section 6 an invariant

Received January 14, 1977 and in revised form February 24, 1977. This research was partially supported by National Research Council of Canada grant No. A4034.

of Burde is considered, which stems from his consideration of representations of knot group in the group of motions of the plane [**3**]. The precise relationship of his invariant to covering linkage invariants is given in Theorem 6.3. It will be seen that Burde's invariant can be calculated immediately from the covering linkage invariants. Riley [**16**] initiated a study of representations of knot groups onto the groups PSL (2, p), which are particularly useful in studying knots with trivial Alexander polynomial. In Section 8 a simple formula is given for the linking number in an important class of such coverings. As a result, certain knots are shown to have property P.

In Section 9 an invariant is defined which is often simpler to use than covering linkage invariants, and which seems to be just as effective in distinguishing knot types. It has the added advantage of being defined in all cases, whereas the covering linkage invariants sometimes fail to be defined. Actually, these invariants are a generalisation of an invariant studied by Reyner [**15**], and can be interpreted as the homology groups of certain topological spaces obtained by performing surgery on the covering space branched over a knot. In the final section of this paper, it is shown that these generalised Reyner's invariants bear a close relation to covering linkage invariants when these are defined, and in fact can often be calculated directly from a knowledge of the linking numbers in the corresponding covering space.

We begin in Section 1 by defining the linking number and deriving an impo tant preliminary proposition.

1. Linking number in a manifold. The terminology used in this section is largely borrowed from Schubert [**18**]. Let M be a manifold and T a finite cellulation of M, that is, M is a cell-complex. If e_p is an (open) p-cell then $\bar{e}_p$, the closure of e_p in M is called a *closed p-cell*. A generator $\mathfrak{e}_p$ of $H_p(\bar{e}_p, \bar{e}_p - e_p)$ is known as an *oriented p-cell*, and to fix a generator $\mathfrak{e}_p$ is known as *fixing an orientation for e_p*. If an orientation is fixed for every cell of M, then one obtains a chain complex

$$\xrightarrow{\partial} C_q(M; Q) \xrightarrow{\partial} C_{q-1}(M; Q) \xrightarrow{\partial}$$

where $C_q(M; Q)$ is the free Q module generated by the oriented q-cells. Also there is a co-chain complex

$$\xleftarrow{\delta'} C^q(M; Q) \xleftarrow{\delta'} C^{q-1}(M; Q) \xleftarrow{\delta'}$$

where $C^q(M; Q) = \text{Hom}_Q\ (C_q(M; Q), Q)$. In future, the Q will be omitted from the notation, and all chain complexes and homology groups will be understood to have rational coefficients.

$C_q(M)$ is isomorphic to $C^q(M)$ by an isomorphism R such that $\langle \mathfrak{e}_q R, \mathfrak{e}_q' \rangle' = 1$ if $\mathfrak{e}_q = \mathfrak{e}_q'$, and 0 otherwise. Here $\mathfrak{e}_q$ and $\mathfrak{e}_q'$ are oriented q-cells, that is, free generators of $C_q(M)$, and $\langle\ ,\ \rangle'$ is the map from $C^q(M) \times C_q(M)$ to Q where $\langle u^q, v_q \rangle'$ is the value of the cochain u^q at v_q. One can then define a *geometric co-boundary operator* δ such that the following diagram commutes:

$$\begin{array}{ccccc} \xrightarrow{\delta} & C_q & \xrightarrow{\delta} & C_{q+1} & \longrightarrow \\ & \wr\!\!\mid R & & \wr\!\!\mid R & \\ \xrightarrow{\delta'} & C^q & \xrightarrow{\delta'} & C^{q+1} & \longrightarrow \end{array}$$

Also, an *inner product* $\langle\ ,\ \rangle : C_q \times C_q \to Q$ is defined by $\langle u_q, v_q \rangle = \langle u_q R, v_q \rangle'$.

Let $u_q = \sum_i r_i \mathfrak{e}_q{}^i$ and $v_q = \sum_i s_i \mathfrak{e}_q{}^i$ be q-chains in $C_q(M)$ expressed in terms of the generators $\{\mathfrak{e}_q{}^i\}$. Then $\langle u_q, v_q \rangle = \sum_i r_i s_i$, and the inner product is symmetric. Other simple properties are

(1.1) $\quad u_q = \sum_i \langle u_q, \mathfrak{e}_q{}^i \rangle \mathfrak{e}_q{}^i$ where the sum is over all generators of $C_q(M)$.

(1.2) $\quad \langle u_q \delta, v_{q+1} \rangle = \langle u_q, v_{q+1} \partial \rangle$.

This last formula follows from a similar property of $\langle\ ,\ \rangle'$, The inner product $\langle u_q, \mathfrak{e}_q \rangle$ is called the *degree* of the chain u_q over $\mathfrak{e}_q$. An n-manifold M with cellulation T and boundary ∂M is said to be *oriented* if an orientation is fixed for all the cells for T such that if $\{\mathfrak{e}_n{}^i\}$ are all the oriented n-cells, then $\sum_i \mathfrak{e}_n{}^i \partial$ is an $n - 1$ chain whose degree is zero over every oriented $(n - 1)$-cell $\mathfrak{e}_{n-1}$ such that e_{n-1} does not lie in the boundary of M. This is equivalent to saying that for every oriented $(n - 1)$-cell not in the boundary of M, $\mathfrak{e}_{n-1}{}^i \delta = \mathfrak{e}_n{}^j - \mathfrak{e}_n{}^k$ for some oriented n-cells $\mathfrak{e}_n{}^j$ and $\mathfrak{e}_n{}^k$.

Now let $T^{(-1)}$ be a triangulation of a closed 3-manifold M in which K is a 1-dimensional subcomplex and submanifold, not necessarily connected. Let T be the first barycentric subdivision of $T^{(-1)}$. Consider M and K as cell complexes, let M and K be oriented. Let T^* be the dual cellulation of M, and let h be the map which takes a q-simplex to its dual $(3 - q)$-cell. That is, if $T^{(1)}$ is the barycentric subdivision of T, and e_q is a q-simplex in T, then $e_q h$ is the union of all open simplexes of $T^{(1)}$ whose closure have non-zero intersection with the barycentre of e_q. The cells in the dual cellulation can be oriented and the map h extended to a map from $C_q(M)$ to $C_{3-q}{}^*(M)$, the Q-module generated by the oriented $(3 - q)$-cells of T^* in such a way that $\mathfrak{e}_q h$ is the oriented $(3 - q)$-cell whose carrier is $e_q h$, and such that the following diagram commutes

$$\begin{array}{ccccc} \xleftarrow{\delta} & C_q\ (M) & \xleftarrow{\delta} & C_{q-1}\ (M) & \xleftarrow{\delta} \\ & h\ \wr\!\!\mid & & h\ \wr\!\!\mid & \\ \xleftarrow{\partial^*} & C^*_{3-q}\ (M) & \xleftarrow{\partial^*} & C^*_{3-q+1}\ (M) & \xleftarrow{\partial^*} \end{array}$$

and hence h induces an isomorphism of $H^q(M)$ onto $H_{3-q}(M)$. Let $\langle\ ,\ \rangle^*$ and δ^* be the inner product and geometric co-boundary operator associated with the cellulation T^*. If $v_q = \sum_i r_i \mathfrak{e}_q{}^i$ and $u_q = \sum_i s_i \mathfrak{e}_q{}^i$, then $v_q h = \sum_i r_i (\mathfrak{e}_q{}^i h)$ and $u_q h = \sum_i s_i(\mathfrak{e}_q{}^i h)$ and it follows that

(1.3) $\langle v_q h, u_q h\rangle^* = \sum_i r_i s_i = \langle v_q, u_q\rangle.$

If $v_q \in C_q(M)$, then $\langle v_q h\, \delta^*, \mathfrak{e}_{q-1}{}^i\, h\rangle^* = \langle v_q h, \mathfrak{e}_{q-1}{}^i\, h\partial^*\rangle^* = \langle v_q h, \mathfrak{e}_{q-1}{}^i\, \delta h\rangle^* = \langle v_q\, \partial\, h, \mathfrak{e}_{q-1}{}^i\, h\rangle^*$. Since this is true for all $\mathfrak{e}_{q-1}{}^i\, h$, that is for all generators of $C_{3-q+1}{}^*\,(M)$, we obtain from (1.1)

(1.4) $h\delta^* = \partial h.$

Now, since $T^{(1)}$ is a second barycentric subdivision of $T^{(-1)}$, a simplicial neighbourhood of K in $T^{(1)}$ is a regular neighbourhood. But, if $\{e_0{}^i\}$ and $\{e_1{}^i\}$ are the simplices of K in the triangulation T, then $\bigcup_i e_0{}^i\, h \cup \bigcup_i e_1{}^i\, h$ is a simplicial neighbourhood $N(K)$ of K in the triangulation $T^{(1)}$, and hence a regular open neighbourhood. Then $M - N(K)$ is a cellular complex, a subcomplex of M in the cellulation T^*, and $M - N(K)$ is a deformation retract of $M - K$, hence $M - K$ and $M - N(K)$ have the same homology groups. We can define $C_q{}^*(M - N(K)), Z_q{}^*(M - N(K)), B_q{}^*(M - N(K))$ and $H_q{}^*(M - N(K))$ to be the chains, cycles, boundary cycles and homology of this complex. Similarly, define $H_q{}^*(M) = Z_q{}^*(M)/B_q{}^*(M)$, which is of course isomorphic to $H_q(M) = Z_q(M)/B_q(M)$. It is easily seen that

(1.5) $\langle u_q h, v_{3-q}{}^*\rangle^* = 0$ if $u_q \in C_q(K)$ and $v_{3-q}{}^* \in C_{3-q}{}^*(M - N(K))$.

Definition 1.1. The *intersection number* Int : $C_2(M) \times C_1{}^*(M) \to Q$ is defined by Int $(u_2, v_1{}^*) = \langle u_2 h, v_1{}^*\rangle^*$.

We show that Int induces a map also called Int from $H_2(M, K) \times H_1{}^*(M - N(K))$ to Q. Let z_2, z_2' be in $Z_2(M, K)$ and $u_3\partial = z_2 - z_2' + u_2$ where $u_2 \in C_2(K)$. Then Int $(u_2, z_1{}^*) = \langle u_2 h, z_1{}^*\rangle^* = 0$ by (1.5). Also Int $(u_3\partial, z_1{}^*) = \langle u_3\partial\, h, z_1{}^*\rangle^* = \langle u_3 h\delta^*, z_1{}^*\rangle^* = \langle u_3 h, z_1{}^*\, \partial^*\rangle^* = 0$. Thus Int $(z_2, z_1{}^*) =$ Int $(z_2', z_1{}^*)$. Similarly, let $z_1{}^*$ and $z_1'{}^*$ be in $Z_1{}^*(M - N(K))$, and $u_2{}^*\, \partial^* = z_1{}^* - z_1'{}^*$ with $u_2{}^*$ in $C_2{}^*(M - N(K))$. Then Int $(z_2, u_2{}^*\, \partial^*) = \langle z_2 h, u_2{}^*\, \partial^*\rangle^* = \langle z_2 h\, \delta^*, u_2{}^*\rangle^* = \langle z_2\partial h, u_2{}^*\rangle^* = 0$, by (1.5) since $z_2\partial \in C_1(K)$.

PROPOSITION 1.1. *Let* $\Lambda : H_1{}^*(M - N(K)) \to Q$ *be a homomorphism. Then there exists* $\alpha \in H_2(M, K)$ *such that for any* $\beta \in H_1{}^*(M - N(K))$, $\beta\Lambda =$ Int (α, β).

Proof. Given Λ, there exists a homomorphism Λ' from $Z_1{}^*(M - N(K))$ to Q such that $z_1{}^*\Lambda' = [z_1{}^*]\Lambda$. Since $C_1{}^*(M - N(K))$ is a free Q module, $Z_1{}^*(M - N(K))$ is a direct summand and so Λ' can be extended to a homomorphism Λ'' from $C_1{}^*(M - N(K))$ to Q. However, the map $\langle\ ,\ \rangle^*$ from $C_1{}^*(M - N(K)) \times C_1{}^*(M - N(K))$ to Q is a non-degenerate bilinear form, and so there exists $u_1{}^*$ in $C_1{}^*(M - N(K))$, and hence u_2 in $C_2(M)$ such that $u_2 h = u_1{}^*$, and $\langle u_2 h, z_1{}^*\rangle^* = z_1{}^*\Lambda'' = [z_1{}^*]\, \Lambda$ for all $z_1{}^*$ in $Z_1{}^*(M - N(K))$.

