BULL. AUSTRAL. MATH. SOC. 43A80
VOL. 63 (2001) [35–58]

AN $(n+1)$–FOLD MARCINKIEWICZ MULTIPLIER THEOREM ON THE HEISENBERG GROUP

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We prove a Marcinkiewicz-type multiplier theorem on the Heisenberg group: for $1 < p < \infty$, we establish the boundedness on $L^p(\mathbb{H}_n)$ of spectral multipliers $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ of the n partial sub-Laplacians $\mathcal{L}_1, \dots, \mathcal{L}_n$ and iT, where m satisfies an $(n+1)$-fold Marcinkiewicz-type condition. We also establish regularity and cancellation conditions which the convolution kernels of these Marcinkiewicz multipliers $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ satisfy.

1. INTRODUCTION

The Marcinkiewicz multiplier theorem in $\mathbb{R}^n$ (see [**1**, **5**, **8**]) establishes the boundedness on L^p of multiplier operators, for a class of multipliers which is invariant under multi-parameter dilations. One can view these operators as functions of $i\dfrac{\partial}{\partial x_1}, \dots, i\dfrac{\partial}{\partial x_n}$, and thus natural corresponding operators to consider on the Heisenberg group are functions $m(\mathcal{L}, iT)$ of iT and the sub-Laplacian $\mathcal{L}$, where m satisfies a two-fold Marcinkiewicz-type condition,

$$\left|(\xi\partial_\xi)^i(\eta\partial_\eta)^j m(\xi,\eta)\right| \leqslant C_{ij},$$

or functions $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ of iT and the partial sub-Laplacians $\mathcal{L}_1, \dots, \mathcal{L}_n$, where m satisfies an $(n+1)$-fold Marcinkiewicz-type condition,

$$\left|(\xi_1\partial_{\xi_1})^{i_1}\cdots(\xi_n\partial_{\xi_n})^{i_n}(\eta\partial_\eta)^j m(\xi,\eta)\right| \leqslant C_{ij}. \tag{1}$$

In [**6**], Müller, Ricci and Stein study the first case. In this and subsequent papers, we use their methods to study the second case.

Here, we prove the boundedness on $L^p, 1 < p < \infty$, of these Marcinkiewicz multiplier operators and establish regularity and cancellation conditions satisfied by their convolution kernels. In the proof, multi-parameter methods cannot be used directly on the Heisenberg group, for, to begin with, multi-parameter scaling is not automorphic. Moreover, unlike the partial derivatives in $\mathbb{R}^n$, the operators $\mathcal{L}_1, \dots, \mathcal{L}_n$ and iT on $\mathbb{H}_n$ do not act independently. However, by lifting to the product group $G = \mathbb{H} \times \cdots \times \mathbb{H} \times \mathbb{R}$,

Received 6th March, 2000

https://doi.org/10.1017/S0004972700019092 Published online by Cambridge University Press

one pulls apart the intertwined actions of these operators, thus bringing the situation to a pure product one. L^p-boundedness and product-type regularity and cancellation conditions on the kernels are known by multi-parameter methods for the lifted Marcinkiewicz multipliers $m(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$ on G. The L^p-boundedness of the operators $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ on $\mathbb{H}_n$ then follows by the method of transference, and the bulk of the work in this paper consists of transferring the conditions for the kernels down to obtain conditions satisfied by kernels of Marcinkiewicz multipliers $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ on $\mathbb{H}_n$.

In fact, these conditions characterise the convolution kernels of Marcinkiewicz multipliers. A subsequent paper will show their sufficiency, making use of the explicit expression of Geller [4] for the Gelfand transform of polyradial functions on $\mathbb{H}_n$, in terms of Laguerre functions.

2. Preliminaries

Let $\mathbb{H}_n$ denote the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication

$$(z,t)(w,s) = (z + w, t + s + 2 \operatorname{Im} z \cdot \overline{w}).$$

The identity for this multiplication is $(0,0)$, and the inverse $(z,t)^{-1}$ of (z,t) is $(-z,-t)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r > 0$, by

$$r(z,t) = (rz, r^2 t) \ .$$

These dilations are group automorphisms. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|h| = \left|(z,t)\right| = \left(|z|^2 + |t|\right)^{1/2} .$$

Using coordinates $h = (z,t) = (x + iy, t)$ for points in $\mathbb{H}_n$, the left-invariant vector fields X_j, Y_j and T on $\mathbb{H}_n$ equal to $\dfrac{\partial}{\partial x_j}$, $\dfrac{\partial}{\partial y_j}$ and $\dfrac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \qquad \text{and} \qquad T = \frac{\partial}{\partial t}$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ with commutation relations

$$[Y_j, X_j] = 4T,$$

for $j = 1, \dots, n$, and all other commutators equal to 0.

A differential operator D on $\mathbb{H}_n$ is called homogeneous of degree d if

$$D\big(f(r\cdot)\big) = r^d (Df)(r\cdot).$$

Thus $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ are homogeneous of degree one, and T is homogeneous of degree two.

The homogeneous dimension of $\mathbb{H}_n$ is $2n+2$, the sum of the degrees of the homogeneous basis elements $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, and T.

The sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_n$ is given by

$$\mathcal{L} = -\frac{1}{4}\sum_{j=1}^{n}(X_j^2 + Y_j^2)$$

and the partial sub-Laplacians $\mathcal{L}_1, \ldots, \mathcal{L}_n$ by

$$\mathcal{L}_j = -\frac{1}{4}(X_j^2 + Y_j^2)\ .$$

The operators $\mathcal{L}_1, \ldots, \mathcal{L}_n$, and iT form a family of commuting self-adjoint operators, and so, by the Spectral Theorem, for $m \in L^\infty\big((\mathbb{R}^+)^n \times \mathbb{R}\big)$, we can define the joint spectral multiplier operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ which is then a bounded operator on $L^2(\mathbb{H}_n)$. Since $\mathcal{L}_1, \ldots, \mathcal{L}_n$, iT are left-invariant, $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ commutes with left translations and is therefore given by convolution with a distribution $K \in \mathcal{S}'(\mathbb{H}_n)$: $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)\varphi = \varphi * K$, for all $\varphi \in \mathcal{S}(\mathbb{H}_n)$.

Given any $\tau \in \mathbb{T}^n$, the n-torus, define the operator ρ_τ on functions f on $\mathbb{H}_n$ by

$$\rho_\tau f(z,t) = f(\tau \cdot z, t)\ .$$

A function f on $\mathbb{H}_n$ will be called *polyradial* if $f = \rho_\tau f$ for all $\tau \in \mathbb{T}^n$. A distribution $K \in \mathcal{S}'(\mathbb{H}_n)$ is said to be *polyradial* if

$$K(\varphi) = K(\rho_\tau \varphi)$$

for all $\tau \in \mathbb{T}^n$ and all $\varphi \in \mathcal{S}(\mathbb{H}_n)$. Since $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and iT commute with all ρ_τ, for $\tau \in \mathbb{T}^n$, so does $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$. Therefore the convolution kernel K of $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ is polyradial.

3. The Lifting Argument

Let $\mathbb{H}^n$ denote the n-fold product $\mathbb{H} \times \cdots \times \mathbb{H}$ of the three-dimensional Heisenberg group $\mathbb{H}$ (not to be confused with the $2n+1$-dimensional Heisenberg group $\mathbb{H}_n$), and set $G = \mathbb{H}^n \times \mathbb{R}$. Elements

$$(z_1, u_1, \ldots, z_n, u_n, t)$$

of G, where $(z_i, u_i) \in \mathbb{H}$ and $t \in \mathbb{R}$, will also be denoted (z,u,t) or (h,t), with $(z,u) = h = (h_1, \ldots, h_n) \in \mathbb{H}^n$, $h_i = (z_i, u_i)$. The group G is a direct product of stratified groups, for which the product group results of [6] hold. In this section, we describe the lifting argument from the Heisenberg group $\mathbb{H}_n$ to the product group $G = \mathbb{H}^n \times \mathbb{R}$.

Each of the partial sub-Laplacians $\mathcal{L}_j$ acts on $\mathbb{H}_n$ in the x_j, y_j and t variables, and so in some sense is acting on just one copy of $\mathbb{H}$ in $\mathbb{H}_n$. However, as they all involve the t variable, the actions of $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and iT are not independent. The lifting argument consists of lifting the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ to the product group $\mathbb{H} \times \cdots \times \mathbb{H} \times \mathbb{R}$, and thus pulling apart the n copies of $\mathbb{H}$, so that the lifted operators $\mathcal{L}_1^\#, \ldots, \mathcal{L}_n^\#$, $iT^\#$ do act independently. We establish in Proposition 3.1 the relation between the kernels of the lifted operators $m(\mathcal{L}_1^\#, \ldots, \mathcal{L}_n^\#, iT^\#)$ and the kernels of Marcinkiewicz multipliers $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ on $\mathbb{H}_n$. This relation will enable us in Sections 5 and 6 to transfer conditions on the kernels from the product group G down to $\mathbb{H}_n$.

We define the homomorphism $\pi\colon G \longrightarrow \mathbb{H}_n$ by

$$\pi(z_1, u_1, \ldots, z_n, u_n, t) = \left(z, t + \sum_{i=1}^{n} u_i\right).$$

The kernel of π is the central subgroup

$$N = \left\{(z, u, t) : z = 0,\ t = -\sum_{i=1}^{n} u_i\right\},$$

which is isomorphic to $\mathbb{R}^n$. The Heisenberg group $\mathbb{H}_n$ can thus be identified with the quotient group G/N.

On the three-dimensional Heisenberg group $\mathbb{H}$ we have the usual left invariant vector fields X, Y, and T and the sub-Laplacian $\mathcal{L}_{\mathbb{H}} = -(1/4)(X^2 + Y^2)$. Denoting by $X_j^\#$, $Y_j^\#$, and $U_j^\#$ the lifted vector fields on G corresponding to the vector fields X, Y, and T for the j^{th} copy of $\mathbb{H}$ in G, then $\mathcal{L}_j^\# = -(1/4)\left[(X_j^\#)^2 + (Y_j^\#)^2\right]$ is the lifted operator corresponding to $\mathcal{L}_{\mathbb{H}}$ on the j^{th} copy of $\mathbb{H}$. We also denote by $T^\#$ the lifted operator on G corresponding to $\dfrac{d}{dt}$ on $\mathbb{R}$. The homomorphism $\pi\colon G \to \mathbb{H}_n$ carries these vector fields to $\mathbb{H}_n$ as follows:

$$d\pi(X_j^\#) = X_j\,, \quad d\pi(Y_j^\#) = Y_j$$
$$d\pi(U_j^\#) = d\pi(T^\#) = T \quad \text{and} \quad d\pi(\mathcal{L}_j^\#) = \mathcal{L}_j$$

when $j = 1, \ldots, n$.

The operators $\mathcal{L}_1^\#, \ldots, \mathcal{L}_n^\#$, and $iT^\#$ are self-adjoint, commuting operators, with commuting spectral measures. For $m \in L^\infty\big((\mathbb{R}^+)^n \times \mathbb{R}\big)$, the joint spectral multiplier $m(\mathcal{L}_1^\#, \ldots, \mathcal{L}_n^\#, iT^\#)$ is then a well-defined, bounded operator on $L^2(G)$.

