A NEW BOUND FOR NIL U-RINGS

R. G. BIGGS

A U-ring is a ring in which every subring is a meta ideal. A meta ideal of a ring R is a subring I of R which lies in a chain of subrings,

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_\beta = R,$$

with the properties:

(1) I_λ is an ideal of $I_{\lambda+1}$ for all $\lambda < \beta$;

(2) If α is a limit ordinal number, then $I_\alpha = \bigcup_{\lambda<\alpha} I_\lambda$.

Freidman [**3**] proved that every nil U-ring is a locally nilpotent ring. Since there are many locally nilpotent rings which are not U-rings, the class of locally nilpotent rings is not a very good bound for the class of nil U-rings. This paper establishes a new bound for nil U-rings based on a property of the multiplicative semigroup of the ring.

Example. Let $B = \{y_s: s \in (0, 1)$ and s is a rational number$\}$. Define multiplication in B by the rule: $y_s y_t = y_{s+t}$ if $s + t < 1$; otherwise $y_s y_t = 0$. Let p be any prime number. The *Zassenhaus Example modulo* p is the algebra over the field of integers modulo p with basis B. More generally, any algebra with basis B will be called a Zassenhaus Example.

The theorem below shows that a Zassenhaus Example is not a U-ring. However, such rings are Baer radical rings (see [**2**]), and hence are locally nilpotent. The following theorem shows that the class of U-rings excludes all rings which have a multiplicative structure similar to a Zassenhaus Example.

THEOREM. *Suppose that a ring* R *has a sequence of elements,* $\{x_i: i \in N\}$, *such that* $x_i^{n_i} = x_{i-1}$ *where* $n_i \geqq 2$ *for all* $i \in N$ *and* $x_1 \neq 0$ *while* $x_0 = 0$. *Then* R *is not a U-ring.*

The following lemmas are needed to establish the proof. In each of the lemmas, S denotes any ring of the type indicated below.

Let W be a subset of $(0, 1) \cap Q$ (Q = rational numbers) which has the properties:

(A) if $s, t \in W$ and $s + t < 1$, then $s + t \in W$,

(B) if $s, t \in W$ and $s - t > 0$, then $s - t \in W$,

(C) 0 is an accumulation point of W (in the usual topology).

Let S be any ring which has the set of generators, $\{y_s: s \in W\}$, which for all $s, t \in W$ satisfy the relations:

Received February 5, 1969.

(1) $y_s y_t = y_{s+t}$ if $s + t < 1$,
(2) $y_s y_t = 0$ if $s + t \geqq 1$.

LEMMA 1. *Suppose that* $y_{s_1}, y_{s_2} \in S$ *and* $s_1 < s_2 < 1$ *and* L_i *is the characteristic of* y_{s_i} *for* $i = 1, 2$. *Then* L_2 *divides* L_1.

Proof. Note that $L_1 y_{s_1} = 0$ implies that $(L_1 y_{s_1}) y_{(s_2 - s_1)} = L_1 y_{s_2} = 0$. Since $L_2 y_{s_2} = 0$, L_3, the greatest common divisor of L_1 and L_2 must be a solution of the equation $X y_{s_2} = 0$. Since L_2 is the smallest positive integral solution of this equation, L_2 must be L_3 and therefore L_2 does divide L_1.

Definition. A *point* in S will be an element of the form y_t.

If the additive characteristic of every or all but one non-zero element in the ring S is 0, define $G = 0$. Otherwise let

$$G^* = \min\{\text{char}(y_s) \colon y_s \in S \text{ and } \text{char}(y_s) > 1\}.$$