We calculate $u_2\partial$. For $\mathfrak{e}_1{}^i$ in $C_1(M)$, $\langle u_2\partial, \mathfrak{e}_1{}^i\rangle = \langle u_2\partial h, \mathfrak{e}_1{}^i h\rangle^* = \langle u_2 h\, \delta^*,$

$\mathfrak{e}_1{}^i h\rangle^* = \langle u_2 h, \mathfrak{e}_1{}^i h\partial^*\rangle^* = [\mathfrak{e}_1{}^i h\partial^*]\,\Lambda$. So $u_2\partial = \sum_{\mathfrak{e}_1{}^i \in C_1(M)} [\mathfrak{e}_1{}^i h\partial^*]\,\Lambda\, \mathfrak{e}_1{}^i$ by (1.1). However, if $e_1{}^i \not\subset K$, then $\mathfrak{e}_1{}^i h \in C_2{}^*(M - N(K))$, so $[\mathfrak{e}_1{}^i h\partial^*] = 0$. Thus

(1.6) $$u_2\partial = \sum_{\mathfrak{e}_1{}^i \in C_1(K)} [\mathfrak{e}_1{}^i h\partial^*]\ \Lambda \cdot \mathfrak{e}_1{}^i$$

So $u_2 \in Z_2(M, K)$. Putting $\alpha = [u_2] \in H_1(M, K)$ we obtain the desired result.

If K is a knot of r components $K_1, \ldots, K_r$ and $e_1{}^i$ is a 1-cell of K_j, then $\mathfrak{e}_1{}^i h\ \partial^*$ is homologous to zero in $N(K_j)$ but not on $\partial N(K_j)$, the boundary of $N(K_j)$, since $e_1{}^i h$ does not separate $N(K_j)$ into two pieces. If $\mathfrak{e}_1{}^i$; $i = 1, \ldots, m$ are the oriented 1-cells of K_j and $\mathfrak{e}_0{}^i$ is an oriented 0-cell, then, since K_j is oriented, we have $\mathfrak{e}_0{}^i\delta = \mathfrak{e}_1{}^i - \mathfrak{e}_1{}^{i+1} + u_1$ for some suitable numbering, where u_1 is a 1-chain with degree zero over the 1-cells of K_j. Then $0 = \mathfrak{e}_0{}^i h\partial^*\partial^* = \mathfrak{e}_0{}^i\delta h\partial^* = \mathfrak{e}_1{}^i h\partial^* - \mathfrak{e}_1{}^{i+1}h\partial^* + u_1 h\partial^*$ and $|u_1 h|$ is contained in $\partial N(K_j)$. Thus $\mathfrak{e}_1{}^i h\partial^* \sim \mathfrak{e}_1{}^{i+1}h\partial^*$ on the boundary of $N(K)$. Let $m_j = \mathfrak{e}_1{}^i h\partial^*$. m_j is a meridian of K_j. Then evaluating (1.6) we get

(1.7) $$u_2\partial = \sum_{i=1}^{r} m_i\Lambda \cdot [K_i]$$

where $[K_i]$ is the sum of those oriented 1-simplexes in K_i. In future we will write simply K_i instead of $[K_i]$.

Now consider the exact sequences:

$$H_2(M, K) \xrightarrow{\partial_*} H_1(K) \xrightarrow{i_*} H_1(M)$$

$$H_2{}^*(M, M - N(K)) \rightarrow H_1{}^*(M - N(K)) \xrightarrow{j_*} H_1{}^*(M)$$

where j_* is induced by the inclusion map. Define $L(K) = \ker (i_*)$ and $L(M - K) = \ker (j_*)$.

Definition 1.2. The *linking number* link: $L(K) \times L(M - K) \rightarrow Q$ is defined by link $(\alpha, \beta) = \text{Int}\ (u_2, v_1{}^*)$ where $[u_2\partial] = \alpha$ and $[v_1{}^*] = \beta$ where $u_2 \in Z_2(M, K)$ and $v_1{}^* \in Z_1{}^*(M - N(K))$.

It is necessary to show that the definition does not depend on the particular choice of u_2, but only on $u_2\partial$. Since $\beta \in L(M - K)$, $v_1{}^*$ can be written as $v_2{}^*\ \partial^*$. Then $\text{Int}\ (u_2, v_1{}^*) = \langle u_2 h, v_2{}^*\ \partial^*\rangle^* = \langle u_2\partial h, v_2{}^*\rangle^*$ in fact depends only on $u_2\partial$.

We collect together the results which will be required in the rest of the paper. We write $H_1(M - K)$ instead of $H_1{}^*(M - N(K))$.

PROPOSITION 1.2. *Let K be an oriented link of r components $K_1, \ldots, K_r$ in M, an oriented 3-manifold, and let m_i be a meridian of K_i. Let $\Lambda : H_1(M - K) \rightarrow Q$ be a homomorphism and $\alpha = \sum_{i=1}^{r} (m_i\Lambda)\ K_i$. Then $\alpha \in L(K)$ and for any $\beta \in L(M - K)$, link $(\alpha, \beta) = \beta\Lambda$.*

The following formulae will also be used.

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

PROPOSITION 1.3.

$$\dim L(K) = \dim H_1(M - K) - \dim H_1(M).$$
$$\dim H_1(M - K) \geqq \dim H_1(K).$$

The dimensions referred to are the dimensions as vector spaces over Q and are equal to the Betti numbers of the corresponding integral homology groups.

Proof. Let k_* be the homomorphism of $H_1^*(M - N(K))$ into $H_1^*(M)$ induced by inclusion. The kernel of k_* is generated by the meridians $\{m_j\}$. Define a homomorphism λ from Hom $(H_1^*(M - N(K)), Q)$ onto $L(K)$ by $\Lambda\lambda = \sum_{i=1}^{r} (m_i\Lambda) K_i$. The kernel of λ consists of all λ such that $m_i\Lambda = 0$ for all i, that is exactly those homomorphisms of $H_1^*(M - N(K))$ which factor through $H_1^*(M)$ via k_*. It follows that ker $(\lambda) \simeq$ Hom $(H_1^*(M), Q)$, and Im $(\lambda) = L(K)$. Therefore

$$\dim \operatorname{Hom} (H_1^* (M), Q) + \dim L(K) = \dim \operatorname{Hom} (H_1^*(M-N(K)), Q)$$

which immediately yields the first formula.

Since $L(K)$ is the kernel of the homomorphism $i_* : H_1(K) \to H_1(M)$, we obtain $\dim H_1(K) \leqq \dim L(K) + \dim H_1(M)$ from which the second formula follows.

2. The Reidemeister-Schreier method. Let J be a set. Except in the present section J will be assumed to be finite and we will write J_n to denote a set of n elements.

Denote by $S(J)$ the group of all permutations of the set J. ϕ will denote a transitive representation of some group G into the group $S(J)$, that is, a homomorphism onto a transitive subgroup of $S(J)$. For $g \in G$, the permutation $g\phi$ will usually be written ϕ_g. However if ϕ occurs as a subscript, this notation will be avoided.

If $i \in J$, the stabilizer of i under ϕ, denoted $\mathrm{St}_\phi(i)$ is the subgroup $\{g \in G: i\phi_g = i\}$ of G.

Group presentations will be used extensively. We will write

$$G = \langle x_i; r_j \rangle_{i \in I, j \in H}$$

to mean that there exists a homomorphism χ from the free group $\langle x_i : \rangle_{i \in I}$ onto G the kernel of which is the normal closure of the set $\{r_j; j \in H\}$. We will frequently consider the natural homomorphism explicitly.

Although the elements x_i are not really in the group G, it is customary to pretend they are, and use the symbol x_i when $x_i\chi$ is meant. We will follow this practice whenever possible, that is, when the context allows only one interpretation. In particular we make statements such as "x_1x_2 has order 3" really meaning $x_1x_2\chi$ has order 3. Similarly if ϕ is a homomorphism of $G = \langle x_i; r_j \rangle$ we will write $x_i\phi$ when we mean $x_i\chi\phi$.

Let $G = \langle x_i; r_j \rangle_{i \in I, j \in H}$ and let ϕ be a transitive permutation representation of G into $S(J)$.

One defines, for each $k \in J$, a map $\mathscr{D}_k{}^{\phi}$, written simply $\mathscr{D}_k$, from the free group $\langle x_i : \rangle_{i \in I}$ to the free group $\langle X_{ik} : \rangle_{i \in I, k \in J}$ by

$$(2.1) \quad \begin{cases} x_i \mathscr{D}_k = X_{ik} \\ uv\mathscr{D}_k = u\mathscr{D}_k \cdot v\mathscr{D}_{k(u\phi)} \quad \text{for } u, v \in \langle x_i : \rangle \end{cases}$$

It is easily verified that $1\mathscr{D}_k = 1$, and that $x_i^{-1}\mathscr{D}_k = (X_{i,k(x_i^{-1}\phi)})^{-1}$ and hence that the $\mathscr{D}_k$ are uniquely defined inductively. The $\mathscr{D}_k$ will be called *rewriting functions* corresponding to ϕ.

Definition 2.1. A *Schreier tree* T for ϕ is a connected, simply connected oriented graph with $|J|$ vertices v_k; $k \in J$, and edges E_l; $l \in L$ labelled with elements of $\langle x_i : \rangle$, the label of the edge E_l being y_l, such that the following condition is satisfied: If E_l is an edge with initial vertex $v_{\iota(l)}$ and terminal vertex $v_{\tau(l)}$ and label y_l, then $\iota(l)\ \phi_{y_l} = \tau(l)$. The element $y_l\mathscr{D}_{\iota(l)}$ of $\langle X_{ij} : \rangle$ will be called the *tree relator* corresponding to the edge E_l.

Let $G^{*\phi}$, written usually G^*, be the group

$$(2.2) \quad G^{*\phi} = \langle X_{ik} : r_j\mathscr{D}_k, y_l\mathscr{D}_{\iota(l)} \rangle_{i \in I, k \in J, j \in H, l \in L}.$$

Since each map $\mathscr{D}_k$ takes relators of G into relators of G^*, $\mathscr{D}_k$ induces a map D_k from G into G^* such that the following diagram commutes:

$$\begin{array}{ccc} \langle x_i : \rangle & \xrightarrow{\mathscr{D}_k} & \langle X_{ij} : \rangle \\ \chi \downarrow & & \chi' \downarrow \\ G & \xrightarrow{D_k} & G^* \end{array}$$

LEMMA 2.1. *If R_k; $k \in J$ are maps from $\langle x_i : \rangle$ to some group H satisfying $uvR_k = uR_k \cdot vR_{k(u\phi)}$ then there is a homomorphism θ from $\langle X_{ij} : \rangle$ to H such that for all k, $R_k = \mathscr{D}_k\theta$.*

The proof is simple enough and consists in defining θ by $X_{ij}\theta = x_iR_j$. Details are omitted.

THEOREM 2.2 (The Reidemeister-Schreier method). *The restriction of the map D_k to $\mathrm{St}_\phi(k)$ is an isomorphism onto G^*.*

Proof. The restriction will also be called D_k. From (2.1) one sees that for x and y in $\mathrm{St}_\phi(k)$, $xyD_k = xD_kyD_k$ and so D_k is a homomorphism on $\mathrm{St}_\phi(k)$. To prove it is an isomorphism we will construct an inverse homomorphism. Define $\mathscr{D}_k^{-1}$ from $\langle X_{ij} : \rangle$ to $\langle x_i : \rangle$ as follows. For each $j \in J$, let α_{kj} be the path through the Schreier tree from vertex v_k to v_j. Let the consecutive edges of α_{kj} be $E_{l_1}{}^{\epsilon_1} \ldots E_{l_n}{}^{\epsilon_n}$ where $\epsilon_j = \pm 1$ and a negative exponent means that the edge is traversed in the direction opposite to its orientation. Let w_{kj} be the element $y_{l_1}{}^{\epsilon_1} \ldots y_{l_n}{}^{\epsilon_n}$ of $\langle x_i : \rangle$, where y_{l_i} is the label on the edge E_{l_i}. Then it can be seen from definition (2.1) and formula (2.2) that

$$(2.3) \quad k(w_{kj}\phi) = j, \quad \text{and} \quad w_{kj}\mathscr{D}_k\chi' = 1.$$

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

Define maps ξ_{kj} from $\langle x_i : \rangle$ to $\langle x_i : \rangle$ by $u\xi_{kj} = w_{kj}u\, w_{kj(u\phi)}^{-1}$. Then, putting $R_j = \xi_{kj}$, we see that the maps R_j satisfy the conditions of Lemma 2.1. Thus there is a homomorphism which we shall call $\mathscr{D}_k^{-1}$ from $\langle X_{ij} : \rangle$ to $\langle x_i : \rangle$ such that

(2.4) $$u\mathscr{D}_j\mathscr{D}_k^{-1} = w_{kj}u\, w_{kj(u\phi)}^{-1}.$$

Now the relators of G^* are of two types. Firstly for relators of the form $r_i\mathscr{D}_j$ we see that $r_i\mathscr{D}_j\mathscr{D}_k^{-1} = w_{kj}r_iw_{kj}^{-1}$. Thus $\mathscr{D}_k^{-1}$ takes $r_i\mathscr{D}_j$ onto a conjugate of a relator of G. Secondly, there are relators of the form $y_i\mathscr{D}_{\iota(i)}$ and $y_i\mathscr{D}_{\iota(i)}\mathscr{D}_k^{-1} = w_{k\iota(i)}y_iw_{k\iota(i)(y_i\phi)}^{-1} = w_{k\iota(i)}y_iw_{k\tau(i)}^{-1} = w_{k\tau(i)}w_{k\tau(i)}^{-1} = 1$. Therefore $\mathscr{D}_k^{-1}$ induces a homomorphism D_k^{-1} of G^* into G such that the following diagram commutes.

$$\begin{array}{ccc} \langle x_i : \rangle & \xleftarrow{\mathscr{D}_k^{-1}} & \langle X_{ij} : \rangle \\ \downarrow \chi & & \downarrow \chi' \\ G & \xleftarrow[D_k^{-1}]{} & G^* \end{array}$$

If $u\chi \in \mathrm{St}_\phi(k)$, then

$$u\chi D_kD_k^{-1} = u\mathscr{D}_k\chi' D_k^{-1} = u\mathscr{D}_k\mathscr{D}_k^{-1}\chi = (w_{kk}u\, w_{kk(u\phi)}^{-1})\chi = u\chi,$$

since $w_{kk} = 1$.