For a function $f \in L^1(G)$, we define the function $f^\flat$ on the quotient $G/N \cong \mathbb{H}_n$ by integrating over cosets:

$$f^\flat(z, t) = \int_{\mathbb{R}^n} f\left(z_1, u_1, \ldots, z_n, u_n, t - \sum_{i=1}^{n} u_i\right) du.$$

Then $f^\flat \in L^1(\mathbf{H}_n)$, with

$$\|f^\flat\|_{L^1(\mathbf{H}_n)} \leqslant \|f\|_{L^1(G)}.$$

If f is a smooth function on G, then $f^\flat$ is smooth on $\mathbf{H}_n$, and for any $X \in \mathfrak{g}$, the Lie algebra of G,

$$d\pi(X)f^\flat = (Xf)^\flat.$$

Thus, $[\partial_{z_j} f]^\flat = \partial_{z_j} f^\flat$.

The kernels of the operators $m(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$ and $m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ are related by the $\cdot^\flat$ operator:

PROPOSITION 3.1. *Let m be a $C_c^k\big((\mathbf{R}^+)^n \times \mathbf{R}\big)$ function, supported away from the axes, and let*

$$m(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)f = f * K$$

for $f \in \mathcal{S}(G)$. Then

$$m(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)g = g * K^\flat$$

for $g \in \mathcal{S}(\mathbf{H}_n)$.

PROOF: Since m is a C_c^k function with support away from the axes, the convolution kernel K on G corresponding to $m(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$ is in $L^1(G)$.

First, consider functions m of the form $m(\xi, \eta) = m_1(\xi_1) \dots m_n(\xi_n) m_0(\eta)$. The convolution kernel on G corresponding to $m_1(\mathcal{L}_1^\#) \dots m_n(\mathcal{L}_n^\#) m_0(iT^\#)$ is

$$K(z, u, t) = k_1(z_1, u_1) \dots k_n(z_n, u_n) k_0(t)\ ,$$

where $k_i \in L^1(\mathbf{H})$ is the convolution kernel of $m_i(\mathcal{L}_{\mathbf{H}})$ on $\mathbf{H}$ for $i = 1, \dots, n$, $\mathcal{L}_{\mathbf{H}}$ is the sub-Laplacian on the three-dimensional Heisenberg group $\mathbf{H}$, and $k_0 \in L^1(\mathbf{R})$ is the convolution kernel of $m_0\Big(i\dfrac{d}{dt}\Big)$ on $\mathbf{R}$. We have

$$\begin{aligned} K^\flat(z, t) &= \int_{\mathbf{R}^n} K\Big(z, u, t - \sum_{i=1}^n u_i\Big)\, du \\ &= \int_{\mathbf{R}^n} k_1(z_1, u_1) \dots k_n(z_n, u_n) k_0\Big(t - \sum_{i=1}^n u_i\Big)\, du\ . \end{aligned}$$

Now, $\mathcal{L}_i$ acts only on the x_i, y_i, and t variables in $\mathbf{H}_n$. For a smooth function f on $\mathbf{H}_n$,

$$\mathcal{L}_i f(z, t) = \mathcal{L}_{\mathbf{H}} f_{z_{\hat{\imath}}}(z_i, t)\ ,$$

where, for $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, $z_{\hat{\imath}} = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \in \mathbf{C}^{n-1}$ and $f_{z_{\hat{\imath}}}$ is the function on $\mathbf{H}$ given by

$$f_{z_{\hat{\imath}}}(w, s) = f(z_1, \dots, z_{i-1}, w, z_{i+1}, \dots, z_n, s)\ .$$

https://doi.org/10.1017/S0004972700019092 Published online by Cambridge University Press

Thus

$$m_i(\mathcal{L}_i)f(z,t) = m_i(\mathcal{L}_{\mathbb{H}})f_{z_{\hat{i}}}(z_i,t) = f_{z_{\hat{i}}} *_{\mathbb{H}} k_i(z_i,t)$$
$$= \int_{\mathbb{H}} f\big((z,t)\{w,u\}_i^{-1}\big)k_i(w,u)\,dw\,du$$

where, for ease of notation, $\{w,u\}_i$ denotes the element $(0,\dots,w,\dots,0,u) \in \mathbb{H}_n$, in which the $w \in \mathbb{C}$ appears in the i^{th} complex entry, and all other complex entries are 0. Also,

$$iTf(z,t) = i\frac{d}{dt}f_z(t)$$

where f_z is the function on $\mathbb{R}$ given by $f_z(s) = f(z,s)$ and so

$$m_0(iT)f(z,t) = m_0\Big(i\frac{d}{dt}\Big)f_z(t) = f_z *_{\mathbb{R}} k_0(t) = \int_{\mathbb{R}} f\big((z,t)(0,s)^{-1}\big)k_0(s)\,ds\,.$$

Therefore, for $f \in \mathcal{S}(\mathbb{H}_n)$, $m_1(\mathcal{L}_1)\dots m_n(\mathcal{L}_n)m_0(iT)f(z,t)$ equals

$$\int_{\mathbb{H}}\cdots\int_{\mathbb{H}}\int_{\mathbb{R}} f\Big((z,t)\{w_1,u_1\}_1^{-1}\dots\{w_n,u_n\}_n^{-1}(0,s)^{-1}\Big)$$
$$k_1(w_1,u_1)\dots k_n(w_n,u_n)k_0(s)\,ds\,dw_n\,du_n\dots\,dw_1\,du_1\,. \tag{2}$$

But,

$$\{w_1,u_1\}_1^{-1}\dots\{w_n,u_n\}_n^{-1}(0,s)^{-1} = \Big(w, s+\sum_{i=1}^n u_i\Big)^{-1}.$$

Thus, changing variables in the s-integration, (2) becomes

$$\int_{\mathbb{C}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}} f\big((z,t)(w,s)^{-1}\big)k_1(w_1,u_1)\dots k_n(w_n,u_n)k_0\Big(s-\sum_{i=1}^n u_i\Big)\,ds\,du\,dw$$
$$= \int_{\mathbb{C}^n\times\mathbb{R}} f\big((z,t)(w,s)^{-1}\big)\int_{\mathbb{R}^n} k_1(w_1,u_1)\dots k_n(w_n,u_n)k_0\Big(s-\sum_{i=1}^n u_i\Big)\,du\,dw\,ds$$
$$= \int_{\mathbb{H}_n} f\big((z,t)(w,s)^{-1}\big)K^\flat(w,s)\,dw\,ds = f * K^\flat(z,t)$$

which proves the proposition for m of the form $m(\xi,\eta) = m_1(\xi_1)\dots m_n(\xi_n)m_0(\eta)$.

For a general C_c^k function, the result holds by approximating m by sums of functions of the product form $m_1(\xi_1)\cdots m_n(\xi_n)m_0(\eta)$. $\square$

4. Product Theory

The lifted group $G = \mathbb{H}\times\dots\mathbb{H}\times\mathbb{R}$ is a product group with automorphic multi-parameter dilations, in which the lifted operators $\mathcal{L}_1^\#,\dots,\mathcal{L}_n^\#$, and $iT^\#$ act independently in different variables. Thus this is a pure product setting in which the product group

https://doi.org/10.1017/S0004972700019092 Published online by Cambridge University Press

results of [6] can be applied. In [6], in analogy to the situation in $\mathbb{R}^n$, a dyadic decomposition on the Fourier transform side is used to show that Marcinkiewicz multiplier operators are bounded on L^p, $1 < p < \infty$, and have product-type kernels. We state this result explicitly in Theorem 4.1 in the context we are interested in here; that is, for the product group $G = \mathbb{H} \times \cdots \times \mathbb{H} \times \mathbb{R}$.

First we define what will be referred to as a product-type kernel on a product group $G = G_1 \times \cdots \times G_N$. We state this in the generality of homogeneous groups (for a definition of homogeneous groups, see for, for example, [3], or Chapter XIII of [9]). However, in this paper we only consider the homogeneous groups $\mathbb{H}_n$ (where $|(z,t)| = \left(|z|^2 + |t|\right)^{1/2}$, and the homogeneous dimension $Q = 2n+2$), $\mathbb{C}$ (where $Q = 2$), and $\mathbb{R}$ ($Q = 1$).

For $j \in \{1, \dots, N\}$, we let G_j be a homogeneous group of homogeneous dimension Q_j. Then G_j is equipped with an automorphic one-parameter dilation (which, for $r_j > 0$, we denote simply by $x_j \mapsto r_j x_j$, for $x_j \in G_j$) and a homogeneous norm $|\cdot|$. Given a basis $\{X_{j,1}, \dots, X_{j,n_j}\}$ of left-invariant vector-fields, for $I \in (\mathbb{Z}^+)^{n_j}$ (where $\mathbb{Z}^+$ denotes the set $\{0, 1, 2, \dots\}$), the degree of the left-invariant differential operator $X_j^I = X_{j,1}^{i_1} \dots X_{j,n_j}^{i_{n_j}}$ on G_j will be denoted by $d_j(I)$.

NOTATION. Throughout this paper, we shall adopt the following notational conventions for product groups. For x in a product group $G = G_1 \times \cdots \times G_N$, we let

$$\mathbf{|}x\mathbf{|} = \left(|x_1|, \dots, |x_N|\right),$$

so that, given $J \in (\mathbb{Z}^+)^N$,

$$\mathbf{|}x\mathbf{|}^J = |x_1|^{j_1} \cdots |x_N|^{j_N}.$$

For $j \in \mathbb{Z}^+$, we denote by $\mathbf{j}$ the multi-index $(j, \dots, j) \in (\mathbb{Z}^+)^m$ for a dimension m which will always be clear from the context. We set $Q = (Q_1, \dots, Q_N)$, and for a multi-index $I = (I_1, \dots, I_N)$, with $I_j \in (\mathbb{Z}^+)^{n_j}, j = 1, \dots, N$, we also set

$$d(I) = \left(d_1(I_1), \dots, d_N(I_N)\right).$$

The differential operator $X^I = X_1^{I_1} \cdots X_N^{I_N}$ on G, with $X_j^{I_j}$ on G_j defined as above, then has degree $\left|d(I)\right|$.

We denote multi-parameter dilation, given $r = (r_1, \dots, r_N) \in (\mathbb{R}^+)^N$, by

$$\delta_r(x) = (r_1 x_1, \dots, r_N x_N)$$

for $x \in G$.

