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EXTENSIONS OF CHARACTERS FROM HALL π-SUBGROUPS OF π-SEPARABLE GROUPS

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1. Introduction

The main result of this paper is the following:

Theorem A. *Let G be a π-separable finite group with Hall π-subgroup H. Suppose $\theta \in \mathrm{Irr}(H)$. Then there exists a unique subgroup M, maximal with the property that it contains H and θ can be extended to a character of M.*

If $H \lhd G$, this result is a trivial consequence of Gallagher's theorem (see Corollary 8.16 of [**3**]). In this case, θ extends to its stabilizer $T = I_G(\theta)$ and any subgroup to which θ extends must stabilize θ and so is contained in T.

In fact, Gallagher's extendibility theorem generalizes (for π-separable groups) to the situation where the Hall π-subgroup H is not necessarily normal. If $H \subseteq U \subseteq G$ and θ is "invariant" in U in the sense that $\theta(x) = \theta(y)$ whenever $x, y \in H$ are conjugate in U, then θ extends to U. (See Theorem 8.1 of [**5**].)

This generalization does not prove Theorem A, however, because the concept of the "stabilizer" of θ is not available when H is not normal. In fact, Theorem A can be viewed as asserting that an irreducible character of H does have a well-defined stabilizer. This is all the more surprising when it is realized that if θ is reducible , then there need not be a unique largest subgroup containing H in which θ is "invariant". We provide an example to illustrate this.

The irreducibility of θ turns out not be be crucial for the generalized Gallagher's theorem, although the proof in [**5**] certainly uses it. By means of the deeper theory in [**4**], it is very easy to prove a stronger result and we do so here.

Theorem B. *Let G be π-separable with Hall π-subgroup H and let $\theta \in \mathrm{Char}(H)$ satisfy $\theta(x) = \theta(y)$ whenever $x, y \in H$ are conjugate in G. Then θ extends to a character of G.*

2. Theorem A

Let G be π-separable. The set $\mathscr{X}_\pi(G)$ of "π-special" irreducible characters of G was defined by D. Gajendragadkar [**1**] and has some quite remarkable properties. We will not give the definition here, but we list a few facts which we shall need.

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Proposition 2.1 *Let G be π-separable and let $H \subseteq G$ be a Hall π-subgroup.*

a) *Restriction defines an injection*

$$\mathscr{X}_\pi(G) \to \mathrm{Irr}(H).$$

b) *The image of the map in* (a) *is exactly the set of $\theta \in \mathrm{Irr}(H)$ which extend to G.*

c) *If $N \lhd G$ and $\chi \in \mathscr{X}_\pi(G)$, then every irreducible constituent of χ_N lies in $\mathscr{X}_\pi(N)$.*

d) *If $N \lhd G$ with G/N a π'-group and $\theta \in \mathscr{X}_\pi(N)$ is invariant in G, then θ extends to G.*

e) *If $N \lhd G$ with G/N a π-group and $\theta \in \mathscr{X}_\pi(N)$, then every irreducible constituent of θ^G lies in $\mathscr{X}_\pi(G)$.*

Proof. All but (b) are in [1] and (b) follows from Theorem 8.1 of [5]. □

Lemma 2.2. *Let G be π-separable with Hall π-subgroup H and suppose $N \lhd G$ with $NH = G$. Let $\theta \in \mathrm{Irr}(H)$ and let φ be an irreducible constituent of $\theta_{N \cap H}$. Then θ extends to G iff φ extends to N.*

Proof. First, suppose θ extends to G. By Proposition 2.1(b), we have $\theta = \chi_H$ for some π-special $\chi \in \mathrm{Irr}(G)$. Now χ_N has some irreducible constituent ψ which lies over φ. By Proposition 2.1(c), ψ is π-special and since $N \cap H$ is a Hall π-subgroup of N, we conclude that $\psi_{N \cap H}$ is irreducible by Proposition 2.1(a) and thus $\psi_{N \cap H} = \varphi$ as desired.