And

$$X_{ij}\chi' D_k^{-1} D_k = X_{ij}\mathscr{D}_k^{-1}\mathscr{D}_k\chi' = (w_{kj}x_iw_{kj(x_i\phi)}^{-1})\mathscr{D}_k\chi'$$
$$= (w_{kj}\mathscr{D}_k \cdot x_i\mathscr{D}_j \cdot (w_{kj}\mathscr{D}_k)^{-1})\chi' = x_i\mathscr{D}_j\chi' = X_{ij}\chi'$$

where we have used (2.3). This completes the proof.

From (2.4) we have

(2.5) If $u \in \mathrm{St}_\phi(j)$, then $uD_jD_k^{-1} = v_{kj}u\, v_{kj}^{-1}$ where v_{kj} is an element of G such that $k(v_{kj}\phi) = j$.

The definition of the rewriting functions is easily remembered if one notes the similarity to the definition of Fox's free derivative. The exact relationship is as follows. Let J be the set of integers and ϕ a representation of G as a cyclic group, $\langle t : \rangle$, of permutations of $Z = J$ generated by the permutation $t : i \rightarrow i + 1$ for all $i \in Z$. That is, for all $x \in G$, $\phi_x = t^k$ for some $k \in Z$. Let ϕ also represent the induced homomorphism of ZG into the integral group ring $Z\langle t : \rangle$, and extend χ also to a homomorphism of $Z\langle x_i : \rangle$ to ZG. Let γ_k be a homomorphism of $\langle X_{ij} : \rangle$ into the group ring $Z\langle t : \rangle$ given by $X_{ij}\gamma_k = \delta_{ik}t^j$, and $UV\gamma_k = U\gamma_k + V\gamma_k$ for all U and V. Then

(2.6) $$\left(\frac{\partial u}{\partial x_j}\right)^{\chi\phi} = (u\mathscr{D}_k\,\gamma_j)\,t^{-k}$$

Then $||(\partial r_i/\partial x_j)^{\chi\phi}||$ is just the Alexander matrix of the group G corresponding to the homomorphism ϕ, so formula (2.6) gives an alternative description of this important matrix. It is recommended that the reader compute an example to see why (2.6) holds. The following, however, is a formal proof.

Proof of (2.6). Putting $uF_j = (\partial u/\partial x_j)^{\chi\phi}$, a mapping from $\langle x_i: \rangle$ to $Z\langle t: \rangle$ we see that $x_iF_j = \delta_{ij}\chi\phi = \delta_{ij} \in Z\langle t: \rangle$, and $uvF_j = uF_j + vF_{j(\mu\phi)}$ and it is well known that $(\partial u/\partial x_j)^{\chi\phi}$ is the unique mapping satisfying these two conditions [**5**]. Now one easily verifies that

$$(x_i\mathscr{D}_k\gamma_j)t^{-k} = X_{ik}\gamma_jt^{-k} = \delta_{ij}, \quad \text{and}$$
$$uv\mathscr{D}_k\gamma_j.t^{-k} = (u\mathscr{D}_kv\mathscr{D}_{k(u\phi)})\gamma_jt^{-k} = u\mathscr{D}_k\gamma_j.t^{-k} + v\mathscr{D}_{k(u\phi)}\gamma_j.t^{-k}$$
$$= u\mathscr{D}_k\gamma_j.t^{-k} + v\mathscr{D}_k\gamma_{j(\mu\phi)}.t^{-k}$$

3. The covering space. This section is mainly notation. Let K be an oriented knot of one component in the oriented triangulated 3-manifold M. Let J_n be a set of n elements and ϕ a transitive representation of $G = \pi_1(M - K, b)$ into the group $S(J_n)$. Corresponding to ϕ is a branched covering space $\tilde{M}$ of M with covering projection p. The base point b in M is covered by n points in $\tilde{M}$. These points can be numbered $\tilde{b}_1, \ldots, \tilde{b}_n$ in such a way that if $\tilde{x}$ is a path from $\tilde{b}_i$ to $\tilde{b}_j$, then $\tilde{x}$ projects to a loop $x \in G$ such that $i\phi_x = j$. Thus p induces isomorphisms p_j^* of $\pi_1(\tilde{M} - \tilde{K}, \tilde{b}_j)$ onto $\mathrm{St}_\phi(j)$ where $\tilde{K} = Kp^{-1}$.

Two permutation representations ϕ and ϕ' will be called *equivalent* if they differ by a renumbering of J_n. That is, if there exists a permutation σ in $S(J_n)$ such that $\phi_x = \sigma\phi_x'\sigma^{-1}$ for all x. The corresponding covering spaces $\tilde{M}$ and $\tilde{M}'$ are homeomorphic by a homeomorphism which takes $\tilde{b}_i$ to $\tilde{b}_{i\sigma}'$.

Let H be a group and let F be a subgroup of finite index, n. One obtains a representation ϕ of H into $S(J_n)$ as follows. Let $\{F_i\}$, $i \in J_n$ be the right cosets of F in H indexed by the set J_n. Then for x in H one defines $i\phi_x = j$ if $F_ix = F_j$. This representation will be called a *representation corresponding to the subgroup F*. If $F = F_{i_0}$, then we see that $F = \mathrm{St}_\phi(i_0)$. The representation thus obtained depends on the particular numbering of the cosets, but any two such representations are equivalent.

The symbol K will be used to represent both a knot and its homology class in $H_1(M)$. The words *meridian* and *parallel* of a knot K will have several meanings. Firstly they will mean elements of the homology group of $M - K$ represented by paths lying in the boundary of a regular neighbourhood $N(K)$ of K. A meridian is a path null-homologous in $N(K)$ but not on its boundary, and a parallel is a path homologous to K in $N(K)$. Secondly, as elements of the homotopy group $\pi_1(M - K, b)$, a meridian is a path $\alpha m\alpha^{-1}$, and a parallel a path $\alpha l\alpha^{-1}$ where α is a path joining b to $\partial N(K)$, and m and l are paths in $\partial N(K)$ representing meridian and parallel respectively. Thus a meridian-parallel pair commute in $\pi_1(M - K, b)$. The symbols m and l will be used to denote meridians and parallels in both the homotopy and homology sense. If M is S^3, the word *longitude* will be used to describe a parallel of K which is null-homologous in $M - K$. If $\tilde{K}$ is a link of r components $\tilde{K}_1, \ldots, \tilde{K}_r$, then $\tilde{m}_i$ represents a meridian of $\tilde{K}_i$.

Let N be a regular neighbourhood of K and suppose b lies on ∂N. Let $\tilde{N}_j$ be the connected component of Np^{-1} containing $\tilde{K}_j$ and let O_j be the set $\{i \in J_n : \tilde{b}_i \in \partial \tilde{N}_j\}$. Now m and l lie on ∂N. Therefore if $i \in O_j$, so are $i\phi_m$ and $i\phi_l$. Conversely, if i and k are in O_j, then $k = i\phi_x$ where $x = m^\mu l^\lambda$. Hence the sets O_j are simply the orbits of J_n under the action of ϕ_m and ϕ_l, and so the number of components of $\tilde{K}$ is equal to the number of orbits.

We define the *branching index* of $\tilde{K}_j$, denoted by br $(\tilde{K}_j)$ to be the length of any cycle of ϕ_m contained in O_j. The *covering index* of $\tilde{K}_j =$ ind $(\tilde{K}_j)$ is the degree of the covering of K by $\tilde{K}_j$ and is equal to $|O_j|/\text{br}\,(\tilde{K}_j)$ where $|O_j|$ is the number of elements in O_j.

We single out the following result which has apparently been shown previously by Montesinos [**11**, Theorem III, 3.2].

Proposition 3.1. *The number of components of the covering link $\tilde{K}$ in a covering space M corresponding to a representation $\phi : G \to S(J_n)$ is the number of orbits of J_n under the action of ϕ_m and ϕ_l.*

For $j \in J_n$, we define $\mu(j)$ and $\lambda(j)$ to be respectively the lengths of the cycles of ϕ_m and ϕ_l containing j. Thus, if $j \in O_i$, then $\mu(j) = \text{br}\,(\tilde{K}_i)$. The isomorphism $p_j{}^*$ maps $\pi_1(\tilde{M} - \tilde{K}, \tilde{b}_j)$ onto $\text{St}_\phi(j)$. If $k \in O_i$, then $p_j{}^*$ maps a meridian of $\tilde{K}_i$ to an element $w_{jk}m^{\mu(k)}w_{jk}{}^{-1}$ where w_{jk} is an element of $G = \pi_1(M, b)$ such that $j(w_{jk}\phi) = k$. Any two such elements $w_{jk}m^{\mu(k)}w_{jk}{}^{-1}$ and $v_{jk}m^{\mu(k)}v_{jk}{}^{-1}$ are conjugate in $\text{St}_\phi(j)$. It follows that $\pi_1(\tilde{M}, \tilde{b}_j)$ is isomorphic to $\text{St}_\phi(j)/N$ where N is the normal closure in $\text{St}_\phi(j)$ of the set $\{w_{jk}m^{\mu(k)}w_{jk}{}^{-1}; k \in J_n\}$. This set is plainly larger than necessary, and we could instead use the set $\{w_{jk}m^{\mu(k)}w_{jk}{}^{-1}; k \in T\}$ where T is a set containing one element from each orbit O_i.

Now the map D_j is an isomorphism of $\text{St}_\phi(j)$ onto G^*. If we choose the elements w_{jk} such that $w_{jk}D_j = 1$ (see the proof of Theorem 2.2, in particular (2.3)) then we see that $(w_{jk}m^{\mu(k)}w_{jk}{}^{-1})D_j = m^{\mu(k)}D_k$. It follows that $\pi_1(\tilde{M}, \tilde{b}_j) \cong G^*/\langle m^{\mu(k)}D_k; k \in T\rangle$ where T is a set containing at least one element from each orbit O_i. The relation $m^{\mu(k)}D_k = 1$ will be called a *branch relation*.

4. Linking functions. According to Proposition 1.2, linking numbers in $\tilde{M}$ could be calculated from homomorphisms of $H_1(\tilde{M} - \tilde{K})$ into Q. For theoretical applications it is convenient to introduce the closely related concept of a *linking function*, which is easier to handle. We will define n functions P_j; $j \in J_n$ from $G = \pi_1(M - K, b)$ into Q. The interpretation of these functions is as follows. Let x be an element of G and $\tilde{x}_j$ its lifting to a path in $\tilde{M}$

starting at the point $\tilde{b}_j$ and terminating at $\tilde{b}_{j(x\phi)}$. For each 2-chain u_2 in $\tilde{M}$ with boundary in $\tilde{K}$, and each $j \in J_n$ we define the functions P_j where xP_j is the intersection number Int $(u_2, \tilde{x}_j)$. Now, if $xy \in G$, then xy lifts to a path $\tilde{x}_i\tilde{y}_{i(x\phi)}$. Thus we obtain a formula $xyP_i = xP_i + yP_{i(x\phi)}$ which will serve in the definition of linking functions. The 2-chain u_2 has boundary $u_2\partial = \sum_{i=1}^r q_i\tilde{K}_i$ which will be called the boundary of $\{P_i\}$. It follows that if $x \in \mathrm{St}_\phi(i)$, so that $\tilde{x}_i$ is a closed path in $\tilde{M}$, and if $\tilde{x}_i \in L(\tilde{M} - \tilde{K})$ so that linking numbers with $\tilde{x}_i$ are defined, then $xP_i = \text{link } (u_2\partial, \tilde{x}_i)$. This is the geometric basis for our theory and it should be borne in mind during the formal treatment to follow.