Frequently it will be necessary to split a variable x in a product group G into two component variables. In such cases we shall write $x = (x_{\underleftarrow{\ell}}, x_{\underrightarrow{\ell}})$, where

$$x_{\underleftarrow{\ell}} = (x_1, \dots, x_\ell), \quad x_{\underrightarrow{\ell}} = (x_{\ell+1}, \dots, x_N)$$

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for $1 \leqslant \ell \leqslant N$, and

$$X^J_{\underleftarrow{\ell}} = X_1^{J_1} \cdots X_\ell^{J_\ell}, \quad X^K_{\underrightarrow{\ell}} = X_{\ell+1}^{K_{\ell+1}} \cdots X_N^{K_N}$$

for $J = (J_1, \dots, J_\ell) \in (\mathbb{Z}^+)^{n_1} \times \cdots \times (\mathbb{Z}^+)^{n_\ell}$ and $K = (K_{\ell+1}, \dots, K_N) \in (\mathbb{Z}^+)^{n_{\ell+1}} \times \cdots \times (\mathbb{Z}^+)^{n_N}$ We also set

$$G_{\underleftarrow{\ell}} = G_1 \times \cdots \times G_\ell, \quad G_{\underrightarrow{\ell}} = G_{\ell+1} \times \cdots \times G_N,$$

with corresponding definitions of $Q_{\underleftarrow{\ell}}$, $Q_{\underrightarrow{\ell}}$, $d_{\underleftarrow{\ell}}$, and $d_{\underrightarrow{\ell}}$.

We note that for $z \in \mathbb{C}^N$, $\boldsymbol{|}z\boldsymbol{|}^2 = |z_1|^2 \cdots |z_N|^2$, while $|z|^2$ is the usual norm squared, $|z_1|^2 + \cdots + |z_N|^2$.

We shall define the product-type kernel conditions in terms of *normalised bump functions*. A C_c^∞ function φ is called a normalised bump function if φ is supported in the unit ball, and φ and all first order partial derivatives of φ are bounded by a fixed, pre-determined constant.

A function K on $G_1 \times \cdots \times G_N$ is said to be a kernel of product-type (or to satisfy product-type kernel conditions) if it satisfies the following conditions:

(a) the *regularity condition:*

$$\left|X^I K(x)\right| \leqslant C_I \boldsymbol{|}x\boldsymbol{|}^{-Q-d(I)}$$

for all $I = (I_1, \dots, I_N)$, $I_j \in (\mathbb{Z}^+)^{n_j}$, $j = 1, \dots, N$;

(b) for each $\ell = 1, \dots, N$, the *cancellation condition in* $x_{\underleftarrow{\ell}}$:

$$\left| \int_{G_{\underleftarrow{\ell}}} X^I_{\underrightarrow{\ell}} K(x) \varphi(\delta_r(x_{\underleftarrow{\ell}}))\, dx_{\underleftarrow{\ell}} \right| \leqslant C_I \boldsymbol{|}x_{\underrightarrow{\ell}}\boldsymbol{|}^{-Q_{\underrightarrow{\ell}} - d_{\underrightarrow{\ell}}(I)}$$

for all $I = (I_{\ell+1}, \dots, I_N)$, $I_j \in (\mathbb{Z}^+)^{n_j}$, $j = \ell+1, \dots, N$, all normalised bump functions φ on $G_{\underleftarrow{\ell}}$, and all $r \in (\mathbb{R}^+)^\ell$.

In addition, for each permutation $\sigma \in S_N$, K must satisfy the *cancellation condition in* $x_{\sigma(\underleftarrow{\ell})}$ obtained from (b) by permuting the indices $1, \dots, N$ by σ.

In the case where K is a tempered distribution, we assume that K is smooth away from the "planes" $\{x \in G : x_j = 0\}$, $j = 1, \dots, N$, and the cancellation conditions are to be understood as follows. Given φ in the Schwartz space $\mathcal{S}(G_{\underleftarrow{\ell}})$, we define the distribution K_φ by

$$K_\varphi(\psi) = K(\varphi \otimes \psi)$$

for all $\psi \in \mathcal{S}(G_{\underrightarrow{\ell}})$, where $\varphi \otimes \psi(x_1, \dots, x_N) = \varphi(x_{\underleftarrow{\ell}})\psi(x_{\underrightarrow{\ell}})$.

The cancellation condition then states that for all normalised bump functions φ on $G_{\underleftarrow{\ell}}$, and for all $r \in (\mathbb{R}^+)^\ell$, the distributions $K_{\varphi \circ \delta_r} \in \mathcal{S}'(G_{\underrightarrow{\ell}})$ are smooth away from the planes $\{x_{\underrightarrow{\ell}} \in G_{\underrightarrow{\ell}} : x_j = 0\}$, $j = \ell+1, \dots, N$, and uniformly satisfy

$$\left| X^I_{\underrightarrow{\ell}} K_{\varphi \circ \delta_r}(x_{\underrightarrow{\ell}}) \right| \leqslant C_I \boldsymbol{|}x_{\underrightarrow{\ell}}\boldsymbol{|}^{-Q_{\underrightarrow{\ell}} - d_{\underrightarrow{\ell}}(I)}$$

for all $I = (I_{\ell+1}, \dots, I_N) \in (\mathbb{Z}^+)^{n_{\ell+1}} \times \cdots \times (\mathbb{Z}^+)^{n_N}$.

Now, and for the remainder of this paper, we once again let G denote the product group $\mathbb{H}^n \times \mathbb{R}$. We recall that elements $(z_1, u_1, \dots, z_n, u_n, t) \in \mathbb{H}^n \times \mathbb{R}$ are also denoted $(h,t) = (z,u,t)$, with $h = (h_1, \dots, h_n)$, $h_i = (z_i, u_i) \in \mathbb{H}$.

Theorem 4.1. *Let m be a function on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying the Marcinkiewicz-type condition (1) for $i_1, \dots, i_n, j \leqslant 9n$. Then $m(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$ is bounded on $L^p(G)$ for $1 < p < \infty$. If (1) holds for all $i_1, \dots, i_n, j \in \mathbb{Z}^+$, then the convolution kernel K on G corresponding to the operator $m(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$ is a product-type kernel. That is, K is smooth away from the planes $\{(z,u,t) : (z_i, u_i) = 0\}$, $i = 1, \dots, n$ and $\{(z,u,t) : t = 0\}$, satisfies the size condition*

$$\left|\partial_z^I \partial_u^J \partial_t^k K(z,u,t)\right| \leqslant C_{I,J,k} |h|^{-4-I-2J} |t|^{-1-k} \tag{3}$$

for all $I, J \in (\mathbb{Z}^+)^n$, $k \in \mathbb{Z}^+$, and the following cancellation conditions: for each ℓ, $1 \leqslant \ell \leqslant n$

$$\left|\int_{\mathbb{H}^\ell} \partial_{z_{\underset{\rightarrow}{\ell}}}^I \partial_{u_{\underset{\rightarrow}{\ell}}}^J \partial_t^k K(z,u,t) \varphi\big(\delta_r(z_{\underset{\leftarrow}{\ell}}, u_{\underset{\leftarrow}{\ell}})\big)\, dz_{\underset{\leftarrow}{\ell}}\, du_{\underset{\leftarrow}{\ell}}\right| \leqslant C_{I,J,k} |h_{\underset{\rightarrow}{\ell}}|^{-4-I-2J} |t|^{-1-k} \tag{4}$$

for all normalised bump functions φ on $\mathbb{H}^\ell$, $r \in (\mathbb{R}^+)^\ell$, $I, J \in (\mathbb{Z}^+)^{n-\ell}$, and $k \in \mathbb{Z}^+$;

$$\left|\int_{\mathbb{H}^\ell \times \mathbb{R}} \partial_{z_{\underset{\rightarrow}{\ell}}}^I \partial_{u_{\underset{\rightarrow}{\ell}}}^J K(z,u,t) \varphi\big(\delta_r(z_{\underset{\leftarrow}{\ell}}, u_{\underset{\leftarrow}{\ell}}, t)\big)\, dz_{\underset{\leftarrow}{\ell}}\, du_{\underset{\leftarrow}{\ell}}\, dt\right| \leqslant C_{I,J,k} |h_{\underset{\rightarrow}{\ell}}|^{-4-I-2J} \tag{5}$$

for all normalised bump functions φ on $\mathbb{H}^\ell \times \mathbb{R}$, $r \in (\mathbb{R}^+)^{\ell+1}$, and $I, J \in (\mathbb{Z}^+)^{n-\ell}$; as well as all conditions obtained from (4) and (5) by permuting the indices $1, \dots, n$; and

$$\left|\int_{\mathbb{R}} \partial_z^I \partial_u^J K(z,u,t) \varphi(rt)\, dt\right| \leqslant C_{I,J} |h|^{-4-I-2J} \tag{6}$$

for all normalised bump functions φ on $\mathbb{R}$, $I, J \in (\mathbb{Z}^+)^n$ and $r > 0$.

Corollary 4.2. *Let m be a function on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying the Marcinkiewicz-type condition (1) for $i_1, \dots, i_n, j \leqslant 9n$. Then m is the almost-everywhere limit of a sequence m_ℓ of C_c^k functions supported away from the axes, uniformly bounded in L^∞, and such that the operators $m_\ell(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$ are uniformly bounded on $L^p(G)$ for $1 < p < \infty$.*

If (1) holds for all $i_1, \dots, i_n, j \in \mathbb{Z}^+$, then the functions m_ℓ are in C_c^∞, and the size and cancellation conditions (3)–(6) of Theorem 4.1 are satisfied uniformly in ℓ by the convolution kernels K_ℓ on G of the operators $m_\ell(\mathcal{L}_1^\#, \dots, \mathcal{L}_n^\#, iT^\#)$.

5. Boundedness of Marcinkiewicz Multiplier Operators on the Heisenberg Group

The boundedness on $L^p(\mathbb{H}_n)$ of the Marcinkiewicz multiplier operators $m_\ell(\mathcal{L}_1, \dots, \mathcal{L}_n, iT)$ follows from Corollary 4.2 by the method of transference.

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Proposition 5.1. (Transference) *Let $K \in L^1(G)$ and $Tf = f * K$. Then the operator $T^\flat$ given by $T^\flat f = f * K^\flat$ is bounded on $L^p(\mathbb{H}_n)$, $1 \leqslant p \leqslant \infty$, and*

$$\|T^\flat\|_{L^p(\mathbb{H}_n)\to L^p(\mathbb{H}_n)} \leqslant \|T\|_{L^p(G)\to L^p(G)}.$$

Proof: See [2], or the method of descent in Chapter XI of [9]. ∎

Theorem 5.2. (Marcinkiewicz Multiplier Theorem on $\mathbb{H}_n$) *Let $m(\xi_1,\dots,\xi_n,\eta)$ be a function on $(\mathbb{R}^+)^n \times \mathbb{R}$, satisfying the Marcinkiewicz condition* (1) *for all $i_1,\dots,i_n$, $j \leqslant 9n$. Then $m(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)$ is a bounded operator on $L^p(\mathbb{H}_n)$, $1 < p < \infty$.*

The method of lifting to a product group necessarily requires a large number of derivatives, though the number $9n$ can perhaps be improved upon slightly. In [7], Müller, Ricci and Stein develop different methods in order to avoid this problem for Marcinkiewicz multipliers $m(\mathcal{L},iT)$. In recent work [10], Veneruso improves Theorem 5.2 above by these other methods.

Proof: From Corollary 4.2, we can write m as the almost-everywhere limit of a sequence m_ℓ of C_c^k functions, supported away from the axes, such that $\|m_\ell\|_{L^\infty}$ are bounded uniformly in ℓ. By the Spectral Theorem, given $f \in L^2(\mathbb{H}_n)$,

$$m(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)f = \lim_{\ell\to\infty} m_\ell(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)f$$

in $L^2(\mathbb{H}_n)$, and consequently, there exists a subsequence $m_{\ell_j}(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)f(z,t)$ converging almost-everywhere on $\mathbb{H}_n$.

