ON AUTOMOBILE INSURANCE RATEMAKING

Estimating relativities in a multiplicative model

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Note: After the manuscript had been distributed to the participants of the Arnhem Colloquium, mr Baily and mr Simon have kindly drawn my attention to some recent works on the problem. Thus mr Bailey presented at the May 1963 Meeting of the Casualty Actuarial Society a paper "Insurance Rates with Minimum Bias", where he recommended an estimation procedure identical with the "Heuristic Method" discussed below. In the discussion following on mr Bailey's paper, mr James R. Bergquist presented numerical results from the application of this method to the original material, which results agree completely with the table D below.

However, some of the arguments and results of this paper are outside the scope of the short 1963 paper by Mr Bailey, and may have an interest of their own.

1. INTRODUCTION

Suppose that an automobile insurance plan is characterized by a double classification. The risks are thus divided into classes, $i = 1, 2, \ldots, p$ (e.g. defined by use of car and age of operator), and groups $j = 1, 2, \ldots, q$ (e.g. defined by licence and by accidents during the last three years). The experience of the company is described by the observed "relative loss ratios" r_{ij} and some measure of exposure n_{ij}. A general model, often used, is that the r_{ij}: s are observations of random variables with the expected values $g_{ij} = g(\alpha_i, \beta_j)$, where the relativities α_i are parameters representing the classes i and the relativities β_j represent the influence of the groups j. One of the ratemaker's problems is to find a realistic function $g(\alpha, \beta)$ and to obtain estimates a_i of α_i and b_j of β_j.

In their paper "Two Studies in Automobile Insurance Ratemaking" (ASTIN Bulletin Vol. I, Part IV, page 192-217) Robert Bailey and LeRoy Simon have thoroughly analyzed this problem

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for private passenger automobiles in Canada. They have principally studied three different types of the function $g(\alpha, \beta)$, namely $g(\alpha, \beta) = \alpha\beta$ (Method 2), $g(\alpha, \beta) = \alpha + \beta$ (Method 3) and $g(\alpha, \beta) = 3\alpha\beta - 2$ (Method 4). The authors show in an appendix, that the variance of r_{ij} is approximately $g(\alpha_i, \beta_j)/Kn_{ij}$ where $K \cong 0.005$ for the Canadian data. They estimate the relativities α_i and β_j by making $\chi^2 = K \,.\, \Sigma\, n_{ij}(r_{ij} - g_{ij})^2/g_{ij}$ a minimum. For the Canadian material, the "method 4" agrees best with the observations. This method gives an observed χ^2 value of about 8 for 11 degrees of freedom.

The pure multiplicative model (Method 2) has been applied to Swedish motor car insurance by the late Bertil Almer. In a lecture to the Swedish Actuarial Society 1954, and in later papers, i.a. his communication to the New York congress of actuaries (Ref. 1), he showed that the claims frequencies and the loss ratios could be fairly well described by a multiplicative model (Almer used the term "factor analysis"), *provided that the small, the medium and the large claims were treated separately.*

In the preparation of the new Swedish automobile insurance rates valid from February 1st 1966, the computational methods given by Bailey-Simon have been applied by G. Andreasson (Ref. 2) to a purely multiplicative model with not less than 8 independent classifications. This application has made actual some practical and theoretical questions connected with the minimum χ^2 method, which will be discussed. To simplify the discussion, mainly the two-dimensional multiplicative case will be treated. The numerical illustrations are based on the Canadian material, wellknown to the readers of ASTIN Bulletin.

2. Three criteria

Bailey-Simon use three types of numerical criteria for the estimation of the relativities.

$$
\begin{aligned}
&\text{1) Class balance factors } B_{i.} &&= \sum_j n_{ij}a_ib_j \Big/ \sum_j n_{ij}r_{ij} \\
&\quad\ \text{Group balance factors } B_{.j} &&= \sum_i n_{ij}a_ib_j \Big/ \sum_i n_{ij}r_{ij} \qquad (1)\\
&\quad\ \text{Total balance factor } B_{..} &&= \sum_{i,j} n_{ij}a_ib_j \Big/ \sum_{i,j} n_{ij}r_{ij}
\end{aligned}
$$

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2) Mean absolute departure $D = \sum_{i,j} n_{ij} \, |r_{ij} - a_i b_j| \, / \sum_{i,j} n_{ij} r_{ij}$ (2)

3) $\chi^2 = K \cdot Q(a, b)$, with $(p - 1)(q - 1)$ degrees of freedom, where

$$Q(a, b) = \sum_{i,j} n_{ij} \, (r_{ij} - a_i b_j)^2 / a_i b_j \tag{3a}$$

$$= \sum_{i,j} \left(\frac{n_{ij} r_{ij}^2}{a_i b_j} - 2 n_{ij} r_{ij} + n_{ij} a_i b_j \right) \tag{3b}$$

As a fourth criterion they want the estimates to reflect the relative credibility of the observed groups, which is attained by using the exposures n_{ij} as weights in the estimation procedure.