Denote by Q^n an n-dimensional vector space over Q with basis $\{e_i;\, i \in J_n\}$. If $\tau \in S(J_n)$ then τ can be extended to a map from Q^n to Q^n by setting

$$(\textstyle\sum_{i\in J_n} q_ie_i)\tau = \sum_{i\in J_n} q_ie_{i\tau}.$$

Definition 4.1. If ϕ is a transitive representation of G into $S(J_n)$, then a map P^ϕ from G to Q^n will be called a *linking function* for ϕ if

$$(4.1)\qquad xyP^\phi = xP^\phi + yP^\phi \cdot \phi_x^{-1}.$$

In most cases we will write simply P. Let P_j be the jth component of P^ϕ, that is the map defined by $xP^\phi = \sum_{j\in J_n} xP_je_j$. We obtain the defining formula:

$$(4.2)\qquad xyP_j = xP_j + yP_{j(x\phi)}$$

The following useful formula is easily shown: If C is the cycle of ϕ_x containing j, and $|C|$ is its length, then:

$$(4.3)\qquad x^{|C|}P_j = \textstyle\sum_{i\in C} xP_i.$$

Definition 4.2. The *boundary*, Bd (P) of a linking function P is the element $\sum_{i=1}^r q_i\tilde{K}_i$ of $H_1(\tilde{K})$ such that $q_i = m^{\mu(j)}P_j$ where j is any element of O_i.

Justification that q_i does not depend on the particular j chosen will be found in the proof of the following theorem.

THEOREM 4.1. *Let P be a linking function for ϕ. Then* Bd (P) *is in* $L(\tilde{K})$. *Let x be in* $\mathrm{St}_\phi(j)$, $\tilde{x}_j$ *its lifting to a loop based at* $\tilde{b}_j$ *and suppose* $[\tilde{x}_j]$ *is in* $L(\tilde{M} - \tilde{K})$. *Then* $xP_j = \text{link (Bd } (P), [\tilde{x}_j])$.

Proof. The map P_j restricted to $\mathrm{St}_\phi(j)$ is a homomorphism of $\mathrm{St}_\phi(j)$ into Q. Indeed, if $x, y \in \mathrm{St}_\phi(j)$, then $xyP_j = xP_j + yP_{j(x\phi)} = xP_j + yP_j$. Therefore, the map $p_j{}^*P_j$ is a homomorphism of $\pi_1(\tilde{M} - \tilde{K}, \tilde{b}_j)$ to Q. Since Q is abelian, this map in turn induces a map Λ_j from $H_1(\tilde{M} - \tilde{K})$ to Q. We show $\Lambda_i = \Lambda_j$ for all i and j. Consider an element $[\tilde{y}]$ of $H_1(\tilde{M} - \tilde{K})$, for convenience represented by a loop y based at $\tilde{b}_i$. Such elements generate $H_1(\tilde{M} - \tilde{K})$ as a Q module. Let $\tilde{x}$ be an arc from $\tilde{b}_i$ to $\tilde{b}_j$. Then $[\tilde{y}] = [\tilde{x}\tilde{y}\tilde{x}^{-1}]$. Therefore $[\tilde{y}]\Lambda_i = [\tilde{x}\tilde{y}\tilde{x}^{-1}]\Lambda_i = (\tilde{x}\tilde{y}\tilde{x}^{-1})p_i{}^*P_i = xyx^{-1}P_i$ where $\tilde{y}p_j{}^* = y \in \mathrm{St}_\phi(j)$ and $i\phi_x = j$, x being the projection of $\tilde{x}$. Thus $[\tilde{y}]\Lambda_i = (xyx^{-1})P_i = xP_i + yP_{i(x\phi)} +$

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

$x^{-1}P_{i(xy\phi)} = xP_i + yP_j + x^{-1}P_{i(x\phi)} = (xx^{-1})P_i + yP_j = yP_j = [\tilde{y}]\Lambda_j$. Since Λ_j does not depend on j it will be denoted by Λ.

Proposition 1.2 now gives that $\alpha = \sum_{i=1}^{r}(\tilde{m}_i\Lambda)\tilde{K}_i$ is in $L(\tilde{K})$ and link $(\alpha, \beta) = \beta\Lambda$ for any $\beta \in L(\tilde{M} - \tilde{K})$. To complete the proof we must show that $\alpha = \text{Bd}(P)$. That is, $\tilde{m}_i\Lambda = m^{\mu(k)}P_k$ where $k \in O_i$. Let $\tilde{m}_i$ be the meridian passing through $\tilde{b}_k$. Then $\tilde{m}_i\Lambda = \tilde{m}_i p_k{}^*P_k = m^{\mu(k)}P_k$.

We are mainly interested in linking numbers between components of the covering link. Let l be a parallel curve of K as defined in Section 3 and let $\lambda(k)$ be the length of the cycle of ϕ_l containing k. If M is S^3, then l is assumed to be a longitude. Let $k \in O_i$. Then $l^{\lambda(k)} \in \text{St}_\phi(k)$ and $l^{\lambda(k)}$ lifts to a closed curve based at $\tilde{b}_k$. Call this curve $\tilde{l}_i$. Clearly, the definition of $\tilde{l}_i$ does not depend on the choice of k in O_i in that any two such curves defined for different k in O_i are free homotopic in $\tilde{M} - \tilde{K}$. If M is S^3 then $\tilde{l}_i$ is well defined. Otherwise it depends on the particular parallel curve, l, chosen. In any case, $\tilde{l}_i$ is homologous to a multiple of $\tilde{K}_i$. In fact $\tilde{l}_i \sim \lambda(k) \cdot \text{ind}(\tilde{K}_i)^{-1} \cdot \tilde{K}_i$. Therefore, if $i \neq j$, and $k \in O_i$

$$\text{(4.4)} \qquad \text{link}(\tilde{K}_i, \tilde{K}_j) = \text{ind}(\tilde{K}_j) \cdot \lambda(k)^{-1} \cdot \text{link}(\tilde{K}_i, \tilde{l}_j)$$

and for completeness we define

$$\text{(4.5)} \qquad \text{link}(\tilde{K}_j, \tilde{K}_j) = \text{ind}(\tilde{K}_j) \cdot \lambda(k)^{-1} \cdot \text{link}(\tilde{K}_j, \tilde{l}_j).$$

As a corollary of Theorem 4.1 we obtain:

COROLLARY 4.2. *Let $\tilde{K}_i$ and $\tilde{K}_j$ be in $L(\tilde{K})$ where $i \neq j$. Suppose P is a linking function with boundary $\tilde{K}_i$. Let $k \in O_j$. Then*

$$\text{link}(\tilde{K}_i, \tilde{K}_j) = l^{\lambda(k)}P_k \cdot \text{ind}(\tilde{K}_j) \cdot \lambda(k)^{-1}.$$

In most cases studied in the literature the representation ϕ sends l to the identity. Thus we single out that particular case:

COROLLARY 4.3. *Let $M = S^3$ and suppose $\phi : \pi_1(M - K) \to S(J_n)$ sends a longitude l to the identity. If $\tilde{K}_i, \tilde{K}_j \in L(\tilde{K})$ and if P is a linking function with boundary $\tilde{K}_i$, then* link $(\tilde{K}_i, \tilde{K}_j) = lP_k$ *where $k \in O_j$.*

We finish this section by exhibiting a linking function which can always be defined in the case where $M = S^3$. Put $xP_i = \text{link}(K, x)$ for all i. It is easily verified that this is a linking function and the boundary is $\sum_{i=1}^{r} \text{br}(\tilde{K}_i) \cdot \tilde{K}_i$. We obtain:

PROPOSITION 4.4. *Let $\alpha = \sum_{i=1}^{r} \text{br}(\tilde{K}_i) \cdot \tilde{K}_i$. Then $\alpha \in L(\tilde{K})$. Suppose that $\tilde{K}_j \in L(\tilde{K})$, then* link $(\alpha, \tilde{l}_j) = 0$, *and*

$$-\text{br}(\tilde{K}_j)\,\text{link}(\tilde{K}_j, \tilde{K}_j) = \text{link}\left(\textstyle\sum_{i \neq j} \text{br}(\tilde{K}_i) \cdot \tilde{K}_i, \tilde{K}_j\right).$$

COROLLARY 4.5. *Let $B_{\tilde{M}-\tilde{K}}$ and $B_{\tilde{M}}$ be the Betti numbers of the first integral homology groups of $\tilde{M} - \tilde{K}$ and $\tilde{M}$. Then $B_{\tilde{M}-\tilde{K}} \geqq r$ and $1 \leqq B_{\tilde{M}-\tilde{K}} - B_{\tilde{M}} \leqq r$*

and $B_{\tilde{M}-\tilde{K}} - B_{\tilde{M}} = r$ *if all linking numbers exist.* (r *is the number of components of* $\tilde{K}$).

The corollary is a direct consequence of Proposition 1.3. This proves the parts concerning the Betti numbers of conjectures A, B and C of Riley [**16**, p. 614]. For instance, in the case of conjecture A, ϕ_m is a permutation $(12 \ldots p)$ $(p + 1)$ and so the number of covering knots is 2.

5. Existence of linking functions. Theorem 4.1 gives a general method of calculating linking numbers if such things as linking functions actually exist. The following theorem shows that indeed linking functions do exist with all possible boundaries.

THEOREM 5.1. *Let* α *be in* $H_1(\tilde{K})$. *Then there exists a linking function with boundary* α *if and only if* $\alpha \in L(\tilde{K})$.

Proof. The only if part of this theorem was proved in Theorem 4.1.

Suppose that $\alpha \in L(\tilde{K})$. Then there is a homomorphism $\Lambda : H_1(\tilde{M} - \tilde{K}) \rightarrow Q$ given by $\beta\Lambda = \text{Int}\,(u_2, \beta)$ where u_2 is a 2-chain with bounadry α. Select a Schreier tree and recall the definition of the group G^* and maps D_k, D_k^{-1} defined in Section 2. We then have maps

$$G \xrightarrow{D_k} G^* \xrightarrow{D_k^{-1}} \text{St}_\phi(k) \xrightarrow{p_k^{*-1}} \pi_1(\tilde{M} - \tilde{K}, \tilde{b}_k) \xrightarrow{[\ \ ]} H_1(\tilde{M} - \tilde{K}) \xrightarrow{\Lambda} Q.$$

All the maps shown are homomorphisms except D_k. Furthermore, $D_k^{-1}p_k^{*-1}[\ \ ]$ is equal to $D_j^{-1}p_j^{*-1}[\ \ ]$ for all k and j as is easily verified. Define $P_k = D_kD_k^{-1}p_k^{*-1}[\ \ ]\Lambda$. Then this defines a linking function P.

Now $\text{Bd}\,(P) = \sum_{i=1}^{r} q_i\tilde{K}_i$ where $q_i = m^{\mu(j)}P_j$ and $j \in O_i$. But $m^{\mu(j)} \in \text{St}_\phi(j)$. So $m^{\mu(j)}P_j = [m^{\mu(j)}p_j^{*-1}]\Lambda = [\tilde{m}_i]\Lambda$, and $\text{Bd}\,(P) = \sum_{i=1}^{r} [\tilde{m}_i]\Lambda \cdot \tilde{K}_i = u_2\partial = \alpha$.

The maps D_k and D_k^{-1} depend on the choice of Schreier tree, and so there are in fact many different linking functions with the same boundary. The proof of the following theorem uses linking functions for which there exists no Schreier tree T with the property that $P_k = D_kD_k^{-1}p_k^{*-1}[\ \ ]\Lambda$, where D_k is the map associated with T.

LEMMA 5.2. *Let* T *be a Schreier tree and let* a_i; $i = 1, \ldots, n - 1$ *be arbitrary rational numbers in one-to-one correspondence with the edges* E_i *of* T. *Then there exists a linking function with boundary zero such that* $y_iP_{\iota(i)} = a_i$ *for* $i = 1, \ldots, n - 1$ *where* E_i *is an edge of* T *with label* y_i *and initial vertex* $v_{\iota(i)}$.

First we observe a lemma, the proof of which is easy and so omitted.

LEMMA 5.3. *If* P *and* P^* *are linking functions with linkage* α *and* α^*, *and* q *is a rational number, then* $qP + P^*$, *defined by* $x(qP + P^*) = q.xP + xP^*$ *is a linking function with boundary* $q\alpha + \alpha^*$.

Proof of Lemma 5.2. We define a linking function $P^{(k)}$ which corresponds geometrically to a small sphere u_2 enclosing $\tilde{b}_k$. Put $xP_j^{(k)} = \delta_{kj} - \delta_{kj(x\phi)}$. It is easily seen that $P^{(k)}$ is a linking function with boundary zero. Hence $P = \sum_{k\in J_n} q_k P^{(k)}$ is a linking function with boundary zero. If P is the required linking function, then for each edge E_i of T we have an equation $y_i P_{\iota(i)} = a_i$. This gives

$$a_i = \sum_{k\in J_n} q_k y_i P_{\iota(i)}{}^{(k)} = \sum_{k\in J_n} q_k(\delta_{k\iota(i)} - \delta_{k,\iota(i)(y_i\phi)}) = q_{\iota(i)} - q_{\tau(i)}$$

We obtain $n - 1$ such equations in the variables q_i; $i \in J_n$. Since T is a tree, there exists a solution to these equations.