Since each m_ℓ is a C_c^k function, supported away from the axes, the convolution kernels K_ℓ corresponding to $m_\ell(\mathcal{L}_1^\#,\dots,\mathcal{L}_n^\#,iT^\#)$ are in L^1 and so we may apply transference to them. Thus, $m_\ell(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)$, which, by Proposition 3.1, have convolution kernels $K_\ell^\flat$, are bounded on $L^p(\mathbb{H}_n)$, with

$$\big\|m_\ell(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)\big\|_{L^p(\mathbb{H}_n)\to L^p(\mathbb{H}_n)} \leqslant \big\|m_\ell(\mathcal{L}_1^\#,\dots,\mathcal{L}_n^\#,iT^\#)\big\|_{L^p(G)\to L^p(G)}$$

for $1 \leqslant p \leqslant \infty$. But by Corollary 4.2, the lifted operators $m_\ell(\mathcal{L}_1^\#,\dots,\mathcal{L}_n^\#,iT^\#)$ are uniformly bounded on $L^p(G)$, $1 < p < \infty$. Thus, for $f \in L^p(\mathbb{H}_n) \cap L^2(\mathbb{H}_n)$, $1 < p < \infty$, by Fatou's lemma

$$\big\|m(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)f\big\|_{L^p(\mathbb{H}_n)} \leqslant \liminf_{j\to\infty}\big\|m_{\ell_j}(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)\big\|_{L^p(\mathbb{H}_n)\to L^p(\mathbb{H}_n)}\|f\|_{L^p(\mathbb{H}_n)}$$

and the result follows. ∎

6. Kernels of Marcinkiewicz Multiplier Operators on the Heisenberg Group

We now obtain regularity and cancellation conditions necessarily satisfied by the kernels of Marcinkiewicz multiplier operators $m(\mathcal{L}_1,\dots,\mathcal{L}_n,iT)$ on $\mathbb{H}_n$.

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THEOREM 6.1. *Let m be a function on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying the Marcinkiewicz-type condition* (1) *for all $i_1, \ldots, i_n, j \in \mathbb{Z}^+$. Then the convolution kernel K on $\mathbb{H}_n$ corresponding to the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ is polyradial, smooth away from the planes $z_i = 0$, and satisfies the size condition*

$$|\partial_z^I \partial_t^k K(z,t)| \leqslant C_{I,k} |z|^{-2-I} \big(|z|^2 + |t|\big)^{-1-k} \tag{7}$$

for all $I \in (\mathbb{Z}^+)^n$, $k \in \mathbb{Z}^+$, as well as the following cancellation conditions: for all ℓ, $1 \leqslant \ell \leqslant n$,

$$\left| \int_{\mathbb{C}^\ell} \partial_{z_{\underrightarrow{\ell}}}^I \partial_t^k K(z,t) \varphi(\delta_r(z_{\underleftarrow{\ell}}))\, dz_{\underleftarrow{\ell}} \right| \leqslant C_{I,j} |z_{\underrightarrow{\ell}}|^{-2-I} \big(|z_{\underrightarrow{\ell}}|^2 + |t|\big)^{-1-k} \tag{8}$$

for all normalised bump functions φ on $\mathbb{C}^\ell$, $r \in (\mathbb{R}^+)^\ell$, $I \in (\mathbb{Z}^+)^{n-\ell}$, and $k \in \mathbb{Z}^+$;

$$\left| \int_{\mathbb{C}^\ell} \int_{\mathbb{R}} \partial_{z_{\underrightarrow{\ell}}}^I K(z,t) \varphi(\delta_r(z_{\underleftarrow{\ell}}, t))\, dz_{\underleftarrow{\ell}}\, dt \right| \leqslant C_I |z_{\underrightarrow{\ell}}|^{-2-I} \tag{9}$$

for all normalised bump functions φ on $\mathbb{C}^\ell \times \mathbb{R}$, $r \in (\mathbb{R}^+)^{\ell+1}$, and $I \in (\mathbb{Z}^+)^{n-\ell}$; all conditions obtained from (8) *and* (9) *by permuting the indices $1, \ldots, n$; and*

$$\left| \int_{\mathbb{R}} \partial_z^I K(z,t) \varphi(rt)\, dt \right| \leqslant C_I |z|^{-2-I} \tag{10}$$

for all normalised bump functions φ on $\mathbb{R}$, $r > 0$, and $I \in (\mathbb{Z}^+)^n$.

The core of the proof consists of the following proposition (proved below) for L^1 kernels, in which the product-type regularity and cancellation conditions on $\mathbb{H}^n \times \mathbb{R}$ are transferred down to conditions (7)–(10) on $\mathbb{H}_n$.

PROPOSITION 6.2. *If $K \in L^1(\mathbb{H}^n \times \mathbb{R})$ satisfies the product kernel conditions* (3)–(6) *on $\mathbb{H}^n \times \mathbb{R}$, then $K^\flat$ satisfies* (7)–(10) *with constants that depend only on the constants in the conditions* (3)–(6) *(for example, they do not depend on $\|K\|_{L^1}$).*

PROOF OF THEOREM 6.1: From Corollary 4.2, the operators $m_\ell(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ converge strongly on $L^2(\mathbb{H}_n)$ to $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$, and each m_ℓ is C^∞, with compact support away from the axes. By Proposition 3.1, the convolution kernel on $\mathbb{H}_n$ of $m_\ell(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$ is $K_\ell^\flat$, where K_ℓ is the convolution kernel on G of $m_\ell(\mathcal{L}_1^\#, \ldots, \mathcal{L}_n^\#, iT^\#)$. Letting K be the distribution convolution kernel on G of $m(\mathcal{L}_1^\#, \ldots, \mathcal{L}_n^\#, iT^\#)$ we denote by $K^\flat$ the distribution convolution kernel on $\mathbb{H}_n$ of $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$. We must show that $K^\flat$ is smooth away from the $z_i = 0$ planes, and satisfies (7)–(10). Since

$$f * K_\ell^\flat \longrightarrow f * K^\flat \text{ in } L^2(\mathbb{H}_n) \text{ as } \ell \to \infty$$

for all $f \in \mathcal{S}(\mathbb{H}_n)$, the Sobolev Embedding Theorem implies that $K_\ell^\flat \longrightarrow K^\flat$ in the sense of distributions, as $\ell \to \infty$. From Corollary 4.2, the kernels K_ℓ satisfy (3)–(6) with

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constants that do not depend on ℓ. Since each K_ℓ is in $L^1(G)$, then by Proposition 6.2, $K_\ell^\flat$ satisfy (7)–(10) uniformly in ℓ.

We consider first the regularity condition, (7). Since $K_\ell^\flat$ satisfy this uniformly in ℓ, then by the Ascoli-Arzelà theorem, $K^\flat$ is equal to a smooth function away from the $z_i = 0$ planes, which satisfies (7).

Next, for the cancellation condition (10) in t, we let φ be a normalised bump function on $\mathbb{R}$ and $r > 0$. Since $K_\ell^\flat$ converge in $\mathcal{S}'(\mathbb{H}_n)$ to $K^\flat$, then $(K_\ell^\flat)_{\varphi\circ\delta_r}$ converge in $\mathcal{S}'(\mathbb{C}^n)$ to $K^\flat_{\varphi\circ\delta_r}$. That $K_\ell^\flat$ satisfy (10) uniformly in ℓ, means precisely that the smooth functions $(K_\ell^\flat)_{\varphi\circ\delta_r}$ satisfy

$$\left|\partial_z^I (K_\ell^\flat)_{\varphi\circ\delta_r}(z)\right| \leqslant C_I |z|^{-2-I} \tag{11}$$

uniformly in ℓ. Thus by the Ascoli-Arzelà theorem, $K^\flat_{\varphi\circ\delta_r}$ is equal to a smooth function away from the $z_i = 0$ planes, which satisfies (11); in other words, $K^\flat$ satisfies the cancellation condition (10). Conditions (8) and (9) on $K^\flat$ follow by similar arguments. ∎

Proof of Proposition 6.2:

Step 1. We prove here that if a kernel K on $\mathbb{H}^n \times \mathbb{R}$ satisfies the regularity condition (3) and the cancellation condition (6) in t, then the derived kernel $K^\flat$ on $\mathbb{H}_n$ satisfies the regularity condition (7) and the cancellation condition (10) in t. We do this by induction, taking the $\cdot^\flat$ operation iteratively. That is, for $\ell = 1, \dots, n$, we define $K^{\flat,\ell}$ on $\mathbb{C}^\ell \times \mathbb{H}^{n-\ell} \times \mathbb{R}$ by

$$\begin{aligned} K^{\flat,\ell}(z_{\underleftarrow{\ell}}, h_{\underrightarrow{\ell}}, t) &= \int_{\mathbb{R}^\ell} K\left(z_1, u_1, \dots, z_n, u_n, t - \sum_{i=1}^{\ell} u_i\right) du_1 \dots du_\ell \\ &= \int_{\mathbb{R}} K^{\flat,\ell-1}(z_{\underleftarrow{\ell-1}}, z_\ell, u_\ell, h_{\underrightarrow{\ell}}, t - u_\ell)\, du_\ell \end{aligned}$$

and we show that $K^{\flat,\ell}$ satisfies the conditions

$$\left|\partial_z^I \partial_t^k \partial_{u_{\underrightarrow{\ell}}}^J K^{\flat,\ell}(z_{\underleftarrow{\ell}}, h_{\underrightarrow{\ell}}, t)\right| \leqslant C_{I,J,k} |z_{\underleftarrow{\ell}}|^{-2-I_{\underleftarrow{\ell}}} |h_{\underrightarrow{\ell}}|^{-4-I_{\underrightarrow{\ell}}-2J} \left(|z_{\underleftarrow{\ell}}|^2 + |t|\right)^{-1-k} \tag{12}$$

for all $I = (i_1, \dots, i_n) \in (\mathbb{Z}^+)^n$, $J = (j_{\ell+1}, \dots, j_n) \in (\mathbb{Z}^+)^{n-\ell}$, $k \in \mathbb{Z}^+$, and

$$\left|\int_{\mathbb{R}} \partial_z^I \partial_{u_{\underrightarrow{\ell}}}^J K^{\flat,\ell}(z_{\underleftarrow{\ell}}, h_{\underrightarrow{\ell}}, t)\varphi(rt)\, dt\right| \leqslant C_{IJ} |z_{\underleftarrow{\ell}}|^{-2-I_{\underleftarrow{\ell}}} |h_{\underrightarrow{\ell}}|^{-4-I_{\underrightarrow{\ell}}-2J} \tag{13}$$

for all $I \in (\mathbb{Z}^+)^n$, $J = (j_{\ell+1}, \dots, j_n) \in (\mathbb{Z}^+)^{n-\ell}$, all $r > 0$, and all normalised bump functions φ on $\mathbb{R}$.

Notice that $K^{\flat,n} = K^\flat$, and if $\ell = n$, conditions (12) and (13) are the required estimates (7) and (10).