Let $y_{s_0} \in S$ be any element with characteristic G^*. Either (1) y_{s_0} is the only point in S which has characteristic G^* or (2) there exists a maximum open interval $(a_1, a_2) \subseteq (0, 1)$ such that $t \in (a_1, a_2)$ implies that y_t has characteristic G^*. In case (1), let

$$G = \min\{\text{char}(y_s) \colon y_s \in S \text{ and } \text{char}(y_s) > G^*\}$$

and let y_{s_1} be a point in S which has characteristic G. Then every point y_t, where $s_1 < t < s_0$, must have characteristic G by Lemma 1. Hence there exists a maximum open interval $(a_1, a_2) \subseteq (0, 1)$ such that $t \in (a_1, a_2)$ implies that y_t has characteristic G. In case (2), let $G = G^*$. Note also that if $G = 0$, then there is a maximum open interval $(a_1, a_2) \subseteq (0, 1)$ such that $t \in (a_1, a_2)$ implies that y_t has characteristic 0.

Definition. G is called the *primary characteristic* of S; (a_1, a_2) is called the *primary interval* of S.

Definition. A *formal additive relationship* in S is an equation of the form $\sum_{i=1}^{h} L_i y_{s_i} = 0$, where $s_i = s_j$ implies that $i = j$, $L_i \in Z$, and $L_i y_{s_i} \neq 0$ for every i in $[1, h]$.

LEMMA 2. *There exists no formal additive relationships in* S *in which every term has subscripts which lie in the primary interval* (a_1, a_2).

Proof. Let h be the least positive number of terms that a formal additive relationship has, when every term has subscripts in (a_1, a_2). Suppose that $\sum_{i=1}^{h} L_i y_{s_i} = 0$ is a formal additive relationship where $s_i \in (a_1, a_2)$ for every i in $[1, h]$. Let $s_m = \max\{s_1, \ldots, s_h\}$ and $s_l = \min\{s_1, \ldots, s_h\}$. Given any $u > 0$ there exists a rational number $s < u$ such that $y_s \in S$.

Due to this fact, there exists $y_t \in S$ such that $t + s_l < a_2 < t + s_m$. Since $L_m y_{(s_m+t)} = 0$,

$$\sum_{i=1}^{h} L_i y_{(s_i+t)} = \left(\sum_{i=1}^{h} L_i y_{s_i}\right) y_t = 0$$

can be rewritten as a formal additive relationship in (a_1, a_2) with fewer than h terms. This is a contradiction.

LEMMA 3. *There exists no formal additive relationships in S in which any term has the form Hy_t, where G does not divide H and $t < g/2$, where g is the length of the primary interval, (a_1, a_2).*

Proof. Suppose that $Hy_t + \sum_{j=1}^{m} L_j y_{s_j} = 0$ is a formal additive relationship where G does not divide H and $t < g/2$. Suppose also that

$$s_1 < \ldots < s_h < t < s_{h+1} < \ldots < s_m.$$

There exists $y_u \in S$ such that $a_1 + g/2 < t + u < a_2$. Then

$$\left(Hy_t + \sum_{j=1}^{m} L_j y_{s_j}\right) y_u = 0$$

is an additive relationship in which every term lies in (a_1, a_2) but not every term is 0 since $Hy_{(t+u)} \neq 0$. Consequently, this can be rewritten as a formal additive relationship in the primary interval, which contradicts Lemma 2.

Definition. A point $y_s \in S$ is an *M-endpoint* if $My_s \neq 0$ but $My_t = 0$ for every $t > s$ where M is an integer.

Definition. If y_s is an M-endpoint for some integer M and L is the smallest positive integer such that y_s is an L-endpoint, then L is the *near characteristic* of y_s.

LEMMA 4. *Every dense subset of an open interval $(b_1, b_2) \subseteq (0, 1)$ contains points s such that y_s is not an M-endpoint for any $M \in Z$ or there is no point y_s in S.*

Proof. If the M-endpoints in S are ordered according to their near characteristics, then no two M-endpoints have the same near characteristics and as the near characteristics of the M-endpoints increase towards infinity, the y-subscripts decrease towards 0. Since the positive integers have only one limit point (plus infinity), the y-subscripts of the M-endpoints in S have at most one limit point. But every dense subset of the interval $(b_1, b_2) \subseteq (0, 1)$ has infinitely many limit points. Hence some of the points in the dense subset of (b_1, b_2) either are not the y-subscripts of any M-endpoints in S or are not the y-subscripts of any points in S at all.

