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THE UNIFORM KADEC-KLEE PROPERTY FOR THE LORENTZ SPACES $L_{w,1}$

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Abstract

In this paper we show that the Lorentz space $L_{w,1}(0,\infty)$ has the weak-star uniform Kadec-Klee property if and only if $\inf_{t>0}(w(\alpha t)/w(t)) > 1$ and $\sup_{t>0}(\phi(\alpha t)/\phi(t)) < 1$ for all $\alpha \in (0,1)$, where $\phi(t) = \int_0^t w(s)\,ds$.

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1. Introduction

For a measurable function f defined on $(0,\infty)$, we define the distribution of $|f|$ by $d_f(t) = |\{x : |f(x)| > t\}|$, $0 < t < \infty$, where $|A|$ denotes the Lebesgue measure of the set A, and we define the decreasing rearrangement of $|f|$ by $f^*(t) = \inf\{s > 0 : d_f(s) \le t\}$.

Let $w : (0,\infty) \to (0,\infty)$ be a decreasing function satisfying $\lim_{t\to 0} w(t) = \infty$, $\lim_{t\to\infty} w(t) = 0$, $\int_0^1 w(t)\,dt = 1$, and $\int_0^\infty w(t)\,dt = \infty$. Define the Lorentz space $L_{w,1}(0,\infty)$ as the space of all (equivalence classes of) measurable functions f on $(0,\infty)$ for which

$$\|f\| = \int_0^\infty f^*(t)w(t)\,dt < \infty.$$

$L_{w,1}$ is sometimes also referred to as Λ_ϕ, where

$$\phi(t) = \int_0^t w(s)\,ds \qquad (t \ge 0).$$

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These spaces were introduced by Lorentz in [15] and were studied recently in [6]. $L_{w,1}$ is a non-reflexive separable dual Banach space. Its natural predual contains the integrable simple functions as a dense subspace.

A dual space has the *weak-star uniform Kadec-Klee* property if, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every sequence $\langle f_n \rangle$ with $\|f_n\| \leq 1$, $\inf_{m \neq n} \|f_n - f_m\| \geq \varepsilon$, and $f_n \to f$ in the weak-star topology, we have $\|f\| \leq 1 - \delta(\varepsilon)$. This property was introduced by Huff in [10]. See [3, 7, 10, 12] for an introduction to the uniform Kadec-Klee and related properties.

Sedaev [16] proved that strict concavity of ϕ is a necessary and sufficient condition for $L_{w,1}$ to have the (non-uniform) weak-star Kadec-Klee property: that is, if $f_n \to f$ weak-star and if $\|f_n\| \to \|f\|$, then $\|f_n - f\| \to 0$. In this paper we give necessary and sufficient conditions for $L_{w,1}$ to have the weak-star uniform Kadec-Klee property. Section 2 gathers together the calculations which are used in Section 3 in the proof of our main result (Theorem 3.2). The proof of the sufficiency of the conditions is based on the proof which is given in [5] for the special case of $L_{p,1}(0, \infty)$. The main result implies a fixed point theorem for non-expansive mappings (Corollary 3.3). See for example [3, 5, 9, 12, 13] for further results about the uniform Kadec-Klee property in classical spaces.

Throughout the paper $I(A)$ will denote the characteristic function of a set $A \subset [0, \infty)$. If $0 < |A| < \infty$, we write $e(A) = I(A)/\phi(|A|)$ (so that $e(A)$ is of norm one in $L_{w,1}$). We also write A^c to denote the complement of the set A.

Finally, we wish to thank Chris Lennard and both referees for their many helpful suggestions.

2. Preliminaries

The proof of the following lemma can be found in [6].

PROPOSITION 2.1. *Let f be a nonnegative function on $(0, \infty)$ with $\|f\| = 1$. Then there exist a collection of Borel sets $\big(A(u)\big)_{u>0}$ and a probability measure μ on $(0, \infty)$ with the following properties:*

(1) *$A(u) \subset A(v)$, except for a set of measure zero, if $u < v$;*

(2) *$|A(u)| = u$;*

(3) $f = \int_0^\infty e\big(A(u)\big)\, d\mu(u)$;

(4) $f^* = \int_0^\infty e\big((0, u)\big)\, d\mu(u)$.

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DEFINITION 2.2. Let C_1 be the class of weight functions w satisfying

$$k_1(\alpha) = \sup_{t>0} \frac{\phi(\alpha t)}{\phi(t)} < 1$$

for all $\alpha \in (0, 1)$. In the literature these are called the *regular* weights (see for example [4, 8]).

