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ON HERMITE-FEJÉR TYPE INTERPOLATION ON THE CHEBYSHEV NODES

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Given $f \in C[-1, 1]$, let $H_{n,3}(f, x)$ denote the $(0, 1, 2)$ Hermite-Fejér interpolation polynomial of f based on the Chebyshev nodes. In this paper we develop a precise estimate for the magnitude of the approximation error $|H_{n,3}(f, x) - f(x)|$. Further, we demonstrate a method of combining the divergent Lagrange and $(0, 1, 2)$ interpolation methods on the Chebyshev nodes to obtain a convergent rational interpolatory process.

1. INTRODUCTION

Suppose f is a continuous real-valued function defined on the interval $[-1, 1]$, and let

$$X = \{x_{k,n} \ : \ k = 1, 2, \ldots, n; \ n = 1, 2, 3, \ldots\}$$

be a triangular matrix such that, for all n,

$$1 \geqslant x_{1,n} > x_{2,n} > \ldots > x_{n,n} \geqslant -1.$$

Then for each integer $m \geqslant 1$ there is a unique polynomial $H_{n,m}(X, f, x)$ of degree at most $mn - 1$ which satisfies

$$\begin{cases} H_{n,m}(X, f, x_{k,n}) &= f(x_{k,n}), \\ H_{n,m}^{(j)}(X, f, x_{k,n}) &= 0, \qquad j = 1, 2, \ldots, m-1, \end{cases}$$

for all k. $H_{n,m}(X, f, x)$ is referred to as the $(0, 1, \ldots, m-1)$ Hermite-Fejér interpolation polynomial of $f(x)$.

$H_{n,1}(X, f, x)$ is the well-known Lagrange interpolation polynomial of $f(x)$. A classic result of Faber [1] states that for any matrix X, there exists an $f \in C[-1, 1]$ so that $H_{n,1}(X, f, x)$ does not tend uniformly to $f(x)$. On the other hand, if T denotes the matrix of Chebyshev nodes

$$T = \{x_{k,n} = \cos\left(\frac{2k-1}{2n}\pi\right) \ : \ k = 1, 2, \ldots, n; \ n = 1, 2, 3, \ldots\},$$

and if the modulus of continuity $\omega(\delta; f) = \omega(\delta)$ of f is defined by

$$\omega(\delta; f) = \omega(\delta) = \max|f(x_1) - f(x_2)|,$$

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where the max is taken over all pairs x_1, x_2 in $[-1, 1]$ for which $|x_1 - x_2| \leqslant \delta$, then there exists a number c_1 (independent of f, n and x) so that

$$(1.1) \qquad |H_{n,1}(T,f,x) - f(x)| \leqslant c_1 \log n\, \omega(1/n)$$

for all $n \geqslant 2$ (see Szabados and Vértesi [9, Chapter 1]). Thus the Chebyshev nodes are a good choice if uniform approximation by Lagrange interpolation polynomials is required, and so they have been much studied. We point out that results sharper than (1.1) are known. For example, Kis [4] showed that there exists a number c_2 such that

$$(1.2) \qquad |H_{n,1}(T,f,x) - f(x)| \leqslant c_2 \left[\log n\, \omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i}\, \omega\left(\frac{i}{n^2}\right)\right].$$

A famous result of Fejér [2] states that if $f \in C[-1, 1]$, the $(0,1)$ Hermite-Fejér interpolation polynomials $H_{n,2}(T,f,x)$ converge uniformly to $f(x)$ as n tends to infinity. A brief history of estimates of the rate of convergence is given in Goodenough and Mills [3], who also proved that there exist absolute constants c_3 and c_4 so that

(1.3)
$$|H_{n,2}(T,f,x) - f(x)| \leqslant \frac{c_3}{n}\, T_n(x)^2 \sum_{k=1}^{n} \left[\omega\left(\frac{\sqrt{1-x^2}}{k}\right) + \omega\left(\frac{1}{k^2}\right)\right] + c_4\, \omega\left(\frac{|T_n(x)|}{n}\right)$$

for all $f \in C[-1, 1]$, $n \geqslant 2$, and $-1 \leqslant x \leqslant 1$. Here $T_n(x)$ denotes the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $-1 \leqslant x \leqslant 1$, whose zeros are $\cos((2k-1)\pi/2n)$, $k = 1, 2, \ldots, n$. Thus the estimate (1.3) reflects the fact that $H_{n,2}(T,f,x)$ interpolates $f(x)$ at the zeros of $T_n(x)$. Note also that because $|T_n(x)| \leqslant 1$, $-1 \leqslant x \leqslant 1$, it follows from (1.3) that the approximation error $\|H_{n,2}(T,f,\cdot) - f\|$ is of order no greater than $n^{-1} \sum_{k=1}^{n} \omega(1/k)$.