The proof of the theorem will now be given.

Since every subring of a U-ring is a U-ring, it is sufficient to show that a subring of R is not a U-ring. Let S be the subring of R generated by $\{x_i: i \in N\}$.

https://doi.org/10.4153/CJM-1970-049-0 Published online by Cambridge University Press

Then S is commutative. Moreover, for all $k, p \in N$, $(x_k)^p$ can be renamed as $y_{(p/d)}$, where $d = \prod_{i=1}^{k} n_i$ if $p/d < 1$; otherwise $(x_k)^p = 0$. Then if $y_s, y_t \in S$, they are both powers of some x_i in the sequence generating S, and therefore $y_s y_t = y_{s+t}$ (which may be 0 if $s + t > (n_1 - 1/n_1)$). Note that

$$W = \{s \in (0, 1): y_s \in S\}$$

has the properties (A), (B), and (C).

Let $E = \{y_{1/k} \in S: k \in N\}$ and let $P(S) = \{\text{primes } p: p \text{ divides } k \text{ for some } k \in N \text{ such that } y_{1/k} \in E\}$.

Case (1). Suppose that $P(S)$ is an infinite set. Then choose $p_0 \in P(S)$ and let $T = \{\sum_{i=1}^{h} L_i y_{l_i/k_i} + \sum_{j=1}^{m} M_j y_{s_j} + \sum_{w=1}^{v} H_w y_{t_w} \in S: L_i \in Z, (l_i, k_i) = 1,$ and $(p_0, k_i) = 1$ for all i in $[1, h]$; $M_j \in Z$, and y_{s_j} is an M_j-endpoint for all j in $[1, m]$; $H_w \in Z$, and either $t_w \geqq g/2$ or G divides H_w for every w in $[1, v]\}$.

Note that the set $\{l/k: k, l \in N$ and p_0 divides $k\}$ is dense in $(0, g/2)$. From the proof of Lemma 4 there exists some $y_t \in S$ such that $t \in (0, g/2)$, $t = l/k$, where p_0 divides k, and y_t is not an M-endpoint for any integer M. By Lemma 3 there exists no formal additive relationships involving elements of the form $H_w y_{t_w}$, where $t_w < g/2$ and G does not divide H_w. Hence $y_t \in S \sim T$, and therefore $T \neq S$. Note that the product of an M-endpoint with any other element in S is 0 and that $(H_w y_{t_w}) \cdot (L y_u) = L H_w y_{t_w + u}$, where either G divides LH_w or $t_w + u > g/2$ for every w in $[1, v]$. If p_0 divides neither k_1 nor k_2, then p_0 does not divide $k_1 k_2$. Consequently,

$$(L_1 y_{l_1/k_1})(L_2 y_{l_2/k_2}) = L_1 L_2 y_{(l_1 k_2 + l_2 k_1)/k_1 k_2}$$

lies in T if $L_i y_{l_i/k_i} \in T$ for $i = 1, 2$. Hence T is a subring of S since it is closed under addition and multiplication. If $L y_{l/k} \in S \sim T$, and $(l, k) = 1$, then p_0 divides k, $l/k < g/2$, L does not divide G, and there exists $t > l/k$ such that $L y_t \neq 0$. Since $P(S)$ is an infinite set, there exists $y_{1/k_1} \in T$ such that $1/k_1 + l/k < \min\{g/2, t\}$. Consequently, $(L y_{l/k})(y_{1/k_1}) = L y_{(lk_1 + k)/kk_1}$ is not 0 and is not in T since p_0 divides kk_1, $(p_0, lk_1 + k) = 1$ and by Lemma 3 this element cannot be expressed as a sum of terms which lie in T. Hence $L y_{l/k}$ is not in the idealizer of T, and T is its own idealizer in S due to the arbitrary nature of this element.