3. Estimation by the χ^2 minimum method

Bailey-Simon chose as estimates for α_i and β_j those values $\hat{a}_i$ and $\hat{b}_j$ which make χ^2 and thus $Q(a, b)$ a minimum.

Minimizing $Q(a, b)$ in (3b) leads to the equations:

$$\sum_{j=1}^{q} (n_{ij} r_{ij}^2 / \hat{a}_i \hat{b}_j - n_{ij} \hat{a}_i \hat{b}_j) = 0; \; i = 1, 2, \ldots, p \tag{4a}$$

$$\sum_{i=1}^{p} (n_{ij} r_{ij}^2 / \hat{a}_i \hat{b}_j - n_{ij} \hat{a}_i \hat{b}_j) = 0; \; j = 1, 2, \ldots, q \tag{4b}$$

Adding the equations (4a) or (4b), we obtain identical equations. Thus the system is indeterminate. This is obvious, because if a set (a_i, b_j) satisfies (4), the set $(c a_i, c^{-1} b_j)$ is also a solution for an arbitrary value of c.

From (3) and (4) it follows, that summing over i, over j or over both i and j

$$\sum n_{ij} (r_{ij} - \hat{a}_i \hat{b}_j)^2 / \hat{a}_i \hat{b}_j = 2 \sum n_{ij} (\hat{a}_i \hat{b}_j - r_{ij}) \tag{5}$$

From (5) it follows, that all the balance factors B of (1) are larger than unity, and that χ^2 may be calculated from twice the sum of the weighted differences between estimated and observed relative loss ratios. As far as the applied model is true, the bias of the total loss for groups and classes could be expected to be of the same order of magnitude as the number of degrees of freedom of the χ^2-values in the left member of (5), but if the model should be false (which is to be expected), the bias could be appreciable.

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Note: The relation (5) is not restricted to the multiplicative model (where it is valid for any number of dimensions). It is characteristic for a wide class of functions $g(\,.\,,.)$, i.a. for all g, which are homogeneous functions of arbitrary functions $A(\alpha)$ and $B(\beta)$. The positive bias resulting from (5), where the difference between estimated and observed sums for groups or classes is equal to a non-negative quadratic form, has in the case of mortality estimation by means of Makeham parameters been pointed out and discussed by S. G. Lindblom (Ref. 5).

4. An heuristic approach to the estimation problem

The quadratic form $Q(a, b)$ of (3a) is identical with the χ^2 expression for the case when the variables $n_{ij}r_{ij}$ are independent and Poisson distributed with parameters $n_{ij}\alpha_i\beta_j$. Assuming for the moment this hypothesis, we obtain for the maximum-likelihood estimates the equations

$$a_i^* \sum_j n_{ij}b_j^* = \sum_j n_{ij}r_{ij}; \; i = 1, 2, \ldots, p \tag{6a}$$

$$b_j^* \sum_i n_{ij}a_i^* = \sum_i n_{ij}r_{ij}; \; j = 1, 2, \ldots, q \tag{6b}$$

These equalities imply that all the balance factors B of (1) are identically 1 and that the resulting χ^2 expression may be written

$$\chi^2 = \sum_{i,j} (n_{ij}r_{ij}^2 \,/\, a_i^*b_j^* - n_{ij}r_{ij}) \tag{7}$$

Maximum-likelihood equations corresponding to (6) and (7) are valid in the multiplicative case with any number of dimensions and for the more general functions $g(.\,,.)$ mentioned in the note on page 3. *Even if these general conditions are not satisfied or if the model used is false, the equations* (6) *always give unbiassed estimates for the totals* $\Sigma\, n_{ij}r_{ij}$ *within groups or classes.*

Returning from the specific Poisson model introduced above to the general model of Bailey-Simon, the χ^2 value is obtained from (7) by multiplying with the factor K. Furthermore it should be noted, that in this more general case, the equations (6) give estimates defined by the "Modified χ^2 minimum method". Cf. Cramér (Ref. 4) p. 425-426 and 506. The assumptions made here are not identical with those in the text mentioned, but the main results hold true, and thus for large n_{ij} the expected value of χ^2 is approximately $(p - 1)(q - 1)$ for both estimation methods considered in this paper.

Comparing the equations (4) and (6), the latter have constant right members, which makes the solution by successive iterations easy to perform.

5. The multiplicative model when one set of parameters is known

In the last section we found, that the estimates a_i^*, b_j^* gave balance factors identically equal to unity and an expected χ^2 value approximately equal to (but of course somewhat larger than) the value in the case of the estimates $\hat{a}_i, \hat{b}_j$. It seems difficult to obtain good approximations of this difference in the general case, but an idea of the general trend could be obtained by studying the case when all β_j:s are (at least practically) known. This case is in no way unimportant in the actual application, as it corresponds to the situation, when new classes or groups are introduced, when new subdivisions are tried or when cars from new geographical areas are included in the portefolio.