For $(0,1,2)$ interpolation, Szabados and Varma [8] proved that, as with Lagrange interpolation, for any system of nodes X there exists $f \in C[-1, 1]$ such that $H_{n,3}(X,f,x)$ does not converge uniformly to $f(x)$. This result has recently been shown to extend to $(0, 1, \ldots, m)$ interpolation for any even m (see Szabados [7]). The first aim of this paper is to obtain an estimate for the error $|H_{n,3}(T,f,x) - f(x)|$ in approximating $f(x)$ by the $(0,1,2)$ interpolation polynomial based on the Chebyshev nodes. Our precise result, to be established in Section 3, is:

Theorem 1. *There exist constants k_1, k_2 such that, for $f \in C[-1, 1]$, $n \geqslant 2$, and $-1 \leqslant x \leqslant 1$,*

(1.4)
$$|H_{n,3}(T,f,x) - f(x)| \leqslant k_1\, T_n(x)^2 \left[\log n\, \omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i}\, \omega\left(\frac{i}{n^2}\right)\right] + k_2\, \omega\left(\frac{|T_n(x)|}{n}\right).$$

We point out that (1.4) is similar to Kis' estimate (1.2) for Lagrange interpolation. However, like the estimate (1.3) for $(0,1)$ interpolation, it also reflects the fact that the approximation error vanishes at the zeros of $T_n(x)$.

For $(0,1,2,3)$ interpolation, it is known (see, for example, Mills [6]) that if $f \in C[-1,1]$, then $H_{n,4}(T,f,x)$ converges uniformly to $f(x)$, and that, as for $(0,1)$ interpolation, the approximation error $\|H_{n,4}(T,f,\cdot)-f\|$ is of the order $n^{-1}\sum_{k=1}^{n}\omega(1/k)$. In his paper [5], A. Meir developed a rational interpolatory process, based on $(0,1)$ and $(0,1,2,3)$ interpolation on the Chebyshev nodes, for which the approximation error is of order $\omega(1/n)$. This error is essentially less than the approximation error in the two convergent interpolation processes on which Meir's technique is based. The second aim of this paper is to use Meir's approach to develop a convergent rational interpolatory approximation method from the *divergent* (0) and $(0,1,2)$ interpolation processes on the Chebyshev nodes. Before stating our main result, we introduce some terminology.

For any system of nodes $X=\{x_{k,n}\}$, and for $k=1,2,\dots,n$, let

$$\ell_{k,n}(X,x)=\frac{\Omega(X,x)}{(x-x_{k,n})\,\Omega'(X,x_{k,n})},$$

where

$$\Omega(X,x)=\prod_{k=1}^{n}(x-x_{k,n}).$$

Then, as is well-known,

$$H_{n,1}(X,f,x)=\sum_{k=1}^{n}f(x_{k,n})\,\ell_{k,n}(X,x). \tag{1.5}$$

Also (Szabados and Varma [8]),

$$H_{n,3}(X,f,x)=\sum_{k=1}^{n}f(x_{k,n})\,A_{k,n}(X,x), \tag{1.6}$$

where

$$A_{k,n}(X,x)=\Big\{1-3\ell'_{k,n}(X,x_{k,n})(x-x_{k,n})+\Big[6\big(\ell'_{k,n}(X,x_{k,n})\big)^2 -\frac{3}{2}\ell''_{k,n}(X,x_{k,n})\Big](x-x_{k,n})^2\Big\}\,(\ell_{k,n}(X,x))^3.$$