PROPOSITION 5.4. *Let T be a Schreier tree and let a_i; $i = 1, \ldots, n - 1$ be arbitrary rational numbers in one-to-one correspondence with the edges E_i of T. Suppose P' is a linking function with linkage α. Then there exists a linking function P with linkage α such that for $i = 1, \ldots, n - 1$, $y_i P_{\iota(i)} = a_i$, where E_i is an edge of T with label y_i and initial vertex $v_{\iota(i)}$.*

This follows straight from Lemmas 5.2 and 5.3.

6. Dihedral covering spaces and an invariant of Burde. Covering linkage has been studied in any depth only in the case of dihedral covering spaces. This is because most knot groups have a representation onto some dihedral group, and this representation is quite easily found.

PROPOSITION 6.1 (Fox [**7**]). *A knot group $G = \pi_1(S^3 - K)$ has a representation onto D_n, the dihedral group of order $2n$ if and only if n divides the highest torsion coefficient of the 2-fold branched covering space of K.*

In particular, n must be odd.

For notational convenience, the set J_n will be taken to be the set $\{0, \ldots, n - 1\}$ when we are dealing with dihedral coverings. We assume that J_n is provided with an addition operation assigning to a pair of elements their sum modulo n. The symbol $||k||$ is k if $0 \leqq k \leqq (n - 1)/2$ and $n - k$ if $(n + 1)/2 \leqq k \leqq n - 1$.

We shall be concerned here with irregular dihedral representations ϕ. This shall always mean a representation of G into $S(J_n)$ where ϕ_m is the permutation $(0)\ (1\quad n - 1)\ (2\quad n - 2) \ldots ((n-1)/2\ (n+1)/2)$ and any other Wirtinger generator is mapped to a conjugate, $\sigma\phi_m\sigma^{-1}$ where σ is a power of the permutation $(012 \ldots n - 1)$.

M will be S^3 and l will mean a longitude of K. Since l is in the second commutator subgroup, and the second commutator subgroup of D_n is trivial, $\phi_l = \text{id}$. Therefore there are just $(n + 1)/2$ components of the covering link which will be labelled $K_0, \ldots, K_{(n-1)/2}$. O_0 is the orbit $\{0\}$ and O_i is $\{i, n - i\}$. Thus $i \in O_j$ if and only if $||i|| = j$.

The main goal of this section is to establish a connection between covering linkage and a certain invariant of Burde. Burde defined [**3**] a representation

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

ϕ^* of G into the group of motions of the complex plane C. Thus $\phi_x^* : z \to (z + xU)\phi_x'$ where ϕ_x' is an isometric linear transformation of C, and U is a map from G into the complex numbers. In the particular case studied by Burde, $xU \in Q(\eta)$ where η is a primitive nth root of unity, and ϕ_x' is such that $z\phi_x' = z\eta^j$ or $\bar{z}\eta^j$ for some j. Also, for some Wirtinger generator m, $z\phi_m^* = \bar{z} + 1$. Since l and m commute, $z\phi_l^* = z + S$ for some real number S. This number is Burde's invariant.

Calculation shows that $z\phi_{xy}^* = (z + xU + yU\phi_{x^{-1}}')\phi_{xy}'$, whence:

$$(6.1) \qquad xyU = xU + yU\phi_{x^{-1}}'.$$

This is similar to the definition of a linking function (4.1). An irregular dihedral representation of G can be defined by $i\phi_x = j$ if $\eta^i\phi_x' = \eta^j$. ϕ^* is said to *extend* the representation ϕ. To identify U as a disguised linking function, we need to define functions $U_i : G \to Q$ associated with U.

Now mU is of the form $\sum_{i=0}^{n-1} q_i\eta^i$ with $\sum_{i=0}^{n-1} q^i = 1$. It is a simple matter of conjugation to show that xU can be written in the same form for any Wirtinger generator x. It follows that if y is an element of G, then yU can be written in the form

$$(6.2) \qquad yU = \sum_{i=0}^{n-1} q_i\eta^i \quad \text{such that} \quad \sum_{i=0}^{n-1} q_i = \text{link } (K, y)$$

Henceforth we will assume that n is prime and write p instead of n. In these circumstances, the numbers q_i in (6.2) are uniquely determined, as follows from the following standard result of number theory.

LEMMA 6.2. *If p is prime, $q_i \in Q$, $\sum_{i=0}^{p-1} q_i\eta^i = 0$ and $\sum_{i=0}^{p-1} q_i = 0$, then $q_i = 0$ for all i.*

Now, let $xU = \sum_{i=0}^{p-1} q_i\eta^i$ be in the form (6.2). Define $xU_i = q_i$. Then $xU = \sum_{i=0}^{p-1} xU_i\eta^i$. So

$$\begin{aligned} xyU &= \sum_{i=0}^{p-1} xU_i\eta^i + \sum_{i=0}^{p-1} yU_i\eta^i \,\phi'_{x^{-1}} \quad \text{by (6.1)} \\ &= \sum_{i=0}^{p-1} xU_i\eta^i + yU_i\eta^{i(x^{-1}\phi)} \\ &= \sum_{i=0}^{p-1} (xU_i + yU_{i(x\phi)})\eta^i \end{aligned}$$

Whence $xyU_i = xU_i + yU_{i(x\phi)}$ and the U_i define a linking function for ϕ. We can now state

THEOREM 6.3. *Let p be an odd prime and ϕ^* a Burde representation of $G = \pi_1(S^3 - K)$ extending the irregular dihedral (D_p) representation ϕ of G into $S(J_p)$, and such that $\phi_m^* : z \to \bar{z} + 1$. Let $\tilde{M}$ be the branched covering space corresponding to ϕ, and let $\tilde{K}_0$ be the component of index 1. Write $z\phi_l^* = z + \sum_{i=0}^{p-1} q_i\eta^i$ where $\sum_{i=0}^{p-1} q_i = 0$. Then $q_i = \text{link } (\tilde{K}_i, \tilde{K}_0)$ for all i, $1 \leqq i \leqq (p-1)/2$.*

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

Proof. We have shown that U is a linking function for ϕ, and it is easily verified that its boundary is $\tilde{K}_0$. Thus the theorem follows straight from corollary 4.3 as long as link $(\tilde{K}_i, \tilde{K}_0)$ is defined. That is, as long as $\tilde{K}_i \in L(\tilde{K})$. This last condition will be verified below.

Assume that ϕ is an irregular dihedral representation and suppose that P' is a linking function for ϕ. Define P by $xP_j = xP'_{j-t} + xP'_{j+t}$ for some fixed $t \leqq (n-1)/2$. Then

$$xyP_j = xyP'_{j-t} + xyP'_{j+t} = xP'_{j-t} + yP'_{(j-t)(x\phi)} + xP'_{j+t} + yP'_{(j+t)(x\phi)} = xP_j + yP_{j(x\phi)}.$$

The last equality holds since $j\phi_x = \pm j + s$ for some s, and so $(j+t)\phi_x = j\phi_x \pm t$. So P is a linking function for ϕ.

Now suppose that $mP_j' = \delta_{0j}$ and so $\mathrm{Bd}(P') = \tilde{K}_0$. Then $mP_j = mP'_{j-t} + mP'_{j+t} = \delta_{t,j} + \delta_{t,-j}$. So P has boundary $\tilde{K}_{||t||} + \tilde{K}_{||-t||} = 2\tilde{K}_t$. This shows that if $\tilde{K}_0 \in L(\tilde{K})$, so is $\tilde{K}_j$ for any j.

From the equation $lP_j = lP'_{j-t} + lP'_{j+t}$ where l is a longitude of K we obtain the following formula due to Perko [**14**].

(6.3) $\quad 2 \text{ link } (\tilde{K}_t, \tilde{K}_j) = \text{link } (\tilde{K}_0, \tilde{K}_{||j-t||}) + \text{link } (\tilde{K}_0, \tilde{K}_{||j+t||})$

This useful formula permits the calculation of all linking numbers between the knots $\tilde{K}_i$ from a knowledge of the linking numbers with the knot of index 1, $\tilde{K}_0$.

7. Calculation of covering linkage. The notion of a linking function is closely related to the Reidemeister-Schreier algorithm. Let $G = \pi_1(M - K, b)$ have presentation $\langle x_i; r_j \rangle$ and let $\phi : G \to S(J_n)$ be a transitive representation. Let $\mathscr{D}_k$; $k \in J_n$ be rewriting functions associated with the representation ϕ.

PROPOSITION 7.1. *If P is a linking function for ϕ, then there exists a homomorphism θ from $\langle X_{ij}: \rangle$ to Q such that for all k in J_n, $\mathscr{D}_k\theta = \chi P_k$ where χ is the natural homomorphism of $\langle x_i: \rangle$ onto G. Conversely, if θ is a homomorphism from $\langle X_{ij}: \rangle$ to Q such that $r_j\mathscr{D}_k\theta = 0$ for all j and k, then there is a linking function P such that $\mathscr{D}_k\theta = \chi P_k$.*

Proof. The maps χP_k satisfy the conditions imposed on the maps R_k of Lemma 2.1. Thus θ exists. The converse is obvious.

Thus, the concept of a linking function is virutally the same as a homomorphism $\theta : \langle X_{ij}: \rangle \to Q$ satisfying $r_j\mathscr{D}_k\theta = 0$ for all j and k. We will refer to such a homomorphism as a *linking homomorphism*.

The following theorem is to be used when covering linkage is to be calculated from a group presentation. It is just a translation of the results of Section 4, and so has already been proved.

THEOREM 7.2. *Let $\langle x_i; r_j \rangle$ be a presentation for $G = \pi_1(M - K)$ in which $x_0 = m$ is a meridian of K, for some generator x_0. Let θ be a linking homomorphism* *for* ϕ. *That is, a homomorphism of* $\langle X_{ij}:\rangle$ *into* Q *such that* $r_j\mathscr{D}_k\theta = 0$ *for all* j *and* k. *Let* $\alpha \in H_1(\tilde{K})$ *be equal to* $\sum_{j=1}^{r} q_j\tilde{K}_j$ *where* $q_j = \sum_{i\in C} X_{0i}\theta$ *where* C *is any cycle of* m *contained in* O_j. *Let* L *be an element of* $\langle x_i:\rangle$ *such that* $L\chi = l$ *is a parallel of* K. *Suppose* $\tilde{K}_i \in L(\tilde{K})$. *If* $k \in O_i$ *then*

$$\text{ind } \tilde{K}_i \cdot \lambda(k)^{-1} \cdot (L^{\lambda(k)}\mathscr{D}_k\theta) = \text{link } (\alpha, \tilde{K}_i).$$

Alternatively stated:

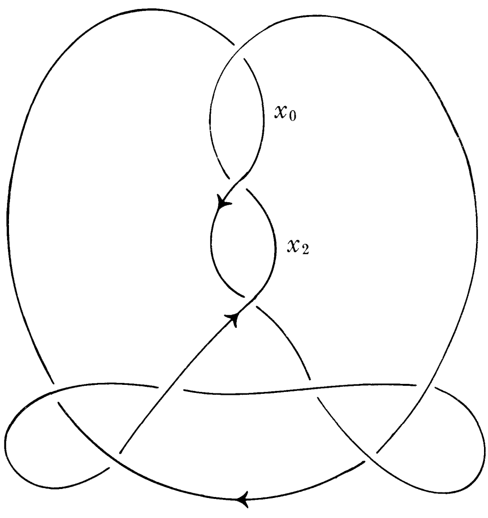
$$\textstyle\sum_{k\in O_i} L\mathscr{D}_k\theta = \text{br } \tilde{K}_i \cdot \text{link } (\alpha, \tilde{K}_i)$$

Once again we single out the case where ϕ_l = identity.

COROLLARY 7.3. *Under the conditions of Theorem* 7.2, *if* $\phi_l = \text{id}$, *then* $\alpha = \sum_{j=1}^{r} q_j\tilde{K}_j$ *where* $q_j = \sum_{i\in O_j} X_{0i}\theta$, *and* $L\mathscr{D}_k\theta = \text{link } (\alpha, \tilde{K}_i)$ *for any* $k \in O_i$.

Since the elements $r_j\mathscr{D}_k$ can be calculated explicitly, the problem of finding linking functions with a given boundary becomes the task of solving a system of linear equations. According to Proposition 5.4, in making calculations we may also assume that $y_i\mathscr{D}_{\iota(i)}\theta = 0$ for $i = 1, \ldots, n - 1$, where T is a Schreier tree for ϕ and edge E_i has label y_i and initial vertex $\iota(i)$. Thus, in effect one seeks a homomorphism of G^* into Q, with G^* defined as in Section 2. Since Q is abelian this induces a homomorphism of $G^*/G^{*\prime}$ which is isomorphic to $H_1(\tilde{M} - \tilde{K})$. The advantage of linking functions in theoretical applications is that one need *not* assume that $y_i\mathscr{D}_{\iota(i)}\theta$ is zero and in fact they can be chosen arbitrarily.