REMARK. If $w \in C_1$, then, clearly, $k_1(\alpha) \to 1$ if and only if $\alpha \to 1$. Moreover, it is easily seen that

$$k_1(\alpha^n) \le \big(k_1(\alpha)\big)^n,$$

and hence $k_1(\alpha) \to 0$ as $\alpha \to 0$. It is also well-known that $w \in C_1$ if and only if $k_1(\alpha) < 1$ for *some* $\alpha < 1$.

DEFINITION 2.3. We say that $L_{w,1}$ has *property P* if whenever we are given two sequences $\langle f_n \rangle$ and $\langle g_n \rangle$ such that $\|f_n\| = 1$, $\|f_n + g_n\| \to 1$ as $n \to \infty$, and f_n, g_n are disjointly supported for each n, then $\|g_n\| \to 0$ as $n \to \infty$.

Note that property P is an abstract form of 'lower p-estimate' (see [14, p.82]).

LEMMA 2.4. *Let $w \in C_1$, and let A, E be sets such that $\|e(A)I(E)\| \ge 1 - \varepsilon$ for some $\varepsilon \ge 0$. Then $\|e(A)I(E^c)\| \le \delta_1(\varepsilon)$ for some $\delta_1(\varepsilon) > 0$, where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

PROOF. Let $|A| = u$. Since $w \in C_1$, we have

$$k_1\Big(\frac{|A \cap E|}{u}\Big) \ge \frac{\phi\big(|A \cap E|\big)}{\phi(u)} = \|e(A)I(E)\| \ge 1 - \varepsilon.$$

Therefore $1 - \dfrac{|A \cap E|}{u} \le \eta(\varepsilon)$, for some $\eta(\varepsilon) > 0$, and so $\dfrac{|A \cap E^c|}{u} \le \eta(\varepsilon)$. Thus we have

$$\begin{aligned} \|e(A)I(E^c)\| &= \frac{\phi\big(|A \cap E^c|\big)}{\phi(u)} \le k_1\Big(\frac{|A \cap E^c|}{u}\Big) \\ &\le k_1\big(\eta(\varepsilon)\big) = \delta_1(\varepsilon), \end{aligned}$$

where $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

LEMMA 2.5. *Suppose $w \in C_1$, $f = f_1 + f_2$, where $\|f\| = 1$ and f_1, f_2 are disjoint. If $\|f_1\| > 1 - \varepsilon^2$, then $\|f_2\| < \delta_2(\varepsilon)$, for some $\delta_2(\varepsilon) > 0$, where $\delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

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PROOF. By Proposition 2.1, $f = \int_0^\infty e\big(A(u)\big)\,d\mu(u)$ for some family of Borel sets $\big(A(u)\big)_{u>0}$ and some probability measure μ on $(0,\infty)$. Since f_1 and f_2 are disjoint there is a set E such that $f_1 = fI(E)$ and $f_2 = fI(E^c)$; it follows that

$$f_1 = \left(\int_0^\infty e\big(A(u)\big)\,d\mu(u)\right)I(E) = \int_0^\infty e\big(A(u)\big)I(E)\,d\mu(u).$$

Since $\|f_1\| > 1-\varepsilon^2$, we have

$$\begin{aligned}\varepsilon^2 > 1 - \|fI(E)\| &\geq 1 - \int_0^\infty \|e\big(A(u)\big)I(E)\|\,d\mu(u)\\ &= \int_0^\infty \big(1 - \|e\big(A(u)\big)I(E)\|\big)\,d\mu(u).\end{aligned}$$

Therefore, by Chebyshev's inequality,

$$\mu\big(\{u : \|e\big(A(u)\big)I(E)\| > 1-\varepsilon\}\big) = \mu\big(\{u : 1 - \|e\big(A(u)\big)I(E)\| \geq \varepsilon\}^c\big) \geq 1-\varepsilon.$$

Let $\delta_1(\varepsilon)$ be as in Lemma 2.4; then

$$\mu\big(\{u : \|e\big(A(u)\big)I(E^c)\| \leq \delta_1(\varepsilon)\}\big) \geq 1-\varepsilon.$$

Thus,

$$\begin{aligned}\|f_2\| = \|\int_0^\infty e\big(A(u)\big)I(E^c)\,d\mu(u)\| &\leq \int_0^\infty \|e\big(A(u)\big)I(E^c)\|\,d\mu(u)\\ &\leq \delta_1(\varepsilon)(1-\varepsilon) + \varepsilon = \delta_2(\epsilon).\end{aligned}$$