For the Chebyshev nodes $x_{k,n}=\cos((2k-1)\pi/2n)$, it is readily verified that

$$\ell_{k,n}(T,x)=\frac{(-1)^{k-1}\sqrt{1-x_{k,n}^2}}{n\,(x-x_{k,n})}\,T_n(x), \tag{1.7}$$

and

$$A_{k,n}(T,x) = P_{k,n}(T,x) + Q_{k,n}(T,x) + R_{k,n}(T,x), \tag{1.8}$$

where

$$P_{k,n}(T,x) = P_k(x) = \frac{(-1)^{k-1}(1 - xx_{k,n})\sqrt{1 - x_{k,n}^2}}{n^3 (x - x_{k,n})^3} T_n(x)^3, \tag{1.9}$$

$$Q_{k,n}(T,x) = Q_k(x) = \frac{(-1)^k x_{k,n}\sqrt{1 - x_{k,n}^2}}{2n^3 (x - x_{k,n})^2} T_n(x)^3, \tag{1.10}$$

$$R_{k,n}(T,x) = R_k(x) = \frac{(-1)^{k-1}(n^2 - 1)\sqrt{1 - x_{k,n}^2}}{2n^3 (x - x_{k,n})} T_n(x)^3. \tag{1.11}$$

Now define $W_n(x)$ by

$$W_n(x) = 1 - \frac{n^2 - 1}{2n^2} T_n(x)^2$$

(so $1/2 < W_n(x) \leqslant 1$ for $-1 \leqslant x \leqslant 1$), and given $f \in C[-1, 1]$, put

$$(\Lambda_n f)(x) = \frac{1}{W_n(x)} \left(H_{n,3}(T,f,x) - \frac{n^2 - 1}{2n^2} T_n(x)^2 H_{n,1}(T,f,x) \right). \tag{1.12}$$

Note that $\Lambda_n f$ is a rational function with numerator of degree at most $3n - 2$, denominator of degree $2n$, and which agrees with f at the Chebyshev nodes. In Section 4 we prove the following result concerning the operator Λ_n.

Theorem 2.. *There exist constants k_3, k_4 such that, for $f \in C[-1, 1]$, $n \geqslant 2$, and $-1 \leqslant x \leqslant 1$,*

(1.13)

$$|(\Lambda_n f)(x) - f(x)| \leqslant k_3 T_n(x)^2 \left[\omega\left(\frac{\sqrt{1 - x^2}}{n} \right) + \frac{\log n}{n} \omega\left(\frac{1}{n} \right) \right] + k_4 \omega\left(\frac{|T_n(x)|}{n} \right).$$

From (1.13) it follows that $\|\Lambda_n f - f\|$ is of order $\omega(1/n)$, and so the divergent (0) and $(0, 1, 2)$ interpolation processes have, indeed, been combined to give a convergent scheme of interpolation. Further, if $x = x_{k,n}$, the right-hand side of (1.13) is zero, and so (1.13) reflects the fact that Λ_n is an interpolatory operator. We also point out that the order of magnitude $\omega(1/n)$ for $\|\Lambda_n f - f\|$ is the best possible, for if $g(x) = |x|$ (so that $\omega(1/n) = 1/n$), it is not too hard to show that

$$|(\Lambda_{2m} g)(0) - g(0)| = \frac{8G}{\pi^2 m} + O(m^{-2})$$

as $m \to \infty$. (Here G is Catalan's constant, $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2$.)

Remark. Since completing the above work, we have become aware of a result due to Yuan Xu [10, Theorem 4, p.149] which is similar to our Theorem 2.

2. Preliminaries

In this section we introduce some preliminary results that will be required for the proofs of Theorems 1 and 2. Because the remainder of this paper is concerned solely with the Chebyshev nodes, we shall henceforth adopt the notation $H_{n,k}(f,x) = H_{n,k}(T,f,x)$, $\ell_k(x) = \ell_{k,n}(T,x)$, $A_k(x) = A_{k,n}(T,x)$, and $x_k = x_{k,n} = \cos((2k-1)\pi/2n)$. Also, define $t_k = (2k-1)\pi/2n$. For each $x \in [-1,1]$ write $x = \cos t$, where $0 \leqslant t \leqslant \pi$, and choose j so that

$$\min\{|t_k - t| : k = 1, 2, \dots, n\} = |t_j - t|.$$

The results of the following lemma are due to Kis [4, Lemmas 1, 2, 3, 4].