Before proceeding, we observe that the conditions (12) and (13) are invariant under 1-parameter dilations of $\mathbb{C}^\ell \times \mathbb{H}^{n-\ell} \times \mathbb{R} = \mathbb{H}_\ell \times \mathbb{H}^{n-\ell}$. That is, if g on $\mathbb{C}^\ell \times \mathbb{H}^{n-\ell} \times \mathbb{R}$

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satisfies (12) and (13), then so does g_r, with the same constants $C_{I,J,k}$ and $C_{I,J}$. Here,

$$g_r(z_{\underleftarrow{\ell}}, h_{\underrightarrow{\ell}}, t) = r^{-2n-2(n-\ell)-2} g\left(\frac{z_1}{r}, \ldots, \frac{z_\ell}{r}, \frac{z_{\ell+1}}{r}, \frac{u_{\ell+1}}{r^2}, \ldots, \frac{z_n}{r}, \frac{u_n}{r^2}, \frac{t}{r^2}\right).$$

Therefore, in order to show (12), it suffices to prove it only for $|t| = 1$.

The proof of (12) and (13) is inductive in ℓ, and we shall see that each step reduces to the following Lemma on $\mathbb{H} \times \mathbb{R}$, which we state now, but prove below.

Lemma 6.3. *Let $\gamma > 0$, $a \geqslant 0$. Suppose $f \in L^1(\mathbb{H} \times \mathbb{R})$ satisfies*

$$\left|\partial_z^i \partial_u^j \partial_t^k f(z,u,t)\right| \leqslant C_{i,j,k} \gamma \left(|z|^2 + |u|\right)^{-2-(i/2)-j} \left(a + |t|\right)^{-1-k} \tag{14}$$

for all $i, j, k \in \mathbb{Z}^+$, and

$$\left|\int_{\mathbb{R}} \partial_z^i \partial_u^j f(z,u,t) \varphi(rt)\, dt\right| \leqslant C_{i,j} \gamma \left(|z|^2 + |u|\right)^{-2-(i/2)-j} \tag{15}$$

for all $i, j \in \mathbb{Z}^+$, all $r > 0$, and all normalised bump functions φ on $\mathbb{R}$. Then

$$f^\flat(z,t) = \int_{\mathbb{R}} f(z,u,t-u)\, du$$

satisfies

$$\left|\partial_z^i \partial_t^k f^\flat(z,t)\right| \leqslant C_{i,k} \gamma |z|^{-2-i} \left(a + |z|^2 + 1\right)^{-1-k} \tag{16}$$

for $|t| = 1$ and for all $i, k \in \mathbb{Z}^+$, and

$$\left|\int_{\mathbb{R}} \partial_z^i f^\flat(z,t) \varphi(rt)\, dt\right| \leqslant C_i \gamma |z|^{-2-i} \tag{17}$$

for all $i \in \mathbb{Z}^+$, all $r > 0$, and all normalised bump functions φ on $\mathbb{R}$. The constants $C_{i,k}$, C_i in (16) and (17) depend only on the constants in (14) and (15).

Now, take ℓ to be 1. To prove (12) with $|t| = 1$ and (13), we fix $(z_{\underrightarrow{1}}, u_{\underrightarrow{1}}) = (z_2, u_2, \ldots, z_n, u_n) \in \mathbb{H}^{n-1}$, and $I, J \in (\mathbb{Z}^+)^{n-1}$. Since K satisfies (3) and (6) on $\mathbb{H}^n \times \mathbb{R}$, then $\partial^I_{z_{\underrightarrow{1}}} D^J_{u_{\underrightarrow{1}}} K$, viewed as a function of z_1, u_1, and t, satisfies (14) and (15), with $a = 0$, and $\gamma = |h_{\underrightarrow{1}}|^{-4-I-2J}$. Thus by Lemma 6.3, (12) and (13) hold for $K^{\flat,1}$.

To show the inductive step, that if (12) (with $|t| = 1$) and (13) hold, then the same conditions also hold when ℓ is replaced by $\ell + 1$, we fix $(z_{\underleftarrow{\ell}}, h_{\underrightarrow{\ell+1}}) \in \mathbb{C}^\ell \times \mathbb{H}^{n-\ell-1}$, $I = (i_1, \ldots, i_\ell, i_{\ell+2}, \ldots, i_n) \in (\mathbb{Z}^+)^{n-1}$, and $J = (j_{\ell+2}, \ldots, j_n) \in (\mathbb{Z}^+)^{n-\ell-1}$, set $z_{\widehat{\ell+1}} = (z_{\underleftarrow{\ell}}, z_{\underrightarrow{\ell+1}})$, and view

$$\partial^I_{z_{\widehat{\ell+1}}} \partial^J_{u_{\underrightarrow{\ell+1}}} K^{\flat,\ell}$$

as a function of $z_{\ell+1}$, $u_{\ell+1}$, and t. Conditions (12) and (13) then give us (14) and (15), with $a = |z_{\underleftarrow{\ell}}|^2$, and $\gamma = |z_{\underleftarrow{\ell}}|^{-2-I_{\underleftarrow{\ell}}} |h_{\underrightarrow{\ell+1}}|^{-4-I_{\underrightarrow{\ell+1}}-2J}$. Hence by Lemma 6.3, (12) and (13) hold for $K^{\flat,\ell+1}$, thus concluding the proof of Step 1.

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STEP 2. We now prove estimates (8), cancellation in some of the z-variables, and (9), cancellation in some of the z-variables and in t. Given ℓ, $1 \leqslant \ell \leqslant n$, by relabelling variables, it suffices to prove the cancellation conditions in $z_{\underleftarrow{\ell}} = (z_1, \dots, z_\ell)$ and in $(z_{\underleftarrow{\ell}}, t) = (z_1, \dots, z_\ell, t)$.

We prove these estimates together by reducing to the previous estimates (7) and (10) as follows. Given any $I \in (\mathbb{Z}^+)^{n-\ell}$, and $r \in (\mathbb{R}^+)^\ell$ we show that, viewed as a function of $(h_{\underrightarrow{\ell}}, t) \in \mathbb{H}^{n-\ell} \times \mathbb{R}$,

$$k(h_{\underrightarrow{\ell}}, t) = k(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}}, t) = \int_{\mathbb{R}^\ell} \int_{\mathbb{C}^\ell} \partial^I_{z_{\underrightarrow{\ell}}} K\Big(z, u, t - \sum_{i=1}^{\ell} u_i\Big) \varphi\big(\delta_r(z_{\underleftarrow{\ell}})\big)\, dz_{\underleftarrow{\ell}}\, du_{\underleftarrow{\ell}}$$

satisfies (3) and (6) on $\mathbb{H}^{n-\ell} \times \mathbb{R}$ with the factor $\gamma = |h_{\underrightarrow{\ell}}|^{-4-I}$ included in the right hand side of each estimate. From Step 1 (with n replaced by $n - \ell$), it then follows that the derived kernel

$$k^\flat(z_{\underrightarrow{\ell}}, t) = \int_{\mathbb{R}^{n-\ell}} k\Big(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}}, t - \sum_{i=\ell+1}^{n} u_i\Big)\, du_{\underrightarrow{\ell}}$$

satisfies (7) and (10) on $\mathbb{H}_{n-\ell}$. But

$$\begin{aligned} k^\flat(z_{\underrightarrow{\ell}}, t) &= \int_{\mathbb{R}^n} \int_{\mathbb{C}^\ell} \partial^I_{z_{\underrightarrow{\ell}}} K\Big(z, u, t - \sum_{i=1}^{n} u_i\Big) \varphi\big(\delta_r(z_{\underleftarrow{\ell}})\big)\, dz_{\underleftarrow{\ell}}\, du \\ &= \int_{\mathbb{C}^\ell} \partial^I_{z_{\underrightarrow{\ell}}} K^\flat(z, t) \varphi\big(\delta_r(z_{\underleftarrow{\ell}})\big)\, dz_{\underleftarrow{\ell}} \end{aligned}$$

and so this amounts to showing (8) and (9) for $K^\flat$ on $\mathbb{H}_n$. (In fact, this proves (9) only for product-type normalised bump functions $\varphi = \varphi_1 \otimes \varphi_2$ on $\mathbb{C}^\ell \times \mathbb{R}$, where φ_1 is a normalised bump function on $\mathbb{C}^\ell$ and φ_2 is a normalised bump function on $\mathbb{R}$. However, the result for any normalised bump function φ on $\mathbb{C}^\ell \times \mathbb{R}$ can then easily be seen, for example, by expanding φ in a Fourier series.)

Therefore (8) and (9) are proved once we have shown that $k(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}}, t)$ satisfies (3) and (6) on $\mathbb{H}^{n-\ell} \times \mathbb{R}$. Fixing $(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}})$, we observe that this is equivalent to showing that $k(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}}, t)$ satisfies, in the t-variable, the standard kernel estimates on $\mathbb{R}$ with factor $\gamma = |h_{\underrightarrow{\ell}}|^{-4-I}$.

Since K satisfies (3)–(6) on $\mathbb{H}^n \times \mathbb{R}$, then, fixing $(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}})$ and viewing $\partial^I_{z_{\underrightarrow{\ell}}} K$ as a function of $(z_{\underleftarrow{\ell}}, u_{\underleftarrow{\ell}}, t)$ only, we see that it is a product-type kernel on $\mathbb{H}^\ell \times \mathbb{R}$, with factor $\gamma = |h_{\underrightarrow{\ell}}|^{-4-I}$. Now, if we integrate a product-type kernel on $\mathbb{H}^\ell \times \mathbb{R}$ against a normalised bump function on $\mathbb{C}^\ell$, we still obtain a product-type kernel on $\mathbb{R}^{\ell+1}$. (Lemma 6.4 in Section 6 below.) Thus as a function of $(u_{\underleftarrow{\ell}}, t)$,

$$\int_{\mathbb{C}^\ell} \partial^I_{z_{\underrightarrow{\ell}}} K(z, u, t) \varphi\big(\delta_r(z_{\underleftarrow{\ell}})\big)\, dz_{\underleftarrow{\ell}}$$

is a product-type kernel on $\mathbb{R}^{\ell+1}$ (with this same factor γ). But integrating a product-type kernel on $\mathbb{R}^{\ell+1}$ over the parallel planes $\sum_{i=1}^{\ell+1} u_i = \text{constant}$ (that is, the flat operation on $\mathbb{R}^{\ell+1}$), yields a standard kernel on $\mathbb{R}$ (Lemma 6.6 below). Consequently,

$$k(z_{\underrightarrow{\ell}}, u_{\underrightarrow{\ell}}, t) = \int_{\mathbb{R}^\ell}\int_{\mathbb{C}^\ell} \partial^I_{z_{\underrightarrow{\ell}}} K\Big(z, u, t - \sum_{i=1}^{\ell} u_i\Big)\varphi(\delta_r(z_{\underleftarrow{\ell}}))\, dz_{\underleftarrow{\ell}}\, du_{\underleftarrow{\ell}}$$

satisfies, in the t-variable, the standard kernel estimates on $\mathbb{R}$ (with factor γ), as required. This concludes Step 2 and hence the proof of Proposition 6.2.