Example. The group of knot 9_{48}



has generators x_0, x_1, x_2 and relators

$$r_1 : x_1(x_2x_1\bar{x}_2\bar{x}_1x_0x_1x_2\bar{x}_1\bar{x}_2x_0) = (x_2x_1\bar{x}_2\bar{x}_1x_0x_1x_2\bar{x}_1\bar{x}_2x_0)x_2 \quad \text{and}$$

$$r_2 : (x_2\bar{x}_0x_1x_2)x_1 = x_0(x_2\bar{x}_0x_1x_2)$$

where a bar represents the inverse. Consider the representation $x_0\phi = (12)$, $x_1\phi = x_2\phi = (02)$, and set $O_0 = \{0\}$ and $O_1 = \{1, 2\}$. Writing R_{ij} for the relation $r_i\mathscr{D}_j = 1$ we obtain

$$\begin{array}{lccccccccccc c ccccccccccc}
 & \multicolumn{11}{c}{X_1X_2X_1\bar{X}_2\bar{X}_1X_0X_1X_2\bar{X}_1\bar{X}_2X_0} & = & \multicolumn{11}{c}{X_2X_1\bar{X}_2\bar{X}_1X_0X_1X_2\bar{X}_1\bar{X}_2X_0X_2} \\
R_{10}: & 0&2&0&0&2&2&1&1&1&1&1 & & 0&2&2&0&0&0&2&2&0&0&0 \\
R_{11}: & 1&1&1&1&1&1&2&0&0&2&2 & & 1&1&1&1&1&2&0&0&2&2&1^{**} \\
R_{12}: & 2&0&2&2&0&0&0&2&2&0&0 & & 2&0&0&2&2&1&1&1&1&1&2
\end{array}$$

and

$$\begin{array}{lccccc c ccccc}
 & \multicolumn{5}{c}{X_2\bar{X}_0X_1X_2X_1} & = & \multicolumn{5}{c}{X_0X_2\bar{X}_0X_1X_2} \\
R_{20}: & 0&1&1&1&1 & & 0&0&1&1&1^{**} \\
R_{21}: & 1&2&2&0&2 & & 1&2&0&0&2 \\
R_{22}: & 2&0&0&2&0 & & 2&1&2&2&0
\end{array}$$

We use shorthand notation here. In full, R_{10} would read $X_{10}X_{22}X_{10}\bar{X}_{20}\ldots$. A Schreier tree is

$$v_1 \xrightarrow{x_0} v_2 \xrightarrow{x_2} v_0,$$

whence we can set $X_{22} = X_{01} = 1$. It is convenient to select a Schreier tree in such a way as to be able to set $X_{0j} = 1$ for all j except one from each cycle of ϕ_m. From the starred relations we obtain, after abelianisation, $X_{00} = X_{11} = X_{21}$. Then a relation matrix for $G^*/G^{*\prime}$ is

$$\begin{array}{l} \\ R_{10}: \\ R_{12}: \\ R_{22}: \end{array}
\begin{array}{c} \begin{array}{cc|ccc} X_{00} & X_{02} & X_{20} & X_{10} & X_{12} \end{array} \\
\left[\begin{array}{cc|ccc} -2 & 1 & -2 & 2 & -1 \\ 2 & -1 & 1 & -1 & 2 \\ -2 & 0 & -1 & 2 & -1 \end{array}\right] \end{array}$$

The row corresponding to R_{21} is the same as R_{12}.

Now, branch relations for $\pi_1(\tilde{M})$ are of the form $x_0^{\mu(k)}\mathscr{D}_k$ with one such relator for each cycle of ϕ_m. With the Schreier tree chosen in the manner suggested above, the result of the branch relations is to set all the remaining X_{0j} equal to 1. Thus, the right part of the matrix is a relation matrix for $H_1(\tilde{M})$, and the whole matrix is a relation matrix for $H_1(\tilde{M} - \tilde{K})$. This remark is irrelevant to the calculation of covering linkage, but it will be discussed in more detail later.

Then it is easily found, setting $X_{02}\theta = 0$ and $X_{00}\theta = 1$ that

$$\begin{array}{lll}
X_{00}\theta = 1 & X_{20}\theta = 0 & X_{10}\theta = 2/3 \\
X_{01}\theta = 0 & X_{21}\theta = 1 & X_{11}\theta = 1 \\
X_{02}\theta = 0 & X_{22}\theta = 0 & X_{12}\theta = -2/3
\end{array}$$

is a homomorphism from $\langle X_{ij}: \rangle$ into Q factoring through G^*, and that the linking function defined by $\chi P_j = \mathscr{D}_j\theta$ has boundary $\tilde{K}_0$, the knot of index 1. An expression for L is

$$x_1x_2\bar{x}_1\bar{x}_2x_0\bar{x}_2\bar{x}_0x_2\bar{x}_1\bar{x}_0x_2x_1\bar{x}_2\bar{x}_1\bar{x}_0x_1x_2\bar{x}_1\bar{x}_2x_1\bar{x}_2\bar{x}_1x_0\bar{x}_2\bar{x}_0x_0^5.$$

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

As before, we calculate $L\mathscr{D}_1$:

X_1	X_2	$\bar{X}_1$	$\bar{X}_2$	X_0	$\bar{X}_2$	$\bar{X}_0$	X_2	$\bar{X}_1$	$\bar{X}_0$	X_2	X_1	$\bar{X}_2$	$\bar{X}_1$	$\bar{X}_0$	X_1	X_2	$\bar{X}_1$	$\bar{X}_2$	X_1	$\bar{X}_2$	$\bar{X}_1$	X_0	$\bar{X}_2$	X_0
1	1	1	1	1	0	0	0	0	0	0	2	2	0	0	0	2	2	0	0	0	2	2	1	1
1	1	-1	-1	0	0	-1	0	$-\frac{2}{3}$	-1	0	$-\frac{2}{3}$	0	$-\frac{2}{3}$	-1	$\frac{2}{3}$	0	$\frac{2}{3}$	0	$\frac{2}{3}$	0	$\frac{2}{3}$	0	-1	0

X_0	X_0	X_0
2	1	2
0	0	0

The sum of the third line is $L\mathscr{D}_1\theta$, and shows that link $(\tilde{K}_0, \tilde{K}_1) = -10/3$. (Cf. [**5**, p. 200]).

8. Representations on PSL (2, Z). Let L_p be the group $PSL(2, p)$ with p a prime. Thus L_p is the group of 2×2 integral matrices with determinant 1 in which a matrix is identified with its negative. Let F be the subgroup of L_p consisting of upper triangular matrices. Now L_p has order $\frac{1}{2}p(p^2 - 1)$ if $p > 2$ and 6 if $p = 2$, whereas F has order $\frac{1}{2}p(p - 1)$ if $p > 2$ and 2 if $p = 2$. Thus F has index $p + 1$ in L_p. Corresponding to the subgroup F is a representation ζ of L_p into $S(J_{p+1})$. Direct calculation shows that ζ is faithful. For $p > 3$ this is also clear because L_p is simple. Let $H = PSL(2, Z)$, sometimes called the unimodular group, and let mod_p be the homomorphism of H onto L_p which reduces each matrix to its residue mod p. Then $\psi = \mathrm{mod}_p\, \zeta$ is a permutation representation of H. The group H has a presentation $\langle S, T : (ST)^3, T^2 \rangle$ where $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $J_{p+1} = \{0, 1, \ldots, p\}$. The following lemma gives the information we require about ψ.

LEMMA 8.1. *Let ψ be a transitive representation of H into $S(J_{p+1})$ such that ψ_S has order p. Then after suitable renumbering ψ satisfies $\psi_S = (01 \ldots p - 1)(p)$ and $0\psi_T = p$, $1\psi_T = p - 1$.*

Proof. By a suitable numbering one may assume $\psi_S = (01 \ldots p - 1)(p)$. Since ψ is transitive, $p\psi_T \neq p$. Thus one may assume $p\psi_T = 0$. Then one calculates that $(p - 1)\psi^3_{ST} = 1\psi_T$. Since ST has order 3 in H, $1\psi_T = p - 1$.

We now state the main theorem of this section.

THEOREM 8.2. *Let η be a representation of $G = \pi_1(S^3 - K)$ onto H such that $\eta_m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and let ψ be a transitive representation of H into $S(J_{p+1})$ such that ψ_S has order p. Let $\phi = \eta\psi$ and $\tilde{M}$ be the covering space corresponding to ϕ. Then K has two components, $\tilde{K}_0$ of index 1 and $\tilde{K}_1$ of index p. Also, η_l is of the form $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ where l is a longitude and link $(\tilde{K}_0, \tilde{K}_1)$ exist and equals $-s/(p + 1)$.*

This result was conjectured by Riley for the case $p = 2$ [**16**].

Proof. η_l must be of the given form since it commutes with η_m.

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

By Lemma 8.1, we may assume that $\psi_S = (01 \ldots p-1)(p)$ and $0\psi_T = p$, $1\psi_T = p-1$, $\psi_T^2 = \text{id}$. Then corresponding to a Schreier tree

$$p \xrightarrow{T} 0 \xrightarrow{S} 1 \xrightarrow{S} 2 \xrightarrow{S} \ldots \xrightarrow{S} p-1$$

one can calculate a presentation for $H^{*\psi}$. Write $S\mathcal{D}_j = S_j$ and $T\mathcal{D}_j = T_j$ then the only relators of $H^{*\psi}$ in which S_p and S_{p-1} occur are $S_0T_1S_{p-1}T_0S_pT_p$; $S_{p-1}T_0S_pT_pS_0T_1$ and $S_pT_pS_0T_1S_{p-1}T_0$, and one sees that there exists a homomorphism θ of $H^{*\psi}$ into Q such that $S_p\theta = 1$ and $S_{p-1}\theta = -1$. Define a linking function for ϕ by $P_i = \eta D_i\theta$. Now $\phi_m = (01 \ldots p-1)(p)$. Let $O_0 = \{p\}$ and $O_1 = \{0, 1, \ldots, p-1\}$. The boundary of P is then $q_0\tilde{K}_0 + q_1\tilde{K}_1$, where $q_0 = mP_p = m\eta D_p\theta = S_p\theta = 1$, and $q_1 = \sum_{i=0}^{p-1} mP_i = \sum_{i=0}^{p-1} S_i\theta = -1$. Therefore $\tilde{K}_0 - \tilde{K}_1 \in L(\tilde{K})$. On the other hand, by Proposition 4.4, $\tilde{K}_0 + p\tilde{K}_1 \in L(\tilde{K})$, so both $\tilde{K}_0$ and $\tilde{K}_1$ are in $L(\tilde{K})$. Then link $(\tilde{K}_0 - \tilde{K}_1, \tilde{l}_0) = lP_p = l\eta D_p\theta = S_p{}^s\theta = s$, and from Proposition 4.4, link $(\tilde{K}_0 + p\tilde{K}_1, \tilde{l}_0) = 0$. We deduce that $(p+1)$ link $(\tilde{K}_1, \tilde{l}_0) = -s$ from which the theorem follows.

As a consequence of this theorem, we can prove that some knots have property P. A knot K in S^3 is said to have *property* P if $\pi_1(S^3 - K)/\langle ml^q \rangle \neq 1$ for all $q \neq 0$. It is easy to see that if K has property P then a counter example to the Poincaré conjecture cannot be constructed by removing a solid torus with core K from S^3 and sewing it back differently. It has been conjectured that every non-trivial knot in S^3 has property P but there are quite a few general results on this conjecture [**2**; **8**; **9**; **17**; **19**]. Now we will prove the following.

PROPOSITION 8.3. *Suppose that the knot group* $G = \pi_1(S^3 - K)$ *has a representation* η *onto* $PSL(2, Z)$ *such that* $\eta_m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *Let* $\tilde{K}_0, \tilde{K}_1$ *be the covering link of* K *in the irregular dihedral* (D_3) *covering space* $\tilde{M}$ *induced from* η. *Then, if* link $(\tilde{K}_0, \tilde{K}_1) \neq 0$, *K has property P.*

Proof. Let $\eta_l = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$. Then by Theorem 8.2, $v = \text{link } (\tilde{K}_0, \tilde{K}_1) = -s/3$. Since $v \neq 0$ and $s \equiv 0 \pmod 6$ [**17**], it follows that $|s| \geqq 6$. Now $\eta_{ml^q} = \begin{bmatrix} 1 & qs+1 \\ 0 & 1 \end{bmatrix}$. Therefore, for any q, there is a prime integer p such that $qs + 1 \equiv 0 \pmod p$, and hence the group $\pi_1(S^3 - K)/\langle ml^q \rangle$ has a representation on $PSL(2, p)$.