Lemma 1. *With notation as introduced in the above discussion, then*

$$|f(x_k) - f(x)| \leqslant 5\omega\left(\frac{i \sin t}{n}\right) + 13\omega\left(\frac{i^2}{n^2}\right), \quad \text{if } i = |k-j| \geqslant 1, \tag{2.1}$$

$$\begin{cases} |f(x_{k+1}) - f(x_k)| \leqslant 4\omega\left(\dfrac{\sin t}{n}\right) + 20\omega\left(\dfrac{i}{n^2}\right), & \text{if } i = k-j \geqslant 1, \\ |f(x_k) - f(x_{k-1})| \leqslant 4\omega\left(\dfrac{\sin t}{n}\right) + 20\omega\left(\dfrac{i}{n^2}\right), & \text{if } i = j-k \geqslant 1, \end{cases} \tag{2.2}$$

$$|\ell_k(x)| \leqslant \begin{cases} 2, & \text{if } k = j, \\ \dfrac{2}{i}, & \text{if } i = |k-j| \geqslant 1, \end{cases} \tag{2.3}$$

$$\begin{cases} |\ell_k(x) + \ell_{k+1}(x)| \leqslant \dfrac{7}{i^2}, & \text{if } i = k-j \geqslant 1, \\ |\ell_k(x) + \ell_{k-1}(x)| \leqslant \dfrac{7}{i^2}, & \text{if } i = j-k \geqslant 1. \end{cases} \tag{2.4}$$

The following elementary inequality will be used.

Lemma 2. *If $0 \leqslant \alpha, \beta \leqslant \pi$, then*

$$\sin\frac{1}{2}(\alpha+\beta) \geqslant \sin\frac{1}{2}|\alpha - \beta|. \tag{2.5}$$

Next, we obtain estimates for the magnitude of the component functions P_k, Q_k and R_k of the fundamental polynomials A_k for $(0,1,2)$ interpolation.

Lemma 3. *For $-1 \leqslant x \leqslant 1$, the functions $P_k(x)$, $Q_k(x)$ and $R_k(x)$, as defined by (1.9), (1.10) and (1.11), satisfy*

$$|P_k(x)| \leqslant \begin{cases} 4, & \text{if } k = j, \\ \dfrac{4}{i^3}\, T_n(x)^2, & \text{if } i = |k-j| \geqslant 1, \end{cases} \tag{2.6}$$

$$|Q_k(x)| \leqslant \begin{cases} \dfrac{1}{2n^2}\, T_n(x)^2 + 2, & \text{if } k = j, \\ \left(\dfrac{1}{2n^2} + \dfrac{2}{i^2}\right) \dfrac{T_n(x)^2}{i}, & \text{if } i = |k-j| \geqslant 1, \end{cases} \tag{2.7}$$

$$|R_k(x)| \leqslant \begin{cases} \dfrac{n^2-1}{n^2}\, T_n(x)^2, & \text{if } k = j, \\ \dfrac{n^2-1}{n^2}\, \dfrac{T_n(x)^2}{i}, & \text{if } i = |k-j| \geqslant 1. \end{cases} \tag{2.8}$$

Proof: From (1.7) and (1.9) it follows that

$$P_k(x) = T_n(x)^2 \frac{\ell_k(x)}{n^2} \frac{1 - x x_k}{(x - x_k)^2}. \tag{2.9}$$

Now, as shown by Kis [4, p.32],

$$|T_n(x)| \leqslant 2n \sin \frac{1}{2} |t_j - t|. \tag{2.10}$$

Further,

$$\begin{aligned} \frac{1 - x x_k}{(x - x_k)^2} &= \frac{1 - \cos t \cos t_k}{(\cos t - \cos t_k)^2} \\ &= \frac{\sin^2 \frac{1}{2}(t_k + t) + \sin^2 \frac{1}{2}(t_k - t)}{4 \sin^2 \frac{1}{2}(t_k + t)\, \sin^2 \frac{1}{2}(t_k - t)} \\ &= \frac{1}{4} \left(\frac{1}{\sin^2 \frac{1}{2}(t_k + t)} + \frac{1}{\sin^2 \frac{1}{2}(t_k - t)} \right) \\ &\leqslant \frac{1}{2 \sin^2 \frac{1}{2}(t_k - t)}, \end{aligned} \tag{2.11}$$

where the last inequality follows from (2.5). Also, if $i = |k - j| \geqslant 1$, then

$$\sin \frac{1}{2} |t_k - t| \geqslant \sin \left(\frac{2i-1}{4n} \pi \right) \geqslant \frac{2i-1}{2n} \geqslant \frac{i}{2n}. \tag{2.12}$$

Thus, by (2.9), (2.10), (2.11), (2.12),

$$|P_k(x)| \leqslant \begin{cases} 2\,|\ell_j(x)|, & \text{if } k = j, \\ \dfrac{2}{i^2}\, |\ell_k(x)|\, T_n(x)^2, & \text{if } i = |k-j| \geqslant 1, \end{cases}$$

and then (2.6) results from (2.3).