Technical details

We now establish Lemmas 6.3, 6.4 and 6.6, which were used in the proofs above.

Proof of Lemma 6.3: We first remark that (15) also holds for translates of dilated normalised bump functions $\varphi\big(r(\cdot + s)\big)$. To see this, we let η be a normalised bump function on $\mathbb{R}$ such that $\eta \equiv 1$ on $[-1/2, 1/2]$, and write

$$\begin{aligned}\int_{\mathbb{R}} \partial^i_z\partial^j_u f(z,u,t)\varphi\big(r(t+s)\big)\,dt &= \int_{\mathbb{R}} \partial^i_z\partial^j_u f(z,u,t)\varphi\big(r(t+s)\big)\eta(rt)\,dt \\ &\quad + \int_{\mathbb{R}} \partial^i_z\partial^j_u f(z,u,t)\varphi\big(r(t+s)\big)\big(1-\eta(rt)\big)\,dt.\end{aligned}$$

Since $\varphi\big(r(t+s)\big)\eta(rt)$ is a dilate by r of the normalised bump function

$$h(t) = \varphi(t + rs)\eta(t)$$

on $\mathbb{R}$, then (15) gives the required estimate for the first term. The second term, by (14), is bounded by

$$C_{i,j}\gamma\big(|z|^2+|u|\big)^{-2-(i/2)-j}\int_{\substack{|t|\geqslant 1/2r\\ |t+s|\leqslant 1/r}} \big(a+|t|\big)^{-1}\,dt \leqslant\ C_{i,j}\gamma\big(|z|^2+|u|\big)^{-2-(i/2)-j},$$

independently of s, as required. The cancellation condition (17) now follows immediately after changing variables in t:

$$\begin{aligned}\int_{\mathbb{R}} \partial^i_z f^\flat(z,t)\varphi(rt)\,dt &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial^i_z f(z,u,t-u)\varphi(rt)\,dt\,du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial^i_z f(z,u,t)\varphi\big(r(t+u)\big)\,dt\,du \\ &\leqslant C_i\gamma\int_{\mathbb{R}}\big(|z|^2+|u|\big)^{-2-(i/2)}\,du\ \leqslant C_i\gamma\ |z|^{-2-i}.\end{aligned}$$

Next, we denote the partial derivative in the u variable on $\mathbb{H}\times\mathbb{R}$ by ∂_1, and the partial derivative in the t variable by ∂_2.

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In the case where $a \geqslant \max\{1, |z|^2\}$, the regularity condition (16) follows immediately by integrating (14) out in u:

$$\begin{aligned}
\left|\partial_z^i \partial_t^k f^\flat(z,t)\right| &\leqslant \int_{\mathbb{R}} \left|\partial_z^i \partial_2^k f(z,u,t-u)\right| du \\
&\leqslant C_{i,k}\gamma \int_{\mathbb{R}} \big(|z|^2 + |u|\big)^{-2-(i/2)} \big(a + |t-u|\big)^{-1-k} du \\
&\leqslant C_{i,k}\gamma |z|^{-2-i} \big(a + |z|^2 + 1\big)^{-1-k} .
\end{aligned}$$

For $a \leqslant \max\{1, |z|^2\}$, we must take into consideration the cancellation in the last component of f. It suffices to show that

$$\left|\partial_z^i \partial_t^k f^\flat(z,t)\right| \leqslant C_{i,k}\gamma \big(|z|^2 + 1\big)^{-2-(i/2)-k}.$$

Letting $r = 1/\max\{1, |z|^2\}$ and letting η be a normalised bump function on $\mathbb{R}$, supported in $[-1/2, 1/2]$ and such that $\eta \equiv 1$ on $[-1/4, 1/4]$, and $|\eta^{(i)}| \leqslant C_k$ for $i = 1, \ldots, k+1$,

$$\begin{aligned}
\partial_z^i \partial_t^k f^\flat(z,t) &= \partial_t^k \int_{\mathbb{R}} \partial_z^i f(z, t-u, u)\, du \\
&= \partial_t^k \int_{\mathbb{R}} \partial_z^i f(z, t-u, u)\eta(ru)\, du + \partial_t^k \int_{\mathbb{R}} \partial_z^i f(z, t-u, u)\big(1 - \eta(ru)\big)\, du \\
&= I_1 + I_2 .
\end{aligned}$$

For I_1, we write

$$I_1 = \int_{\mathbb{R}} \Big[\partial_1^k \partial_z^i f(z, t-u, u) - \partial_1^k \partial_z^i f(z,t,u)\Big]\eta(ru)\, du + \int_{\mathbb{R}} \partial_1^k \partial_z^i f(z,t,u)\eta(ru)\, du.$$

Then by (15) we obtain the required estimate for the second term. For the first term, by the mean value theorem and (14),

$$\begin{aligned}
\left|\partial_1^k \partial_z^i f(z, t-u, u) - \partial_1^k \partial_z^i f(z,t,u)\right| &\leqslant |u| \sup_{0<|s|<|u|} \left|\partial_1^{k+1} \partial_z^i f(z, t-s, u)\right| \\
&\leqslant C_{i,k}\gamma |u| \sup_{0<|s|<|u|} \big(|z|^2 + |t-s|\big)^{-3-(i/2)-k} \big(a + |u|\big)^{-1} \\
&\leqslant C_{i,k}\gamma \big(|z|^2 + 1 - |u|\big)^{-3-(i/2)-k}.
\end{aligned}$$

But for $|u| \leqslant 1/2r = \max\{1, |z|^2\}/2$, $|z|^2 + 1 - |u| \geqslant \big(|z|^2+1\big)/2$. Thus, since η is supported in $[-1/2, 1/2]$, the first term of I_1 is bounded by

$$C_{i,k}\gamma \int_{|u|\leqslant 1/2r} du\ r\ \big(|z|^2 + 1\big)^{-2-(i/2)-k} \leqslant\ C_{i,k}\gamma\ \big(|z|^2 + 1\big)^{-2-(i/2)-k},$$

as required.

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For I_2, if $|z|^2 \geqslant 1$, we can simply integrate the rough estimate (14),

$$\begin{aligned}|I_2| &= \left| \int_{\mathbb{R}} \partial_1^k \partial_z^i f(z, t-u, u)\left(1 - \eta\left(\frac{u}{|z|^2}\right)\right) du \right| \\ &\leqslant C_{i,k}\gamma \int_{|u| \geqslant |z|^2/4} \left(|z|^2 + |t-u|\right)^{-2-(i/2)-k} (a + |u|)^{-1}\, du \\ &\leqslant C_{i,k}\gamma \int_{\mathbb{R}} \left(|z|^2 + |u|\right)^{-2-(i/2)-k} du\, |z|^{-2} \\ &\leqslant C_{i,k}\gamma |z|^{-2-i}\left(|z|^2+1\right)^{-1-k}.\end{aligned}$$

If $1 \geqslant |z|^2$, however, this gives too many powers of $|z|^{-1}$. We therefore first change variables in u, so that the t-derivatives will fall on the final component of f:

$$\begin{aligned}I_2 &= \partial_t^k \int_{\mathbb{R}} \left[\partial_z^i f(z, u, t-u)\left(1 - \eta(t-u)\right)\right] du \\ &= \sum_{j=0}^{k} \binom{k}{j} \int_{\mathbb{R}} \partial_2^j \partial_z^i f(z, u, t-u)\psi_{k-j}(t-u)\, du\end{aligned}$$

where $\psi_0(y) = 1 - \eta(y)$, and $\psi_\ell(y) = \eta^{(\ell)}(y)$ for $\ell \in \mathbb{N}$. As all of the ψ_ℓ are supported in $\{|y| \geqslant 1/4\}$, (14) gives

$$\begin{aligned}&\left| \int_{\mathbb{R}} \partial_2^j \partial_z^i f(z, u, t-u)\psi_{k-j}(t-u)\, du \right| \\ &\qquad \leqslant C_{i,k}\gamma \int_{|t-u| \geqslant 1/4} \left(|z|^2 + |u|\right)^{-2-(i/2)} \left(a + |t-u|\right)^{-1-j} du \\ &\qquad \leqslant C_{i,k}\gamma \int_{\mathbb{R}} \left(|z|^2 + |u|\right)^{-2-(i/2)} du \;\leqslant C_{i,k}\gamma |z|^{-2-i}\left(|z|^2+1\right)^{-1-k},\end{aligned}$$

which concludes the proof of Lemma 6.3. ∎

LEMMA 6.4. *If $K \in L^1(\mathbb{H}^n \times \mathbb{R})$ satisfies the product kernel estimates on $\mathbb{H}^n \times \mathbb{R}$ (with factor $\gamma > 0$ included in the right-hand side of each estimate), then given a normalised bump function φ on $\mathbb{C}^n$, and $r \in (\mathbb{R}^+)^n$, the function $K^{(n)}$ on $\mathbb{R}^{n+1}$ defined by*

$$K^{(n)}(u,t) = \int_{\mathbb{C}^n} K(z,u,t)\varphi(\delta_r(z))\, dz$$

satisfies the product kernel estimates on $\mathbb{R}^{n+1}$ (with factor γ), with constants that depend only on those in the product kernel conditions K satisfies.

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Proof of Lemma 6.4: The regularity condition on $K^{(n)}$ is easily obtained by integrating out in the z-variables the regularity condition on K:

$$
\begin{aligned}
\left|\partial_u^J \partial_t^k K^{(n)}(u,t)\right| &= \left|\int_{\mathbb{C}^n} \partial_u^J \partial_t^k K(z,u,t)\varphi(\delta_r(z))\,dz\right| \\
&\leqslant C\int_{\mathbb{C}^n} \left|\partial_u^J \partial_t^k K(z,u,t)\right| dz \\
&\leqslant C_{I,J,k}\int_{\mathbb{C}^n} |h|^{-4-2J}\,dz\ |t|^{-1-k} \\
&\leqslant C_{I,J,k}|u|^{-1-J}|t|^{-1-k}\,.
\end{aligned}
$$

Next, we show the product-type cancellation conditions for $K^{(n)}$ on $\mathbb{R}^{n+1}$. Given $1 \leqslant \ell \leqslant n$, it suffices to prove cancellation in $u_{\underleftarrow{\ell}}$ and in $(u_{\underleftarrow{\ell}}, t)$.