It is known that if K is a 2-bridge knot, then link $(\tilde{K}_0, \tilde{K}_1) \equiv 2 \pmod 4$ in an irregular D_3 covering space. Therefore, we obtain from Proposition 8.3 the following

COROLLARY 8.4. *If the group of a* 2-*bridged knot K has a representation on* $PSL(2, Z)$ *such that* $\eta_m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, *then K has property P.*

The condition that such a representation exist is quite restrictive but the following classes of 2-bridge knots have such a representation onto $PSL(2, Z)$.

$$(18ab - 3a \pm 3b, 6a \pm 1), \quad a, b \geqq 1 \text{ and } a + b \equiv 1 \pmod 2,$$
$$(54ab + 3a - 12b, 18a + 1), \quad a, b \geqq 1 \text{ and } b \equiv 1 \pmod 2.$$

9. A generalisation of an invariant of Reyner. So far we have not really considered the use of linking numbers as knot invariants. In order to obtain an invariant from linking numbers one must consider the covering linkage invariants for all the covering spaces of a given type. The reader is referred to Perko [**13**] for examples of the use of such invariants.

One of the most troublesome aspects of covering linkage invariants is that they do not always exist. Furthermore, the calculation is a nuisance if ϕ_l is not the identity. In this section, certain invariants will be defined which have most of the advantages of covering linkage invariants (for instance, they are extremely effective in detecting non-amphicheiral knots) while avoiding the disadvantages of covering linkage invariants.

It should be clear from the preceding sections of this paper that linking numbers, since they may be computed without reference to the topological aspects, may be defined for arbitrary groups. That is, given a group G, two elements x and y in G taking the place of meridian and longitude, and a transitive permutation representation of G, one may define some sort of pseudo-linking numbers. This approach is not emphasised, since the real interest is in the topological interpretation. However, the invariants to be defined below will be described purely algebraically. First, we make some remarks applicable also to covering linkage.

Let G be a group, x, y two elements in G and ϕ a transitive representation of G into $S(J_n)$. Let F be some function assigning to a quadruple (G, x, y, ϕ) some object in some category. We will suppose that F has the following properties:

i) $F(G, x, y, \phi) = F(G, x, y, \phi')$ if ϕ is equivalent to ϕ'.
ii) If $\tau: G \to G'$ is an isomorphism, then $F(G, x, y, \phi) = F(G', x\tau, y\tau, \tau^{-1}\phi)$.

We say that two representations $\phi: (G, x, y) \to S(J_n)$ and $\phi': (G', x', y') \to S(J_n)$ are of the *same type* if ϕ' is equivalent to a representation ϕ'' such that $G'\phi'' = G\phi$, $x'\phi'' = x\phi$ and $y'\phi'' = y\phi$.

One defines a function F^* by

$$F^*(G, x, y) = \{F(G, x, y, \phi_i); \phi_i \in A\},$$

where A is a set containing one representation from each equivalence class of representations of a given type. Then, $F^*(G, x, y) = F^*(G\tau, x\tau, y\tau)$, so F is an isomorphism invariant of the triple (G, x, y).

Given a knot K with knot group G and a meridian longitude pair, m, l we can define a function F' by $F'(K) = F^*(G, m, l)$. This is independent of the particular meridian longitude pair chosen, since any two pairs are related by an inner automorphism of the knot group.

If K and K' are (ambient)-isotopic, then there is an isomorphism taking (G, m, l) to (G', m', l'). Thus, $F'(K) = F^*(G, m, l) = F^*(G', m', l') = F'(K')$ and so F' is an invariant of ambient isotopy type for knots.

If K is amphicheiral, then there exists τ taking m to m^{-1} and l to l. Therefore

$$F^*(G, m, l) = F^*(G, m^{-1}, l).$$

If K is invertible then τ takes m to m^{-1} and l to l^{-1}. Thus $F^*(G, m, l) = F^*(G, m^{-1}, l^{-1})$. The particular functions F to be considered are particularly effective in distinguishing non-amphicheiral knots, but unfortunately we always have $F(G, m, l, \phi) = F(G, m^{-1}, l^{-1}, \phi)$ so that they are (not surprisingly) useless for proving knots non-invertible. We now proceed with the definitions.

Let G be any group, not necessarily a knot group, and let x be some particular element of G. Let ϕ: $G \rightarrow S(J_n)$ be a transitive representation and let $S = \mathrm{St}_\phi(a)$ where a is some element of J_n. Define elements $\{x_i: i \in J_n\}$ of S as follows. Let v_{ai} be some element of G such that $a(v_{ai}\phi) = i$. Then x_i is the element $v_{ai}x^{\sigma(i)}v_{ai}^{-1}$ where $\sigma(i)$ is the smallest positive integer such that $x^{\sigma(i)} \in \mathrm{St}_\phi(i)$. Of course, x_i depends on the choice of elements v_{ai}, but the conjugacy class of x_i in S is independent of this choice. Define the group $\pi(G, \phi, x)$ to be $S/\langle\{x_i; i \in J_n\}\rangle^S$ and $H(G, \phi, x)$ to be the commutator quotient group of $\pi(G, \phi, x)$. Here $\langle\{x_i; i \in J_n\}\rangle^S$ means the normal closure in S of the set $\{x_i; i \in J_n\}$.

The normal subgroup $\langle\{x_i; i \in J_n\}\rangle^S$ can be the normal closure of subset of the x_i. In fact, if $y \in C(x)$, the centraliser of x in G, and $i\phi_y = j$ then x_i is conjugate to x_j as is easily verified. Thus if A is a subset of J_n containing one element from each orbit of J_n under the action of $C(x)\phi$, then $\langle\{x_i; i \in J_n\}\rangle^S = \langle\{x_i; i \in A\}\rangle^S$.

Now if G^* is the group defined in Section 2 and D_i are rewriting functions corresponding to ϕ, then D_a is an isomorphism of $\mathrm{St}_\phi(a)$ onto G^*. Then

$$x_iD_a = (v_{ai}x^{\sigma(i)}v_{ai}^{-1})D_a = v_{ai}D_a \cdot x^{\sigma(i)}D_i \cdot (v_{ai}D_a)^{-1}$$

which is conjugate to $x^{\sigma(i)}D_i$. Thus we have

$$\pi(G, \phi, x) = G^*/\langle\{x^{\sigma(i)}D_i; i \in A\}\rangle^{G*}$$

whence one can easily find a presentation for $\pi(G, \phi, x)$, and $H(G, \phi, x)$ is easily calculated.

In the case where G is the group of a knot K in S^3 one may consider such groups as $\pi(G, \phi, m^al^b)$ where m and l are meridian and longitude. Since we have maps

$$\pi_1(\tilde{M} - \tilde{K}, \tilde{b}_j) \xrightarrow{p_j^*} G \xrightarrow{D_j} G^*$$

in which $p_j^*D_j$ is an isomorphism, we will identify $\pi_1(\tilde{M} - \tilde{K})$ with G^* and

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

$H_1(\tilde{M} - \tilde{K})$ with $G^*/G^{*\prime}$. Thus in particular, it will be convenient to use geometric language and talk of $m^{\mu(j)}D_j$ as being a meridian of the component $\tilde{K}_i$ of $\tilde{K}$ where $j \in O_i$. Thus one sees that $\pi(G, \phi, m)$ is just the fundamental group of the branched covering space corresponding to ϕ. Similarly, $\pi(G, \phi, \text{id}) = \pi_1(\tilde{M} - \tilde{K})$. The special case of $\pi(G, \phi, l)$ was considered by Reyner [**15**] for the case where $\phi_l = \text{id}$ in his Ph.D. thesis. He calculated many examples where ϕ is a dihedral representation and collected the results in tables.

In general, if a and b are coprime then $\pi(G, \phi, m^a l^b)$ can be interpreted as the fundamental group of a manifold obtained from $\tilde{M} - N(\tilde{K})$ by sewing $N(\tilde{K})$ back in "wrongly". To be precise, let M be obtained from S^3 by removing a tubular neighbourhood of K and sewing it back so that a meridian corresponds to $m^a l^b$. Then we obtain a knot K in M, and $S^3 - K \simeq M - K$. (We use here the same symbol K for the knots in M and S^3). Thus corresponding to ϕ: $\pi_1(S^3 - K) \rightarrow S(J_n)$ there is a homomorphism of $\pi_1(M - K)$ into $S(J_n)$. If $M^*(a, b)$ is the covering space of M branched over K, then $\pi_1(M^*(a, b))$ is isomorphic to $\pi(G, \phi, m^a l^b)$.

In the next section we will be considering the group $H(G, \phi, m^a l^b)$. We will not need the above geometrical interpretation of this group, nor the assumption that a and b are coprime. However it will be convenient to use geometrical language and refer to the above group as $H_1(M^*(a, b))$ or more simply, $H_1(M^*)$. If $s = m^a l^b$ and $j \in O_i$ then we will write $\tilde{m}_i$, $\tilde{l}_i$ and $\tilde{s}_i$ instead of $m^{\mu(j)}D_j$, $l^{\lambda(j)}D_i$ and $s^{\sigma(j)}D_j$ respectively. In general, if $x \in \text{St}_\phi(j)$, then xD_j can be considered as the lifting to the base point $\tilde{b}_j$ in $\tilde{M} - \tilde{K}$ of the path x.

Since we are dealing with an abelian group, additive notation will be used.

10. The connection with covering linkage invariants. In this section, the connection between the group $H_1(M^*)$ and the linking numbers in the covering space $\tilde{M}$ corresponding to ϕ will be considered. We therefore make the assumption that linking numbers are defined between all components of the link $\tilde{K}$ in $\tilde{M}$. All homology groups will be integral homology groups unless otherwise stated.

It is necessary to define torsion coefficients and the Betti number of a matrix of rational numbers. Let Q be considered as a module over Z. It is easily seen that a finitely generated submodule of Q is generated by a single element. Define the greatest common divisor of a finite set of rational numbers to be the generator of the submodule they generate.

Let V be an $m \times n$ matrix of rational numbers. Define for $i = 0, \ldots, n - 1$, the ith elementary ideal, E_i of V to be the Z-submodule of Q generated by the $(n - i) \times (n - i)$ subdeterminants of V if $n - i \leqq m$, and the zero submodule if $n - i > m$. The Betti number of V, $B(V)$, is the number of zero ideals and the sequence of torsion coefficients of V is the generators of the non-zero ideals, which will be numbered so that $T_1(V)$ is the generator of the first non-zero ideal.

If A is an abelian group then one defines the Betti number and torsion coefficients of A to be the Betti number and torsion coefficients of a relation matrix for A. (These will be integers.) Then $T_1(A)$ is the order of the torsion subgroup of A. We will also write $B(\tilde{M})$, $B(M^*)$, $T_i(\tilde{M})$ and $T_i(M^*)$ to mean the Betti number and torsion coefficients of $H_1(\tilde{M})$ and $H_1(M^*)$ respectively.

Now, we may choose a set of generators for $H_1(\tilde{M} - \tilde{K})$ of the form $\tilde{u}_1, \ldots, \tilde{u}_k, \tilde{m}_1, \ldots, \tilde{m}_r$ where the $\tilde{m}_i$ are meridians of the components of $\tilde{K}$. Then a relation matrix for $H_1(M^*)$ is of the form:

$$
\begin{array}{c} \\ \\ \tilde{s}_1 \\ \vdots \\ \tilde{s}_r \end{array}
\overset{\begin{array}{ccc|ccc} \tilde{u}_1 & \ldots & \tilde{u}_k & \tilde{m}_1 & \ldots & \tilde{m}_r \end{array}}{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]} = F \tag{10.1}
$$

where A is a relation matrix for $H_1(\tilde{M})$, and $(A|B)$, is a relation matrix for $H_1(\tilde{M} - \tilde{K})$. The horizontal and vertical lines will be referred to as *the* horizontal and vertical lines. A matrix F divided into blocks in this way will be called a partitioned matrix. Consider the following types of row operations on partitioned matrices.

R1: Add an integral multiple of a row above the (horizontal) line to a different row above the line.

R2: Add an integral multiple of a row below the line to a different row below the line.

R3: Add a rational multiple of a row above the line to a row below the line.

R4: Add or eliminate a row of zeros.

Similarly define column operations C1, C2, C3, analogously to the row operations with the word column replacing the word row, vertical line replacing horizontal line, left replacing above and right replacing below. By a rational row or column operation will be meant an operation of this sort, and an integral row (column) operation will mean an operation of this type in which addition only of integral multiples of rows (columns) to other rows (columns) is allowed. Clearly the Betti number of a matrix is invariant under rational row and column operations.

Let $j \in O_i$ and define $\tilde{s}_i = (m^a l^b)^{\sigma(j)} D_j$. For geometrical reasons $\tilde{s}_i$ ought to be expressible in terms of $\tilde{m}_i$ and $\tilde{l}_i$. In fact

$$\sum_{j \in O_i} (m^a l^b) D_j = \sum_{j \in O_i} (m^a D_j + l^b D_j)$$

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

Whence

$$\tilde{s}_i = \tilde{m}_i \cdot a \cdot \sigma(j)/\mu(j) + \tilde{l}_i \cdot b \cdot \sigma(j)/\lambda(j).$$

Furthermore, if all linking number are defined, then as elements of $H_1(\tilde{M} - \tilde{K}, Q)$ we have $\tilde{l}_i = \sum_{k=1}^{r} q_{ik}\tilde{m}_k$ where $q_{ik} = \lambda(j) \cdot \text{ind } (K_i)^{-1} \cdot \nu_{ik}$ by (4.4). Here, $\nu_{ik} = \text{link } (\tilde{K}_i, \tilde{K}_k)$.