In order to obtain the comparison intended, it is necessary to specify the hypothesis. Let us assume, that $Kn_{ij}r_{ij}$ are independent Poisson variables with mean $Kn_{ij}\alpha_i\beta_j$ (which assumption leaves the moments $E(r_{ij}) = \alpha_i\beta_j$ and $D^2(r_{ij}) = \alpha_i\beta_j / Kn_{ij}$ unaltered), and introduce the notations

$$S_i = K \sum_j n_{ij}\alpha_i\beta_j; \; \sigma_i = K^{-1} \sum_j (n_{ij}\alpha_i\beta_j)^{-1} \tag{8}$$

For the estimates $\hat{a}_i$ we obtain from (4a) after elementary calculations

$$E(\hat{a}_i^2) = \alpha_i^2(1 + q/S_i); \; D^2(\hat{a}_i^2) = \alpha_i^4 [4/S_i + (6q + \sigma_i) / S_i^2].$$

With wellknown approximations we obtain

$$E(\hat{a}_i) \cong \alpha_i[1 + (q - 1)/2S_i - (q^2 + \sigma_i)/8S_i^2 + o(S_i^{-2})]$$

Thus $\hat{a}_i$ has usually a positive bias.

Further $E[K \sum_j n_{ij}(\hat{a}_i\beta_j - r_{ij})] \cong (q - 1)/2 - (q^2 + \sigma_i)/8S_i$,

whence it follows, that the balance factors $B_{i.}$ have a bias of magnitude $(q - 1)/2S_i$ and corresponding for $B_{.j}$.

Finally we obtain for $\chi^2 \equiv 2K \sum_{i,j} n_{ij} (\hat{a}_i\beta_j - r_{ij})$:

$$E\chi^2 \cong p(q-1) - \sum_i (q^2 + \sigma_i)/4S_i. \tag{9}$$

For the estimates a_i^* defined by (6a) we get

$$E(a_i^*) = \alpha_i,\ D^2(a_i^*) = \alpha_i^2/S_i \text{ and } B_{i.} = B_{.j} = 1,$$
$$\chi^2 = K \sum_{i,j} (n_{ij}r_{ij}^2/a_i^*\beta_j - n_{ij}r_{ij},$$

where a_i^* stands in the denominator and is statistically dependent of the r_{ij}:s. Under the Poisson hypotheses the variables $Kn_{ij}r_{ij}$, conditioned by a_i^*, are binomial random variables, and trivial computations give the result (which for other reasons is almost evident) that

$$E(\chi^2) \equiv p(q-1) \tag{10}$$

As q is a constant number, while S_i has the order of magnitude of the total loss in the class i, the difference between (9) and (10) is of slight importance in our application. It seems likely that this applies even to the case when also the β_j:s (or still more parameters) are estimated.

6. A numerical comparison

In their ASTIN paper of 1960, Bailey-Simon have given some numerical results in tables C, D and E (p. 213-214).

Starting from the class relativities a_i ("method 1" in table C), I have used the equation (6b) to obtain a first set of estimates $b_j^{(1)}$, and then made two further iterations giving $a_i^{(2)}$ and finally $b_j^{(3)}$. In the following tables Bailey-Simon's "method 2" is compared to the results obtained by the "heuristic" method, after 3 steps of iteration.

Table D

Estimated loss ratios a_ib_j

Method 2 ($\hat{a}_i \hat{b}_j$)					Heuristic Method ($a_i^*b_j^*$)				
i/j	1	2	3	4	i/j	1	2	3	4
1	0.798	0.981	1.070	1.288	1	0.797	0.978	1.068	1.284
5	1.052	1.292	1.411	1.697	5	1.051	1.290	1.408	1.693
3	1.186	1.457	1.590	1.914	3	1.186	1.455	1.589	1.911
2	1.239	1.521	1.661	1.999	2	1.237	1.517	1.657	1.993
4	1.925	2.365	2.582	3.107	4	1.927	2.363	2.580	3.103

The estimates $a_i^* b_j^*$ thus seem to converge rapidly to values very little different from the estimates $\hat{a}_i \hat{b}_j$.

In the following table E the values of the criteria 1-3 are compared. That the class balance factors in the right column are not identically 1 depends of course on the use of only 3 steps of iteration.

TABLE E

Tests of Criteria	Method 2	Heuristic Method
1. Balance. Class 1	1.0007	0.9993
5	1.0014	1.0002
3	1.0006	1.0006
2	1.0027	1.0004
4	1.0027	1.0026
Group 1	1.0006	1.
2	1.0026	1.
3	1.0015	1.
4	1.0025	1.
Total	1.0011	1.
2. Average error	0.0317	0.0315
3. χ^2 (12 degrees of freedom)	34	34.12

7. SUMMARY

For the multiplicative model used to estimate the relative loss ratios in automobile insurance, the author recommends an estimation method, slightly different from the method discussed in ASTIN Bulletin I: IV by Bailey and Simon. Without seriously affecting the χ^2 value for "goodness of fit", the proposed modified method always gives unbiassed estimates for the total loss within groups and classes of a portefolio. This quality is not changed if the applied model should be false, in which case the Minimum-χ^2 estimates earlier used may have an appreciable positive bias.

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