Given any normalised bump functions η on $\mathbb{R}^\ell$ and η_0 on $\mathbb{R}$, $J = (j_{\ell+1},\ldots,j_n) \in (\mathbb{Z}^+)^{n-\ell}$, $k \in \mathbb{Z}^+$, $s \in (\mathbb{R}^+)^\ell$ and $s_0 > 0$, we must estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^\ell} \partial^J_{u_{\underrightarrow{\ell}}} \partial_t^k K^{(n)}(u,t)\eta(\delta_s(u_{\underleftarrow{\ell}}))\,du_{\underleftarrow{\ell}} \\
&\qquad = \int_{\mathbb{C}^{n-\ell}} \int_{\mathbb{H}^\ell} \left[\partial^J_{u_{\underrightarrow{\ell}}} \partial_t^k K(z,u,t)\right] \varphi(\delta_r(z))\eta(\delta_s(u_{\underleftarrow{\ell}}))\,du_{\underleftarrow{\ell}}\,dz_{\underleftarrow{\ell}}\,dz_{\underrightarrow{\ell}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^{\ell+1}} \partial^J_{u_{\underrightarrow{\ell}}} K^{(n)}(u,t)\eta(\delta_s(u_{\underleftarrow{\ell}}))\eta_0(s_0 t)\,du_{\underleftarrow{\ell}}\,dt \\
&\qquad = \int_{\mathbb{C}^{n-\ell}} \int_{\mathbb{H}^\ell} \left[\int_{\mathbb{R}} \partial^J_{u_{\underrightarrow{\ell}}} K(z,u,t)\eta_0(s_0 t)\,dt\right] \varphi(\delta_r(z))\eta(\delta_s(u_{\underleftarrow{\ell}})\,du_{\underleftarrow{\ell}}\,dz_{\underleftarrow{\ell}}\,dz_{\underrightarrow{\ell}}\,.
\end{aligned}
$$

We observe first that for fixed $z_{\underrightarrow{\ell}}$, $\varphi(\delta_r(z))$ is a dilate of the normalised bump function $\widetilde{\varphi}(z_{\underleftarrow{\ell}}) = \varphi(z_{\underleftarrow{\ell}}, \delta_{r_{\underrightarrow{\ell}}}(z_{\underrightarrow{\ell}}))$ on $\mathbb{C}^\ell$. Now, from the product kernel conditions satisfied by K on $\mathbb{H}^n \times \mathbb{R}$, it follows that, viewed as functions of $(z_{\underleftarrow{\ell}}, u_{\underleftarrow{\ell}})$,

$$
\partial^J_{u_{\underrightarrow{\ell}}} \partial_t^k K(z,u,t), \qquad \text{and} \qquad \int_{\mathbb{R}} \partial^J_{u_{\underrightarrow{\ell}}} K(z,u,t)\eta_0(s_0 t)\,dt
$$

satisfy the product kernel conditions on $\mathbb{H}^\ell$, with factors $\gamma_1 = |h_{\underrightarrow{\ell}}|^{-4-2J}|t|^{-1-k}$ and $\gamma_2 = |h_{\underrightarrow{\ell}}|^{-4-2J}$ respectively. Therefore, applying the following lemma, and then integrating in $z_{\underrightarrow{\ell}}$, we obtain the required estimates. ∎

Lemma 6.5. *If $f \in L^1(\mathbb{H}^n)$ satisfies the product kernel conditions on $\mathbb{H}^n$ with a factor $\gamma > 0$ included in the right-hand side of each estimate, then*

$$
\left|\int_{\mathbb{H}^n} f(z,u)\varphi(\delta_r(z))\eta(\delta_s(u))\,dz\,du\right| \leqslant C\gamma
$$

for all normalised bump functions φ on $\mathbb{C}^n$, η on $\mathbb{R}^n$, and all $r, s \in (\mathbb{R}^+)^n$. The constant C depends only on the constants in the product kernel conditions satisfied by f.

PROOF OF LEMMA 6.5: Relabelling coordinates if necessary, we assume

$$r_i^2 \leqslant s_i \text{ for } i = 1, \dots, n_1, \qquad \text{and} \qquad r_i^2 > s_i \text{ for } i = n_1+1, \dots, n\,.$$

Then

$$\varphi\Big(\frac{r_1}{\sqrt{s_1}}z_1, \dots, \frac{r_{n_1}}{\sqrt{s_{n_1}}}z_{n_1}, z_{n_1+1}, \dots, z_n\Big)\eta\Big(u_1, \dots, u_{n_1}, \frac{s_{n_1+1}}{r_{n_1+1}{}^2}u_{n_1+1}, \dots, \frac{s_n}{r_n{}^2}u_n\Big)$$

has bounded derivatives, but it is not a normalised bump function, as its support in the first set of z-variables, and the second set of u-variables is too large. We therefore split the integral up according to the size of these variables, by introducing a normalised bump function ψ on $\mathbb{C}$, with $\psi \equiv 1$ on $\{w \in \mathbb{C} : |w| \leqslant 1/2\}$, and a normalised bump function μ on $\mathbb{R}$ such that $\mu \equiv 1$ on $[-1/2, 1/2]$. Then inserting the factors $1 = \psi(\sqrt{s_i}z_i) + \big(1 - \psi(\sqrt{s_i}z_i)\big)$, for $i = 1, \dots, n_1$ and $1 = \mu(r_j^2 u_j) + \big(1 - \mu(r_j^2 u_j)\big)$, for $j = n_1+1, \dots, n$ in the integrand, we can write

$$\int_{\mathbb{H}^n} f(z,u)\varphi(r_1 z_1, \dots, r_n z_n)\eta(s_1 u_1, \dots, s_n u_n)\,dz\,du$$

as a sum, for $1 \leqslant n_0 \leqslant n_1 \leqslant n_2 \leqslant n$ of $\binom{n_1}{n_0}\binom{n-n_1}{n_2-n_1}$ terms obtained by permuting the variables $z_1, \dots, z_{n_1}$, and the variables $u_{n_1+1}, \dots, u_n$ in

$$\begin{aligned} I = \int_{\mathbb{H}^n} & f(z,u)\varphi(r_1 z_1, \dots, r_n z_n)\eta(s_1 u_1, \dots, s_n u_n) \\ & \psi\big(\sqrt{s_1}z_1\big)\dots\psi\big(\sqrt{s_{n_0}}z_{n_0}\big)\Big(1 - \psi\big(\sqrt{s_{n_0+1}}z_{n_0+1}\big)\Big)\dots\Big(1 - \psi\big(\sqrt{s_{n_1}}z_{n_1}\big)\Big) \\ & \mu(r_{n_1+1}^2 u_{n_1+1})\dots\mu(r_{n_2}^2 u_{n_2})\big(1 - \mu(r_{n_2+1}^2 u_{n_2+1})\big)\dots\big(1 - \mu(r_n^2 u_n)\big)\,dz\,du\,. \end{aligned}$$

Without loss of generality, we consider only this term.

Setting $h = (h', h'')$, with $h' = (h_1, \dots, h_{n_0}, h_{n_1+1}, \dots, h_{n_2})$, $h_i = (z_i, u_i)$ then

$$|I| \leqslant \int_\Omega \Big| \int_{\mathbb{H}^{k_1}} f(z,u)\Phi_{(z'',u'')}\big(\delta_R(z',u')\big)dz'\,du'\Big|\,dz''\,du''$$

where $R = (\sqrt{s_1}, \dots, \sqrt{s_{n_0}}, r_{n_1+1}, \dots, r_{n_2}) \in (\mathbb{R}^+)^{k_1}$, $k_1 = n_0 + n_2 - n_1$, $k_2 = n - k_1$,

$$\begin{aligned} \Omega = \big\{(z'',u'') \in (\mathbb{C}\times\mathbb{R})^{k_2} : & |z_i| \geqslant 1/2\sqrt{s_i},\ |u_i| \leqslant 1/s_i,\ \text{for } i = n_0+1, \dots, n_1, \\ & |z_j| \leqslant 1/r_j,\ |u_j| \geqslant 1/2r_j^2,\ \text{for } j = n_2+1, \dots, n\big\} \end{aligned}$$

and given $(z'', u'') \in (\mathbb{C}\times\mathbb{R})^{k_2}$, we define $\Phi_{(z'',u'')}$ on $\mathbb{H}^{k_1}$ by

$$\begin{aligned} \Phi_{(z'',u'')}(z',u') = \varphi\Big(&\frac{r_1}{\sqrt{s_1}}z_1, \dots, \frac{r_{n_0}}{\sqrt{s_{n_0}}}z_{n_0}, r_{n_0+1}z_{n_0+1}, \dots, r_{n_1}z_{n_1}, \\ & z_{n_1+1}, \dots, z_{n_2}, r_{n_2+1}z_{n_2+1}, \dots, r_n z_n\Big) \\ \eta\Big(&u_1, \dots, u_{n_0}, s_{n_0+1}u_{n_0+1}, \dots, s_{n_1}u_{n_1}, \\ & \frac{s_{n_1+1}}{r_{n_1+1}^2}u_{n_1+1}, \dots, \frac{s_{n_2}}{r_{n_2}^2}u_{n_2}, s_{n_2+1}u_{n_2+1}, \dots, s_n u_n\Big) \\ \psi(&z_1)\dots\psi(z_{n_0})\mu(u_{n_1+1})\dots\mu(u_{n_2}). \end{aligned}$$

Then we observe that $\Phi_{(z'',u'')}$ is a normalised bump function on $\mathbb{H}^{k_1}$, since $r_i/\sqrt{s_i} \leqslant 1$ for $i = 1, \ldots, n_0$ and $s_j/r_j^2 < 1$ for $j = n_1+1, \ldots, n_2$. Thus by cancellation in (z', u') on f, we estimate the inner integral by

$$\left| \int_{\mathbb{H}^{k_1}} f(z,u)\Phi_{(z'',u'')}\big(\delta_R(z',u')\big)dz'\,du' \right| \leqslant C\,\gamma\,|h''|^{-4}$$

where the constant C is independent of the normalised bump function $\Phi_{(z'',u'')}$ and hence of (z'', u''). Thus $|I|$ is bounded by

$$\begin{aligned} C\gamma \prod_{i=n_0+1}^{n_1} \int_{|z_i|\geqslant(2/\sqrt{s_i})} \int_{|u_i|\leqslant 1/s_i} |h_i|^{-4}\,du_i\,dz_i \prod_{j=n_2+1}^{n} \int_{|z_j|\leqslant 1/r_j} \int_{|u_j|\geqslant 1/(2r_j^2)} |h_j|^{-4}\,du_j\,dz_j \\ \leqslant C\gamma \prod_{i=n_0+1}^{n_1} \frac{1}{s_i} \int_{|z_i|\geqslant 1/(2\sqrt{s_i})} |z_i|^{-4}\,dz_i \prod_{j=n_2+1}^{n} \frac{1}{r_j^2} \int_{|u_j|\geqslant 1/(2r_j^2)} |u_j|^{-2}\,du_j \\ \leqslant C\gamma \prod_{i=n_0+1}^{n_1} \frac{1}{s_i} 4s_i \prod_{j=n_2+1}^{n} 1/(r_j^2)2r_j^2 \quad \leqslant C\gamma\,. \end{aligned}$$

∎

Lemma 6.6. *Let $\gamma > 0$. Suppose $K \in L^1(\mathbb{R}^{n+1})$ satisfies the product kernel conditions on $\mathbb{R}^{n+1}$ with the factor γ included in the right-hand side of each estimate. Then the function $K^\flat$ on $\mathbb{R}$ given by*

$$K^\flat(t) = \int_{\mathbb{R}^n} K\left(u_1, \ldots, u_n, t - \sum_{i=1}^{n} u_i\right) du$$

is a standard kernel on $\mathbb{R}$, that is,

$$\left| \left(\frac{d}{dt}\right)^k K^\flat(t) \right| \leqslant C_k\gamma\,|t|^{-1-k}, \text{ and } \left| \int_{\mathbb{R}} K^\flat(t)\varphi(rt)\,dt \right| \leqslant C\gamma,$$

for all normalised bump functions φ on $\mathbb{R}$, all $r > 0$, and $k \in \mathbb{Z}^+$.