Consequently, remembering that $\mu(j) = \text{br } (\tilde{K}_i); j \in O_i$ we obtain

$$\tilde{s}_i = \sigma(j) \cdot \sum_{k=1}^{r} \tilde{m}_k[\delta_{ik} \cdot a/\text{br } (\tilde{K}_i) + \nu_{ik} \cdot b/\text{ind } (\tilde{K}_i)]$$

Let Ξ be the matrix defined by:

(10.2) $\quad \Xi_{ik} = \sigma(j)[\delta_{ik} \cdot a/\text{br } (\tilde{K}_i) + \nu_{ik} \cdot b/\text{ind } (\tilde{K}_i)]$

When it is necessary for clarity, this matrix will be denoted by $\Xi(a, b)$. We have shown that by rational row operations, the relation matrix $F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ can be transformed to $\left[\begin{array}{c|c} A & B \\ \hline 0 & \Xi \end{array}\right]$. Furthermore, by Corollary 4.5, rank (A) = rank $(A|B)$. Therefore by rational column operations, we can obtain $\begin{bmatrix} A & 0 \\ 0 & \Xi \end{bmatrix}$. The following theorem is immediate.

THEOREM 10.1. $B(M^*) = B(\tilde{M}) + B(\Xi)$.

This gives a more general answer to a question of Reyner [**16**] who asked: "If the linking number of the two branch curves [in a D_3 covering space] is 0, does $H_1(M^*)$ contain a free abelian group of rank two?" He was considering the case where $m^a l^b = l$, and $\phi_l = \text{id}$, in which case Ξ is just the matrix of linking numbers, assumed to be a 2×2 zero matrix.

In order to study the torsion coefficients of $H_1(M^*)$, we include a further column operation:

C4: Add or eliminate a column of zeros.

We will call two partitioned matrices *P-equivalent* if one can be obtained from the other by a sequence of row and column operations of type R1 to R4 and C1 to C4. If only integral row and column operations are used then the matrices will be said to be *integral-P-equivalent*. The torsion co-efficients of a matrix are invariant under integral-P-equivalence. Now in the matrix F given in (10.1),

$$\text{rank } (A|B) = \text{rank } (A) \quad \text{and} \quad \text{rank} \left(\frac{A}{C}\right) = \text{rank } (A).$$

Therefore we may assume that A is a non-singular diagonal matrix. We obtain immediately

THEOREM 10.2. *If $H_1(\tilde{M})$ is free abelian then the torsion coefficients of $H_1(M^*)$ are just the torsion coefficients of Ξ.*

The proof of the following lemma is straightforward.

LEMMA 10.3. *If* $\begin{bmatrix} A' & 0 \\ 0 & D' \end{bmatrix}$ *is P-equivalent to* $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ *then* $T_i(A) = T_i(A')$ *and* $T_i(D) = T_i(D')$ *for all* i.

Lemma 10.3 is used in the proof of the following

LEMMA 10.4. *If* F *is integral-P-equivalent to a non-singular matrix, then* $T_1(M^*) = T_1(\tilde{M}) \cdot T_1(\Xi)$.

Proof. F is P-equivalent to $\begin{bmatrix} A & 0 \\ 0 & \Xi \end{bmatrix}$. Suppose that $F' = \begin{bmatrix} A & B' \\ C' & D' \end{bmatrix}$, F' is non-singular and F is integral-P-equivalent to F'. Since A is non-singular, F' is P-equivalent to a matrix $E = \begin{bmatrix} A & 0 \\ 0 & D'' \end{bmatrix}$ of the same size such that $\det(F') = \det(E) = \det(A) \cdot \det(D'')$. Then $T_1(A) \cdot T_1(\Xi) = T_1(A) \cdot T_1(D'') = \det(A) \cdot \det(D'') = \det(F') = T_1(F)$. The first equality is a consequence of Lemma 10.3.

Clearly, if $\det(\Xi) \neq 0$ then F itself is non-singular. This gives immediately:

THEOREM 10.5. *If* Ξ *is non-singular, then* $T_1(M^*) = T_1(\tilde{M}) \cdot \det(\Xi)$.

Example 1. If $\phi_l = \text{id}$, then the matrix $\Xi(0, 1)$ is just the matrix of linking numbers, $||\nu_{ij}||$. If ϕ is a D_3 representation and $\tilde{M}$ is a Z-homology sphere, (this occurs if the knot group G is generated by two Wirtinger elements) then a relation matrix for $H_1(M^*(0, 1))$ is of the form $\begin{bmatrix} -4\alpha & 2\alpha \\ 2\alpha & -\alpha \end{bmatrix}$ where $2\alpha = \text{link}(\tilde{K}_0, \tilde{K}_1)$. Thus $H_1(M^*(0, 1)) = Z + Z_\alpha$. So, $H_1(M^*)$ is completely determined by the covering linkage.

Example 2. Let ϕ be a D_3 representation, then $\Xi(2, 1)$ is the matrix $\begin{bmatrix} -4\alpha + 2 & 2\alpha \\ 2\alpha & -\alpha + 1 \end{bmatrix}$ where $2\alpha = \text{link}(\tilde{K}_0, \tilde{K}_1)$. Then, $\det(\Xi(2, 1)) = 2 - 6\alpha$. On the other hand, $\Xi(-2, 1) = \begin{bmatrix} -4\alpha - 2 & 2\alpha \\ 2\alpha & -\alpha - 1 \end{bmatrix}$ which has determinant $6\alpha + 2$. It follows that $H_1(M^*(2, 1)) \neq H_1(M^*(-2, 1))$ unless $\alpha = 0$. The non-amphicheirality is thus clearly shown.

When the invariant $H_1(M^*(0, 1))$ is to be considered, Theorem 10.5 is not applicable, since the matrix $\Xi(0, 1)$ is never non-singular, because of Proposition 4.4. In the following, we will assume for convenience that $\phi_l = \text{id}$ so that $\Xi(0, 1)$ is simply the matrix of linking numbers. The row of the matrix F shown in (10.1) corresponding to $\tilde{l}_i$ will be denoted as (l_i) and the column corresponding to $\tilde{m}_i$ will be denoted by (m_i).

LEMMA 10.6. *If* $\sum_{i=1}^{r} a_i(l_i)$ *is a rational (resp. integral) linear combination of rows above the line, then* $\sum_{i=1}^{r} a_i(m_i)$ *is a rational (resp. integral) linear combina-*

https://doi.org/10.4153/CJM-1977-132-2 Published online by Cambridge University Press

tion of columns to the left of the vertical line.

Proof. If $\sum_{i=1}^{r} a_i(l_i)$ is a rational linear combination, then $c = \sum_{i=1}^{r} a_i \tilde{K}_i \sim 0$ in $H_1(\tilde{M} - \tilde{K}; Q)$. This means that there is a rational 2-chain V_2 with $V_2\partial = c$. Then there is defined a homomorphism Λ from $H_1(\tilde{M} - \tilde{K})$ to Q given by $\alpha\Lambda = \text{Int}(V_2, \alpha)$. Furthermore, $\tilde{m}_i\Lambda = a_i$ and $\tilde{l}_i\Lambda = 0$. It follows that $\sum_{i=1}^{r} \tilde{m}_i\Lambda \cdot (m_i) + \sum_{i=1}^{r} \tilde{u}_i\Lambda \cdot (u_i) = 0$ where (u_i) represents the column corresponding to the generator $\tilde{u}_i$.

If $\sum_{i=1}^{r} a_i(l_i)$ is an integral linear combination of the rows above the line, then V_2 is an integral 2-chain, and so $\tilde{u}_i\Lambda$ is an integer.

We give two cases in which Lemma 10.4 is satisfied.

THEOREM 10.7. *Let* $\Xi = \Xi(0, 1)$ *and* $M^* = M^*(0, 1)$. *Suppose* $\phi_l = \text{id}$. *If either*

i) $H_1(\tilde{M} - \tilde{K}; Z)$ *is free abelian, or*

ii) $B(\Xi) = 1$ *and the greatest common divisor of the lengths of the cycles of* ϕ_m *is* 1, *then*

$$T_1(M^*) = T_1(\tilde{M}) \cdot T_1(\Xi).$$

Proof. First observe that it is possible by integral row operations on a matrix to replace a row r_i by $\sum a_j r_j$; $a_j \in Z$, if and only if $a_i \neq 0$ and g.c.d. $(a_j) = 1$.

i) Consider $F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where we assume that A is non-singular. The rows above the line are linearly independent. Thus if F is singular, then there exist integers a_i with g.c.d. $(a_i) = 1$ such that $\sum_{i=1}^{r} a_i(l_i)$ is a linear combination of rows above the line. That means that $\sum_{i=1}^{r} a_i\tilde{l}_i \sim 0$ in $H_1(\tilde{M} - \tilde{K}; Q)$. Therefore $\sum_{i=1}^{r} a_i\tilde{l}_i \sim 0$ in $H_1(\tilde{M} - \tilde{K}; Z)$ since $H_1(\tilde{M} - \tilde{K}; Z)$ is free abelian. That is, $\sum_{i=1}^{r} a_i(l_i)$ is an integral linear combination of the rows above the line. By Lemma 10.6, this means that $\sum_{i=1}^{r} a_i(m_i)$ is an integral linear combination of the columns to the left. Thus, we may eliminate a row and a column by integral row and column operations and continue in this way until a non-singular matrix is obtained.

ii) Since $l \sim 0$ in $H_1(S^3 - K; Z)$ it follows that $\sum_{i=1}^{r} \text{br}(\tilde{K}_i)\tilde{K}_i \sim 0$ in $H_1(\tilde{M} - \tilde{K}; Z)$ by lifting a surface spanning l to the covering space (This is the idea behind Proposition 4.4). The greatest common divisor of br $(\tilde{K}_i)$ is one by the assumption on the lengths of the cycles of ϕ_m. So, one can replace one of the rows below the line by $\sum_{i=1}^{r}$ br $(\tilde{K}_i)$ (l_i) and then eliminate it by integral row operations. Similarly, one column can be eliminated by Lemma 10.6.

The assumptions $B(\Xi) = 1$ and g.c.d. $\{\text{br}(\tilde{K}_i)\} = 1$ are natural enough. In particular, in the case of dihedral covering spaces, the second condition is always met, and it seems probable that unless all the linking numbers vanish, that the first condition is met. Thus in most cases, the covering linkage invariants if they exist determine the Betti number and the product of the torsion coefficients of $H_1(M^*)$ for dihedral coverings.

REFERENCES

1. Bankwitz and Schumann, *Uber Viergeflechte*, Abh. Math. Sem. Univ. Hamburg *10* (1934), 263–284.

2. R. Bing and M. Martin, *Cubes with knotted holes*, Trans. Amer. Math. Soc. *155* (1971), 217–231.

3. G. Burde, *Darstellungen von Knotengruppen*, Math. Annalen *173* (1967), 24–33.

4. S. E. Cappell and J. L. Shaneson, *Invariants of 3-manifolds*, Bull. A.M.S. *81* (1975), 559–562.

5. Crowell and Fox, *An introduction to knot theory* (Ginn and Co., 1963).

6. R. Fox, *Metacyclic invariants of knots and links*, Can. J. Math. *22* (1970), 193–201.

7. ——— *A quick trip through knot theory*, Topology of Three Manifolds and Related Topics (Prentice Hall, 1962), 120–167.

8. F. Gonzalez-Acuna, *Dehn's construction on knots*, Boletin de la Scoiedad Mathematica Mexicana *15* (1970), 58–79.

9. J. Hempel, *A simply connected 3-Manifold is S^3 if it is the sum of a solid torus and the complement of a torus knot*, Proc. Amer. Math. Soc. *15* (1964), 154–158.

10. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory* (Interscience, 1966).

11. J. M. Montesinos, *Sobre la Conjetura de Poincaré y los recubridores ramificados sobre un nudo*, Tesis doctoral, Madrid, 1971.

12. K. Perko, *On covering spaces of knots*, Glasnik Matematicki, *9* (29) (1974), 141–145.

13. ——— *On the classification of knots*. Proc. Amer. Math. Soc. *45* (1974), 262–266.

14. ——— *On dihedral covering spaces of knots*, Inventiones Math. *34* (1976), 77–82.

15. S. W. Reyner, *On metabelian and related invariants of knots*, Ph.D. thesis, Princeton, 1972.

16. R. Riley, *Homomorphisms of knot groups on finite groups*, Mathematics of Computation *25* (1971), 603–619.

17. ——— *Knots with the parabolic property P*, Oxford Quarterly Journal of Math. *25* (1974), 273–283.

18. H. Schubert, *Topology* (Allyn and Bacon, 1968).

19. H. Seifert, *Topologie dreidimensionaler gefaserter Raume*, Acta Math. *60* (1933), 147–238.

University of Toronto,
Toronto, Ontario