Now assume φ extends to N and choose a π-special extension ψ by Proposition 2.1(b). Then $(\psi^G)_H = (\psi_{N \cap H})^H = \varphi^H$ and hence some irreducible constituent χ of ψ^G lies over θ. However, χ is π-special by Proposition 2.1(e) and therefore $\chi_H = \theta$ by Proposition 2.1(a). □

Proof of Theorem A. Work by induction on $|G|$. Let $H \subseteq M \subseteq G$ with M maximal such that θ extends to M. Let $N \lhd G$ be a maximal normal subgroup of G. We shall complete the proof by showing how to determine M from a knowledge of θ and N. This will be sufficient since the choice of N did not depend on M.

First suppose G/N is a π-group so that $NH = G$. For each irreducible constituent φ of $\theta_{N \cap H}$, let $V(\varphi) \supseteq N \cap H$ denote the (unique) subgroup of N maximal such that φ extends to $V(\varphi)$. (We are, of course, using the inductive hypothesis in N.) Since φ uniquely determines $V(\varphi)$ and the various φ are conjugate under H, it follows that

$$D = \bigcap_{\varphi} V(\varphi)$$

is normalized by H, where φ runs over all irreducible constituents of $\theta_{N \cap H}$.

By Lemma 2.2, since $M = (N \cap M)H$ and θ extends to M, it follows that all of the φ extend to $N \cap M$ and thus $N \cap M \subseteq D$. We conclude that DH is a group containing M. Applying Lemma 2.2 in this group, since $D \lhd DH$ and $D \cap H = N \cap H$ and φ extends to D, we see that θ extends to DH. By the maximality of M, we have $M = DH$ and so M is determined, as required.

Now assume G/N is a π'-group so that $H \subseteq N$. Let $V \supseteq H$ be the unique subgroup of N maximal with the property that θ extends to it. By Proposition 2.1(b), there is a

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unique π-special extension $\hat{\theta} \in \mathrm{Irr}(V)$ of θ. Let T be the stabilizer of $\hat{\theta}$ in $\mathbf{N}_G(V)$ and observe that $\hat{\theta}$ (and therefore also θ) extends to T by Proposition 2.1(d). If we can show that $M \subseteq T$, then by the maximality of M, we have $M = T$ and M is determined as desired.

Now let $K = \mathbf{N}_M(H)$. Since θ extends to M, it certainly extends to K and thus K stabilizes θ. Since θ uniquely determines V (by the inductive hypothesis) and θ determines $\hat{\theta}$, we conclude that K stabilizes $\hat{\theta}$ and thus $K \subseteq T$.

By the Frattini argument, $M = (M \cap N)K$. Now θ certainly extends to $M \cap N$ and thus $M \cap N \subseteq V \subseteq T$. Since also $K \subseteq T$, we have $M \subseteq T$ and the proof is complete. □

3. An example

Let $H \subseteq G$ be a Hall π-subgroup where G is π-separable, and let θ be a class function of H. If $H \subseteq K \subseteq G$, we shall say that θ is *invariant in* K if $\theta(x) = \theta(y)$ whenever $x, y \in H$ are conjugate in K. We aim to show that in general there is no unique largest subgroup $K \supseteq H$ in which θ is invariant even if $\theta \in \mathrm{Char}(G)$. We shall do this by constructing an example where $H \subseteq K \subseteq G$, $H \subseteq L \subseteq G$, $\langle K, L \rangle = G$, θ is invariant in each of K and L and is not invariant in G. By Theorem A and either Theorem 8.1 of [**5**] or Theorem **B**, no such example could exist if $\theta \in \mathrm{Irr}(H)$.