Clearly, $\delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

PROPOSITION 2.6. *$L_{w,1}$ has property P if and only if $w \in C_1$.*

PROOF. The fact that if $w \in C_1$ then $L_{w,1}$ has property P follows easily from Lemma 2.5. Conversely, suppose $w \notin C_1$; then $\sup_{t>0}(\phi(\alpha t)/\phi(t)) = 1$ for some $\alpha \in (0,1)$. Let $\langle t_n\rangle$ be a sequence such that $\lim_{n\to\infty}(\phi(\alpha t_n)/\phi(t_n)) = 1$, let $f_n = e\big((0,\alpha t_n)\big)$, and let $g_n = I\big((\alpha t_n, t_n)\big)/\phi(t_n)$. Then $\|f_n\| = 1$ and

$$\|g_n\| = \frac{\phi\big((1-\alpha)t_n\big)}{\phi(t_n)} \geq 1-\alpha$$

by the concavity of ϕ, and f_n, g_n are disjoint. But

$$\|f_n + g_n\| = 1 + \frac{\phi(t_n) - \phi(\alpha t_n)}{\phi(t_n)},$$

which converges to 1 as $n \to \infty$. Thus $L_{w,1}$ does not have property P.

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REMARK. For related results in Lorentz sequence spaces see [1, 2].

DEFINITION 2.7. Let C_2 be the class of weight functions w satisfying

$$k_2(\alpha) = \inf_{t>0} \frac{w(\alpha t)}{w(t)} > 1$$

for all $\alpha \in (0, 1)$.

REMARK. It is clear that, for each $w \in C_2$, $k_2(\alpha) \to 1$ if and only if $\alpha \to 1$.

EXAMPLE 2.8. Neither C_1 nor C_2 contains the other.

(a) $w(t) = \dfrac{1}{\sqrt{\log(1+t)}} \in C_1 \setminus C_2$.

PROOF. It is easy to see that $\lim_{t\to\infty}(\phi(\alpha t)/\phi(t)) = \alpha$, and that $\lim_{t\to 0}(\phi(\alpha t)/\phi(t)) = \sqrt{\alpha}$. So $\sup_{t>0}(\phi(\alpha t)/\phi(t)) = k_1(\alpha) < 1$, and thus $w \in C_1$. But $w \notin C_2$ since $\lim_{t\to\infty}(w(\alpha t)/w(t)) = 1$, which implies that $\inf_{t>0}(w(\alpha t)/w(t)) = 1$ for all $\alpha \in (0, 1)$.

(b) $\phi'(t) \in C_2 \setminus C_1$ for $\phi(t) = \sqrt{\log(1+t)}$.

PROOF. Clearly, $\lim_{t\to\infty}(\sqrt{\log(1+\alpha t)}/\sqrt{\log(1+t)}) = 1$, and so $\sup_{t>0}(\phi(\alpha t)/\phi(t)) = 1$ for all $\alpha \in (0, 1)$. Hence $\phi \notin C_1$.

Since $\lim_{t\to\infty}(w(\alpha t)/w(t)) = 1/\alpha > 1$, and $\lim_{t\to 0}(w(\alpha t)/w(t)) = 1/\sqrt{\alpha} > 1$, we have $\inf_{t>0}(w(\alpha t)/w(t)) > 1$. Thus $\phi \in C_2$.

LEMMA 2.9. *Let $w \in C_1 \bigcap C_2$ and $\varepsilon > 0$. Suppose that $A \subset [0, \infty)$, that $|A| = u > 0$, and that $\int_0^\infty e(A)(t)\,d\phi(t) > 1 - \varepsilon$. Then $|A \setminus [0, u]| \le \delta_3(\varepsilon)u$ for some $\delta_3(\varepsilon) > 0$, and hence $\|e(A) - e\big((0, u)\big)\| \le \delta_4(\varepsilon)$ for some $\delta_4(\varepsilon) > 0$. Moreover, $\delta_3(\varepsilon), \delta_4(\varepsilon) \to 0$ as $\varepsilon \to 0$ (through positive values).*