For $Q_k(x)$, we note from (1.7) and (1.10) that

$$Q_k(x) = -T_n(x)^2 \frac{\ell_k(x)}{2n^2} \frac{x_k}{x - x_k}. \tag{2.13}$$

Consider

(2.14)

$$\begin{aligned}\left|\frac{x_k}{x - x_k}\right| &= \left|\frac{\cos t_k}{\cos t - \cos t_k}\right| \\ &= \frac{1}{2}\left|\frac{\cos\frac{1}{2}(t_k + t)\cos\frac{1}{2}(t_k - t) - \sin\frac{1}{2}(t_k + t)\sin\frac{1}{2}(t_k - t)}{\sin\frac{1}{2}(t_k + t)\sin\frac{1}{2}(t_k - t)}\right| \\ &\leqslant \frac{1}{2}\left(1 + \frac{1}{\sin\frac{1}{2}(t_k + t)\sin\frac{1}{2}\left|t_k - t\right|}\right) \\ &\leqslant \frac{1}{2}\left(1 + \frac{1}{\sin^2\frac{1}{2}(t_k - t)}\right),\end{aligned}$$

where (2.5) has been used to obtain the final inequality. Thus, by (2.10), (2.12), (2.13), (2.14),

$$|Q_k(x)| \leqslant \begin{cases} \left(\dfrac{T_n(x)^2}{4n^2} + 1\right) |\ell_j(x)|, & \text{if } k = j, \\ \left(\dfrac{1}{4n^2} + \dfrac{1}{i^2}\right) |\ell_k(x)|\, T_n(x)^2, & \text{if } i = |k - j| \geqslant 1, \end{cases}$$

from which (2.7) follows by applying (2.3).

Finally, (2.8) is an immediate consequence of the identity

$$R_k(x) = T_n(x)^2 \frac{n^2 - 1}{2n^2} \ell_k(x), \tag{2.15}$$

and the inequalities (2.3). $\square$

We shall also need a lemma of Goodenough and Mills [3, Lemma 3].

Lemma 4. *For $x = \cos t$, $0 \leqslant t \leqslant \pi$, and with j chosen so that* $\min\{|t_k - t| : k = 1, 2, \ldots, n\} = |t_j - t|$, *then*

$$|t_j - t| \leqslant \frac{\pi}{2n} |\cos nt| = \frac{\pi}{2n} |T_n(x)|. \tag{2.16}$$

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3. Proof of Theorem 1

Because $\sum_{k=1}^{n} H_{n,3}(1,x) = 1$, it follows from (1.6) that

$$|H_{n,3}(f,x) - f(x)| = \left|\sum_{k=1}^{n} (f(x_k) - f(x))A_k(x)\right|,$$

where $A_k(x)$ is given by (1.8), (1.9), (1.10) and (1.11). On putting

$$U_k(x) = (f(x_k) - f(x))A_k(x),$$

we obtain

$$\begin{aligned}(3.1)\qquad |H_{n,3}(f,x) - f(x)| &\leqslant \left|\sum_{k=1}^{j-1} U_k(x)\right| + |U_j(x)| + \left|\sum_{k=j+1}^{n} U_k(x)\right| \\ &= I_1 + I_2 + I_3, \quad \text{say.}\end{aligned}$$

(Clearly if $j = 1$ or n, then one of these terms will not be present.)

First, we estimate I_3. If $n - j$ is odd, then

$$\begin{aligned}(3.2)\qquad I_3 \leqslant\ & |U_{j+1}(x) + U_{j+2}(x)| + |U_{j+3}(x) + U_{j+4}(x)| + \dots \\ &+ |U_{n-2}(x) + U_{n-1}(x)| + |U_n(x)|.\end{aligned}$$

(If $n - j$ is even, the final term of this sum is $|U_{n-1}(x) + U_n(x)|$.) Now for $k = j + i$, $i = 1, 3, 5, \dots$, $i < n - j$,