Construction 3.1. Let E be extra-special of order 3^3 and exponent 3 and let $\sigma \in \mathrm{Aut}(E)$ invert $E/\mathbf{Z}(E)$ and centralize $\mathbf{Z}(E)$ with $\sigma^2 = 1$. Let $R = E \rtimes \langle \sigma \rangle$. Now R has a faithful absolutely irreducible representation of degree 3 over $GF(5^2)$ in which the eigenvalues of σ are -1, -1 and 1. Let V be the corresponding module so that V is elementary abelian of order 5^6. Let $G = V \rtimes R$. Let $\pi = \{2, 5\}$ and $H = V\langle \sigma \rangle$. Choose subgroups $A, B \subseteq E$, of order 3, inverted by σ, with $\langle A, B \rangle = E$. Let $K = VA\langle \sigma \rangle$ and $L = VB\langle \sigma \rangle$.

Proposition 3.2. *Assume the notation of* 3.1. *Then there exists* $\theta \in \mathrm{Char}(H)$ *such that* θ *is invariant in each of* K *and* L *but is not invariant in* G.

Proof. Note that H contains elements of order 10. Let C be a cyclic subgroup of H with $\sigma \in C$ and $|C| = 10$. Let $a, b \in \mathbb{Z}$ and define the class function $\theta = \theta_{a,b}$ of H as follows:

$$\theta(x) = \begin{cases} a & \text{if } x = 1 \\ b & \text{if } \langle x \rangle \text{ is } H\text{-conjugate to } C \\ 0 & \text{otherwise.} \end{cases}$$

We claim that θ is invariant in K and L and if $b \neq 0$, then θ is not invariant in G. Afterwards, we shall show that θ is a character for suitable choices of a and b.

Suppose $x, y \in H$ are conjugate in K. We wish to show that $\theta(x) = \theta(y)$. The only nontrivial item to check is that if $\langle x \rangle$ is H-conjugate to C then $\langle y \rangle$ is also. Without loss, we can replace x and y by H-conjugates and assume that σ is the 2-part of each of x and y. Since $y = x^k$ for some $k \in K$, we conclude that $k \in \mathbf{C}_K(\sigma)$. Since $K = VA\langle \sigma \rangle$ and σ inverts A, it is easy to see that $\mathbf{C}_K(\sigma) \subseteq V(\sigma) = H$ and so $k \in H$ and hence $\theta(x) = \theta(y)$. A similar argument, of course, works for L.

Now let $x \in H$ with $\langle x \rangle = C$ and let $z \in \mathbf{Z}(E)$ of order 3. Then z centralizes σ and so $H^z = H$ and $y = x^z \in H$. We claim that $\theta(y) = 0$ and so $\theta(x) \neq \theta(y)$ and θ is not invariant in G. We need to show that $\langle y \rangle$ is not H-conjugate to $\langle x \rangle$. If $\langle y \rangle = \langle x \rangle^h$ for $h \in H$, then $\sigma^h = \sigma$ and $h \in \mathbf{C}_H(\sigma)$ and this is an abelian group. Thus $\langle y \rangle = \langle x \rangle$ and z normalizes C. It follows that z centralizes C since $3 \nmid |\mathrm{Aut}(C)|$. This is a contradiction since, in fact, z acts on V by a nontrivial scalar multiplication (over $GF(5^2)$).

Finally, we need to show how to choose $a, b \in \mathbb{Z}$ with $b \neq 0$ so that $\theta = \theta_{a,b}$ will be a character. Let $\varphi = \theta_{0,1}$ and note that since φ is a class function of H, it is some complex linear combination of $\mathrm{Irr}(H)$. We assert that, in fact, the coefficients are rational. To see this, let $\alpha \in \mathrm{Aut}(\mathbb{C})$ and let ε be a primitive $|H|$ root of unity in $\mathbb{C}$. Then $\varepsilon^\alpha = \varepsilon^r$ for some integer r with $(r, |H|) = 1$ and $1 \leqq r \leqq |H|$. It suffices (by Lemma 3.2 of [**2**]) to show that $\varphi(x)^\alpha = \varphi(x^r)$ for $x \in H$. However, φ has values 0 and 1 and so $\varphi(x)^\alpha = \varphi(x)$. Also, $\langle x \rangle = \langle x^r \rangle$ and so $\varphi(x) = \varphi(x^r)$ by the definition of φ. Our assertion now follows.