Proof of Lemma 6.6: For $n = 1$, this is proved in Lemma 6.7 below. The general result follows by induction on n, since

$$\begin{aligned} K^\flat(t) &= \int_{\mathbb{R}^n} K\left(u_1, \ldots, u_n, t - \sum_{i=1}^{n} u_i\right) du \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} K\left(\underleftarrow{u_{n-1}}, u_n, \left(t - \sum_{i=1}^{n-1} u_i\right) - u_n\right) du_n\,d\underleftarrow{u_{n-1}} \\ &= \int_{\mathbb{R}^{n-1}} K^{\flat,1}\left(\underleftarrow{u_{n-1}}, t - \sum_{i=1}^{n-1} u_i\right) d\underleftarrow{u_{n-1}} \end{aligned}$$

and by Lemma 6.7 below, $K^{\flat,1}$ is a product-type kernel on $\mathbb{R}^n$ with factor γ. ∎

LEMMA 6.7. *Suppose $K \in L^1(\mathbb{R}^{n+1})$ satisfies the product kernel conditions on $\mathbb{R}^{n+1}$, with a factor γ included in the right-hand side of each estimate. Then the function $K^{\flat,1}$ on $\mathbb{R}^n$ given by*

$$K^{\flat,1}(u) = \int_{\mathbb{R}} K(\underleftarrow{u_{n-1}}, x, u_n - x)\, dx$$

satisfies the product kernel conditions on $\mathbb{R}^n$, with the factor γ included in the right-hand side of each estimate, and constants depending only on the constants in the product kernel conditions satisfied by K on $\mathbb{R}^{n+1}$.

PROOF OF LEMMA 6.7: For the regularity condition, along with the cancellation condition in the final variable u_n of $K^{\flat,1}$, given $I = (i_1, \dots, i_{n-1}) \in (\mathbb{Z}^+)^{n-1}$ and fixing $\underleftarrow{u_{n-1}}$, we observe that

$$\partial^I_{\underleftarrow{u_{n-1}}} K(\underleftarrow{u_{n-1}}, x, y)$$

is a product-type kernel in (x, y) on $\mathbb{R}^2$, with factor $\gamma = |\underleftarrow{u_{n-1}}|^{-1-I}$. The result then follows from the fact shown in Lemma 6.8 below, that a product-type kernel on $\mathbb{R}^2$ when integrated over parallel lines $x + y = t$ yields a standard kernel in t.

For the cancellation conditions on $K^{\flat,1}$, given $1 \leqslant \ell \leqslant n-1$, by relabelling of variables, it suffices to obtain cancellation in $\underleftarrow{u_\ell}$, and in $(\underleftarrow{u_\ell}, u_n)$. Let η a normalised bump function on $\mathbb{R}^\ell$, $I = (i_{\ell+1}, \dots, i_{n-1}) \in (\mathbb{Z}^+)^{n-\ell-1}$, and $r \in (\mathbb{R}^+)^\ell$. Then setting $u' = (u_{\ell+1}, \dots, u_{n-1})$,

$$\int_{\mathbb{R}^\ell} \partial^I_{u'} K(\underleftarrow{u_{n-1}}, x, y)\eta(\delta_r(\underleftarrow{u_\ell}))\, d\underleftarrow{u_\ell}$$

is a product-type kernel in (x, y) on $\mathbb{R}^2$, with factor $\gamma = |u'|^{-1-I}$, and the result follows from Lemma 6.8. □

LEMMA 6.8. *Let $\gamma > 0$. Suppose $f \in L^1(\mathbb{R}^2)$ satisfies the product kernel conditions on $\mathbb{R}^2$, with the factor γ included in the right-hand side of each estimate. That is, f satisfies*

$$\left|\partial^i_x \partial^j_y f(x, y)\right| \leqslant C_{i,j}\gamma |x|^{-1-i}|y|^{-1-j} \tag{18}$$

for all i, $j \in \mathbb{Z}^+$,

$$\left|\int_{\mathbb{R}} \partial^j_y f(x, y)\varphi(rx)\, dx\right| \leqslant C_j \gamma |y|^{-1-j} \tag{19}$$

for all $j \in \mathbb{Z}^+$, all $r > 0$, and all normalised bump functions φ on $\mathbb{R}$,

$$\left|\int_{\mathbb{R}} \partial^i_x f(x, y)\varphi(ry)\, dy\right| \leqslant C_i \gamma |x|^{-1-i} \tag{20}$$

for all $i \in \mathbb{Z}^+$, all $r > 0$, and all normalised bump functions φ on $\mathbb{R}$,

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$$\left| \int_{\mathbb{R}^2} f(x,y)\varphi(r_1x, r_2y)\, dx\, dy \right| \leqslant C\gamma \tag{21}$$

for all r_1, $r_2 > 0$, and all normalised bump functions φ on $\mathbb{R}^2$. Then

$$f^\flat(t) = \int_{\mathbb{R}} f(x, t-x)\, dx$$

is a standard kernel on $\mathbb{R}$, with the factor γ included in the right-hand side of each estimate:

$$\left|\partial_t^k f^\flat(t)\right| \leqslant C_k\gamma\, |t|^{-1-k}\ , \text{ and } \qquad \left| \int_{\mathbb{R}} f^\flat(t)\varphi(rt)\, dt \right| \leqslant C\gamma$$

for all $k \in \mathbb{Z}^+$, $r > 0$, and all normalised bump functions φ on $\mathbb{R}$. The constants C_k and C depend only on the constants in the conditions (18)–(21).

PROOF: By homogeneity, it suffices to prove the regularity condition for $|t| = 1$. Letting η be a normalised bump function on $\mathbb{R}$, supported in $[-1/2, 1/2]$, with $\eta \equiv 1$ on $[-1/4, 1/4]$, and $|\eta^{(i)}| \leqslant C_k$ for $i = 0, \ldots, k+1$, we write

$$\begin{aligned} \partial_t^k f^\flat(t) &= \partial_t^k \int_{\mathbb{R}} f(x, t-x)\, dx \\ &= \partial_t^k \int_{\mathbb{R}} f(x, t-x)\eta(x)\, dx + \partial_t^k \int_{\mathbb{R}} f(x, t-x)\big(1-\eta(x)\big)\, dx = I_1 + I_2. \end{aligned}$$

Now,

$$I_1 = \int_{\mathbb{R}} \Big[\partial_2^k f(x, t-x) - \partial_2^k f(x,t)\Big]\eta(x)\, dx + \int_{\mathbb{R}} \partial_2^k f(x,t)\eta(x)\, dx\ .$$

We estimate the second term directly, using (19). Next, by the mean value theorem and (18),

$$\begin{aligned} \Big|\partial_2^k f(x, t-x) - \partial_2^k f(x,t)\Big| &\leqslant |x| \sup_{0<|s|<|x|} \left|\partial_2^{k+1} f(x, t-s)\right| \\ &\leqslant C_k\gamma|x| \sup_{0<|s|<|x|} |x|^{-1}|t-s|^{-k-2} \leqslant C_k\gamma, \end{aligned}$$

for $|x| \leqslant 1/2$, $|t| = 1$, and so the estimate for the first term of I_1 follows, since η is supported in $[-1/2, 1/2]$.

Next, for I_2, changing variables in x to ensure that no t-derivatives fall on the second component of f (so that cancellation can be used in this second variable), we obtain

$$I_2 = \int_{\mathbb{R}} \partial_t^k \Big[f(t-x, x)\big(1-\eta(t-x)\big)\Big]\, dx = \sum_{i=0}^{k} \binom{k}{i} \int_{\mathbb{R}} \partial_1^i f(t-x, x)\psi_{k-i}(t-x)\, dx$$

https://doi.org/10.1017/S0004972700019092 Published online by Cambridge University Press

where $\psi_0(y) = 1 - \eta(y)$, and $\psi_j(y) = -\eta^{(j)}(y)$ for $j \in \mathbb{N}$. Each ψ_j is thus supported in $\{|y| \geqslant 1/4\}$. We now split the integrand of the i^{th} term according to the size of the second component of f:

$$\int_{\mathbb{R}} \partial_1^i f(t-x,x)\psi_{k-i}(t-x)\,dx = \int_{\mathbb{R}} \partial_1^i f(t-x,x)\eta(x)\psi_{k-i}(t-x)\,dx + \int_{\mathbb{R}} \partial_1^i f(t-x,x)\big(1-\eta(x)\big)\psi_{k-i}(t-x)\,dx.$$

The function $\eta(x)\psi_{k-i}(t-x)$ is a normalised bump function in x, and thus we estimate the first term exactly as we did I_1. For the second term, using (18)

$$\left|\int_{\mathbb{R}} \partial_1^i f(t-x,x)\big(1-\eta(x)\big)\psi_{k-i}(t-x)\,dx\right| \leqslant C \int_{\substack{|t-x|\geqslant 1/4 \\ |x|\geqslant 1/4}} \big|\partial_1^i f(t-x,x)\big|\,dx \leqslant C_k\gamma \int_{\substack{|t-x|\geqslant 1/4 \\ |x|\geqslant 1/4}} |t-x|^{-1-i}|x|^{-1}\,dx\,.$$

But the last integral is bounded, thus proving the regularity condition.

Next, by homogeneity, it suffices to show the cancellation condition for $r = 1$. By Fubini, and a change of variables in t,

$$I = \int_{\mathbb{R}} f^\flat(t)\varphi(t)\,dt = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,s)\varphi(s+x)\,ds\,dx.$$

Letting η be a normalised bump function on $\mathbb{R}$, such that $\eta \equiv 1$ on $[-1/2,1/2]$, and including the factor $1 = \eta(x) + \big(1-\eta(x)\big)$ in the integrand, we can split I correspondingly into two terms: $I = I_1 + I_2$. Observing that $\eta(x)\varphi(s+x)$ is a dilate by $1/4$ of the normalised bump function

$$h(x,s) = \eta(4x)\varphi(4s+4x)$$

on $\mathbb{R}^2$, then by (21), we obtain the required estimate for I_1. For I_2, we split the integral further, according to the size of $|s|$, and write

$$I_2 = \int_{\mathbb{R}^2} f(x,s)\big(1-\eta(x)\big)\varphi(s+x)\eta(s)\,ds\,dx + \int_{\mathbb{R}^2} f(x,s)\big(1-\eta(x)\big)\varphi(s+x)\big(1-\eta(s)\big)\,ds\,dx.$$

Now, $\big(1-\eta(x)\big)\ \varphi(s+x)\ \eta(s)$ is a dilate of a normalised bump function in x and s, and so we estimate the first term using (21). Using the regularity condition (18), we bound the second term by

$$\int_{\substack{|x|,|s|\geqslant 1/2 \\ |s+x|\leqslant 1}} \big|f(x,s)\big|\,dx\,ds \leqslant C\gamma \int_{\substack{|x|,|s|\geqslant 1/2 \\ |s+x|\leqslant 1}} |x|^{-1}|s|^{-1}\,dx\,ds \leqslant C\gamma\,.$$

which concludes the proof of Lemma 6.8. ∎

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