Case (2). Suppose that $P(S)$ is a finite set. Then choose $p_1 \in P(S)$ such that p_1 divides an infinite number of terms in the sequence $\{n_i: i \in N\}$. Note that every power of p_1 divides some k such that $y_{1/k} \in E$. Let

$$Q = \left\{ \sum_{i=1}^{u} L_i y_{l_i/k_i} \in S: L_i \in Z,\ (l_i, k_i) = 1, \text{ and } k_i = p_1{}^n \right.$$
$$\left. \text{for some } n \in N \text{ for all } i \text{ in } [1, h] \right\}.$$

https://doi.org/10.4153/CJM-1970-049-0 Published online by Cambridge University Press

Let q be a prime such that $q \notin P(S)$ and let

$$Q^* = \left\{ \sum_{i=1}^{h} L_i y_{q l_i/k_i} + \sum_{j=1}^{m} M_j y_{s_j} + \sum_{w=1}^{v} H_w y_{t_w} \in Q: L_i y_{l_i/k_i} \in Q \text{ for all } i \text{ in } [1, h]; \right.$$
$$M_j \in Z, \text{ and } y_{s_j} \text{ is an } M_j\text{-endpoint for all } j \text{ in } [1, m];$$
$$\left. H_w \in Z, \text{ and either } t_w \geqq g/2 \text{ or } G \text{ divides } H_w \text{ for all } w \text{ in } [1, v] \right\}.$$

Note that the set $\{l/p_1^n: l, n \in N$ and $(l, p_1 q) = 1$ is dense in $(0, g/2)$. From the proof of Lemma 4, it follows that there exists a point $y_t \in S$ such that $t \in (0, g/2)$, y_t is not an M-endpoint for any integer M, and $t = l/p_1^n$, where $(l, p_1 q) = 1$. By Lemma 3 there exists no formal additive relationships involving elements of the form $H_w y_{t_w}$, where $t_w < g/2$ and G does not divide H_w. Hence $y_t \in Q \sim Q^*$ and therefore $Q \neq Q^*$. Now, note that if $L_1 y_{q l_1/k_1}$ and $L_2 y_{q l_2/k_2}$ are elements in Q^*, their product, $L_1 L_2 y_d$, where

$$d = q(l_1 k_2 + l_2 k_1)/k_1 k_2,$$

is an element in Q^*. Since the statements found in Case (1) on M_j-endpoints and elements of the form $H_w y_{t_w}$, where either $t_w \geqq g/2$ or G divides H_w apply in this case also, Q^* is a subring of Q.

If $L y_{l/k} \in Q \sim Q^*$ and $(l, k) = 1$, then $(q, l) = 1$, G does not divide L, $l/k < g/2$, and there exists a rational number $t > l/k$ such that $L y_t \neq 0$. Note that $\min\{t, g/2\} < (l/k + q/p_1^n)$ for some natural number n and there exists a point $y_{1/k_1} \in E$ such that p_1^n divides k_1. Consequently,

$$(L y_{l/k})(y_{q/p_1^n}) = L y_{(l p_1^n + qk)/k p_1^n}$$

which is not 0 and does not lie in Q^* since $(q, l p_1^n + qk) = 1$ and by Lemma 3 this element cannot be expressed as a sum of terms which lie in Q^*. Hence $L y_{l/k}$ is not in the idealizer of Q^*, and Q^* is its own idealizer in Q due to the arbitrary nature of this element.

References

1. R. Baer, *Meta ideals*, Report of a conference on linear algebras, June, 1956, pp. 33–52 (National Academy of Sciences–National Research Council, Washington, Publ., 1957).
2. N. J. Divinsky, *Rings and radicals* (Univ. Toronto Press, Toronto, Ontario, 1965).
3. P. A. Freĭdman, *Rings with an idealizer condition*. I, Izv. Vysš. Učebn. Zaved. Matematika *1960*, no. 2 (15), 213–222. (Russian)

The University of Western Ontario,
London, Ontario