(3.3)
$$\begin{aligned}&|U_k(x) + U_{k+1}(x)| \\ &\quad = |(f(x_k) - f(x))A_k(x) + (f(x_{k+1}) - f(x))A_{k+1}(x)| \\ &\quad \leqslant |A_k(x) + A_{k+1}(x)|\ |f(x_k) - f(x)| + |A_{k+1}(x)|\ |f(x_{k+1}) - f(x_k)|.\end{aligned}$$

Consider

(3.4)
$$\begin{aligned}|A_k(x) + A_{k+1}(x)| &\leqslant (|P_k(x)| + |P_{k+1}(x)|) + (|Q_k(x)| + |Q_{k+1}(x)|) \\ &\qquad + |R_k(x) + R_{k+1}(x)| \\ &= (|P_k(x)| + |P_{k+1}(x)|) + (|Q_k(x)| + |Q_{k+1}(x)|) \\ &\qquad + \frac{n^2-1}{2n^2} T_n(x)^2\ |\ell_k(x) + \ell_{k+1}(x)| \\ &\leqslant T_n(x)^2 \left(\frac{8}{i^3} + \left(\frac{1}{n^2 i} + \frac{4}{i^3}\right) + \frac{7}{2i^2}\right), \text{ by (2.4), (2.6), (2.7),} \\ &\leqslant \frac{33}{2}\, \frac{T_n(x)^2}{i^2}.\end{aligned}$$

Also,

(3.5)
$$\begin{aligned}|A_k(x)| &\leqslant |P_k(x)| + |Q_k(x)| + |R_k(x)| \\ &\leqslant T_n(x)^2\left(\frac{4}{i^3} + \frac{1}{2n^2 i} + \frac{2}{i^3} + \frac{n^2-1}{n^2 i}\right), \text{ by (2.6), (2.7), (2.8),} \\ &\leqslant 7\,\frac{T_n(x)^2}{i}.\end{aligned}$$

Thus, by (2.1), (2.2), (3.3), (3.4) and (3.5), we deduce that

$$\begin{aligned}|U_k(x) + U_{k+1}(x)| &\leqslant T_n(x)^2\left[\frac{33}{2i^2}\left(5\omega\left(\frac{i\sin t}{n}\right) + 13\omega\left(\frac{i^2}{n^2}\right)\right)\right. \\ &\qquad \left. + \frac{7}{i}\left(4\omega\left(\frac{\sin t}{n}\right) + 20\omega\left(\frac{i}{n^2}\right)\right)\right] \\ &= O(1)\,\frac{T_n(x)^2}{i}\left[\omega\left(\frac{\sin t}{n}\right) + \omega\left(\frac{i}{n^2}\right)\right].\end{aligned} \tag{3.6}$$

(Here, and elsewhere, $O(1)$ terms can be made independent of i, f and x.)

If $n-j$ is odd, we need an estimate for $|U_n(x)|$. Now, by (2.1) and (3.5),

$$\begin{aligned}|U_n(x)| &= |f(x_n) - f(x)|\,|A_n(x)| \\ &\leqslant 7\,\frac{T_n(x)^2}{n-j}\left(5\omega\left(\frac{(n-j)\sin t}{n}\right) + 13\omega\left(\frac{(n-j)^2}{n^2}\right)\right) \\ &= O(1)\,T_n(x)^2\left[\omega\left(\frac{\sin t}{n}\right) + \omega\left(\frac{1}{n}\right)\right].\end{aligned} \tag{3.7}$$

However,

$$\begin{aligned}\omega\left(\frac{1}{n}\right) &= \sum_{i=1}^{n}\frac{1}{n}\,\omega\left(\frac{n}{i}\times\frac{i}{n^2}\right) \\ &\leqslant \sum_{i=1}^{n}\frac{1}{n}\left(\frac{n}{i}+1\right)\omega\left(\frac{i}{n^2}\right) \\ &= O(1)\sum_{i=1}^{n}\frac{1}{i}\,\omega\left(\frac{i}{n^2}\right).\end{aligned} \tag{3.8}$$

Since $\sin t = \sqrt{1-x^2}$, we conclude from (3.2), (3.6), (3.7), (3.8),

$$I_3 = O(1)\,T_n(x)^2\left[\log n\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n}\frac{1}{i}\,\omega\left(\frac{i}{n^2}\right)\right]. \tag{3.9}$$

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Similarly, we can write

$$(3.10)\qquad I_1 = O(1)\,T_n(x)^2 \left[\log n\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i}\,\omega\left(\frac{i}{n^2}\right)\right],$$

and so it remains to estimate I_2.