We can now clear denominators and we have $b\varphi$ is a $\mathbb{Z}$-linear combination of $\mathrm{Irr}(H)$. If ρ is the regular character of H, then $b\varphi + m\rho \in \mathrm{Char}(H)$ for sufficiently large $m \in \mathbb{Z}$. We have $\theta_{m|H|,b} = b\varphi + m\rho$ and we are done. □

4. Theorem B

In [**4**], we defined for π-separable groups G a certain set $B_\pi(G)$ of irreducible characters. We list some facts.

Proposition 4.1. *Let G be π-separable and let χ^* denote the restriction of the class function χ of G to the set of π-elements of G. Then the functions χ^* for $\chi \in B_\pi(G)$ are distinct and form a basis for the complex vector space of class functions on the π-elements of G.*

Proof. This is Theorem 9.3 of [**4**]. □

Proposition 4.2. *Let G be π-separable with Hall π-subgroup H. For each $\chi \in B_\pi(G)$, there is an irreducible constituent α of χ_H such that $[\chi_H, \alpha] = 1$ and $[\xi_H, \alpha] = 0$ for $\chi \neq \xi \in B_\pi(G)$.*

Proof. This is part of Theorem 8.1 of [**4**]. □

Note that the linear independence of $\{\chi^* | \chi \in B_\pi(G)\}$ in Proposition 4.1 is immediate from Proposition 4.2.

The next result includes Theorem B.

Theorem 4.3. *Let G be π-separable with Hall π-subgroup H. Suppose $\theta \in \mathrm{Char}(H)$ satisfies $\theta(x) = \theta(y)$ whenever $x, y \in H$ are G-conjugate. Then θ has a unique extension $\psi \in \mathrm{Char}(G)$ with the property that each irreducible constituent of ψ lies in $B_\pi(G)$.*

Proof. Define the function φ on the π-elements of G by setting $\varphi(g) = \theta(x)$ where $x \in H$ is conjugate to G. Note that φ is well defined since every π-element $g \in G$ is conjugate to some $x \in H$ by property D_π and if g is also conjugate to $y \in H$, then $\theta(x) = \theta(y)$ by hypothesis.

An extension ψ of θ is just a character such that $\psi^*=\varphi$. By Proposition 4.1, we can write

$$\varphi=\sum_{\chi\in B_\pi(G)} a_\chi\chi^*$$

for some uniquely defined complex coefficients a_χ. Write

$$\psi=\sum a_\chi\chi.$$

Then $\psi^*=\varphi$ and ψ is the unique linear combination of $B_\pi(G)$ with this property.

What remains in showing that ψ is a character is that each a_χ is a non-negative integer. For $\chi\in B_\pi(G)$, choose an irreducible constituent α of χ_H as in Proposition 4.2. Then by Proposition 4.2,

$$a_\chi=[\psi_H,\alpha]=[\theta,\alpha]$$

and the result follows. □

REFERENCES

1. D. Gajendragadkar, A characteristic class of characters of finite π-separable groups, *J. of Algebra* **59** (1979), 237–259.

2. D. Gluck and I. M. Isaacs, Tensor induction of generalized characters and permutation characters, *Illinois J. of Math.* **27** (1983), 514–518.

3. I. M. Isaacs, *Character theory of finite groups* (Academic Press, New York, 1976).

4. I. M. Isaacs, Characters of π-separable groups, *J. of Algebra* **86** (1984), 98–128.

5. I. M. Isaacs, Induction and restriction of π-special characters, *Canad. J. of Math.* (To appear).

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