We have

$$(3.11)\qquad \begin{aligned} |f(x_j) - f(x)| &\leqslant \omega(|x_j - x|) \\ &\leqslant \omega(|t_j - t|) \\ &\leqslant \omega\left(\frac{\pi}{2n}\,|T_n(x)|\right), \qquad \text{by (2.16)}, \\ &= O(1)\,\omega\left(\frac{|T_n(x)|}{n}\right). \end{aligned}$$

Also, by (2.6), (2.7), (2.8),

$$|A_j(x)| \leqslant 6 + \frac{2n^2-1}{2n^2}\,T_n(x)^2 < 7.$$

Therefore

$$(3.12)\qquad I_2 = |f(x_j) - f(x)|\,|A_j(x)| = O(1)\,\omega\left(\frac{|T_n(x)|}{n}\right).$$

The proof of the theorem is now completed by substituting the results (3.9), (3.10) and (3.12) into (3.1). $\square$

4. Proof of Theorem 2

We begin by noting that $H_{n,3}(1,x) = H_{n,1}(1,x) = 1$, and so $(\Lambda_n 1)(x) = 1$. Also, by (1.5), (1.6), (1.8) and (2.15), the definition (1.12) of Λ_n can be rewritten as

$$(\Lambda_n f)(x) = \frac{1}{W_n(x)}\left(\sum_{k=1}^{n}(P_k(x) + Q_k(x))f(x_k)\right).$$

Therefore, since $W_n(x) > 1/2$, $-1 \leqslant x \leqslant 1$, we obtain

$$(4.1)\qquad \begin{aligned} &|(\Lambda_n f)(x) - f(x)| \\ &\quad < 2\left|\sum_{k=1}^{n}(P_k(x) + Q_k(x))(f(x_k) - f(x))\right| \end{aligned}$$

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$$\leqslant 2\left(\sum_{k=1}^{j-1}|P_k(x)+Q_k(x)|\,|f(x_k)-f(x)|+|P_j(x)+Q_j(x)|\,|f(x_j)-f(x)|\right.$$
$$\left.+\sum_{k=j+1}^{n}|P_k(x)+Q_k(x)|\,|f(x_k)-f(x)|\right)$$
$$=2(J_1+J_2+J_3),\qquad \text{say.}$$

Consider J_3. If $i=k-j$, $i\geqslant 1$, then by (2.6) and (2.7) we have

$$|P_k(x)+Q_k(x)|\leqslant\left(\frac{6}{i^3}+\frac{1}{2n^2i}\right)T_n(x)^2=O(1)\,\frac{T_n(x)^2}{i^3}.$$

From (2.1) it then follows that

$$J_3=O(1)\,T_n(x)^2\left(\sum_{i=1}^{n}\frac{1}{i^3}\,\omega\!\left(\frac{i\sin t}{n}\right)+\sum_{i=1}^{n}\frac{1}{i^3}\,\omega\!\left(\frac{i^2}{n^2}\right)\right) \tag{4.2}$$
$$=O(1)\,T_n(x)^2\left(\sum_{i=1}^{n}\frac{1}{i^2}\,\omega\!\left(\frac{\sin t}{n}\right)+\sum_{i=1}^{n}\frac{1}{i^3}\left(\frac{i^2}{n}+1\right)\omega\!\left(\frac{1}{n}\right)\right)$$
$$=O(1)\,T_n(x)^2\left[\omega\!\left(\frac{\sqrt{1-x^2}}{n}\right)+\frac{\log n}{n}\,\omega\!\left(\frac{1}{n}\right)\right].$$

Similarly,

$$J_1=O(1)\,T_n(x)^2\left[\omega\!\left(\frac{\sqrt{1-x^2}}{n}\right)+\frac{\log n}{n}\,\omega\!\left(\frac{1}{n}\right)\right]. \tag{4.3}$$

Finally, by (2.6) and (2.7) we have

$$|P_j(x)+Q_j(x)|\leqslant 6+\frac{T_n(x)^2}{2n^2}\leqslant\frac{13}{2},$$

and so (by (3.11)),

$$J_2=O(1)\,\omega\!\left(\frac{|T_n(x)|}{n}\right). \tag{4.4}$$

Upon substituting (4.2), (4.3) and (4.4) in (4.1), the desired result (1.13) is obtained. ▯

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