PROOF. Suppose that $|A \setminus [0, u]| = \alpha u$. Then

$$\begin{aligned}
\varepsilon &> 1 - \int_0^\infty e(A)(t)\,d\phi(t) \\
&= \phi(u)^{-1}\int_0^u d\phi(t) - \phi(u)^{-1}\int_A d\phi(t) \\
&\ge \phi(u)^{-1}\left(\int_{u(1-\alpha)}^u d\phi(t) - \int_u^{u(1+\alpha)} d\phi(t)\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \phi(u)^{-1}\left(\int_{u(1-\alpha)}^{u} w(t)\,dt - \int_{u}^{u(1+\alpha)} w(t)\,dt\right)\\
&= \phi(u)^{-1}\left(\int_{u(1-\alpha)}^{u} w(t)\,dt - \int_{u(1-\alpha)}^{u} w(t+\alpha u)\,dt\right)\\
&\geq \phi(u)^{-1}\int_{u(1-\alpha)}^{u} \big(w(t) - w((1+\alpha)t)\big)\,dt\\
&\geq \phi(u)^{-1}\int_{u(1-\alpha)}^{u} \left(w(t) - k_2\big(\frac{1}{1+\alpha}\big)^{-1} w(t)\right)\,dt\\
&= \left(1 - k_2\big(\frac{1}{1+\alpha}\big)^{-1}\right)\phi(u)^{-1}\Big(\phi(u) - \phi\big(u(1-\alpha)\big)\Big)\\
&\geq \left(1 - k_2\big(\frac{1}{1+\alpha}\big)^{-1}\right)\phi(u)^{-1}\Big(\phi(u) - k_1(1-\alpha)\phi(u)\Big)\\
&= \left(1 - k_2\big(\frac{1}{1+\alpha}\big)^{-1}\right)\Big(1 - k_1(1-\alpha)\Big).
\end{aligned}
$$

Hence $\alpha \leq \delta_3(\varepsilon)$, where $\delta_3(\epsilon) \to 0$ as $\varepsilon \to 0$. Therefore

$$
\begin{aligned}
\|e(A) - e\big((0,u)\big)\| &= \int_0^\infty |e(A) - e\big((0,u)\big)|^*(t)\,d\phi(t) \leq \int_0^\infty \frac{I\big((0, 2u\delta_3(\varepsilon)\big)(t)}{\phi(u)}\,d\phi(t)\\
&= \frac{\phi\big(2\delta_3(\varepsilon)u\big)}{\phi(u)} \leq k_1\big(2\delta_3(\varepsilon)\big) = \delta_4(\varepsilon).
\end{aligned}
$$

It is easy to see that $\delta_4(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We can now deduce the main technical ingredient in the proof of Theorem 3.2.

PROPOSITION 2.10. *Suppose that* $w \in C_1 \bigcap C_2$ *and that* $\|f\| = 1$. *Let* $\varepsilon > 0$. *If* $\int_0^\infty f(t)\,d\phi(t) > 1 - \varepsilon^2$, *then* $\|f - f^*\| < \delta_5(\varepsilon)$ *for some* $\delta_5(\varepsilon) > 0$, *where* $\delta_5(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ (*through positive values*).

PROOF. Let f^+ and f^- denote the positive and negative parts of f. We have

$$
\|f^+\| \geq \int_0^\infty f^+(t)\,d\phi(t) \geq \int_0^\infty f(t)\,d\phi(t) > 1 - \varepsilon^2.
$$

By Lemma 2.5, there exists $\delta_6(\varepsilon) > 0$ such that $\|f^-\| < \delta_6(\varepsilon)$, whence $\|f - |f|\| \leq 2\delta_6(\varepsilon)$. We can associate with $|f|$ a Borel probability measure μ and a collection of sets $\big(A(u)\big)_{u>0}$ having the properties described in Proposition 2.1.

Thus

$$\begin{aligned}\varepsilon^2 &> 1 - \int_0^\infty f(t)\,d\phi(t) \geq 1 - \int_0^\infty |f(t)|\,d\phi(t) = \int_0^\infty \big(f^*(t) - |f(t)|\big)\,d\phi(t)\\ &= \int_0^\infty \int_0^\infty \big(e\big((0,u)\big) - e\big(A(u)\big)\big)(t)\,d\mu(u)\,d\phi(t)\\ &= \int_0^\infty \int_0^\infty \big(e\big((0,u)\big) - e\big(A(u)\big)\big)(t)\,d\phi(t)\,d\mu(u).\end{aligned}$$

Hence $\varepsilon^2 > \int_0^\infty g(u)\,d\mu(u)$, where $g(u) = \int_0^\infty [e(0,u) - e\big(A(u)\big)](t)\,d\phi(t)$. Observe that $0 \leq g(u) \leq 2$, and so $\mu\big(\{u : g(u) \geq \varepsilon\}\big) < \varepsilon$ by Chebyshev's inequality. Let $\delta_4(\varepsilon)$ be as in Lemma 2.9. Then

$$\mu\big(\{u : \|e\big((0,u)\big) - e\big(A(u)\big)\| > \delta_4(\varepsilon)\}\big) < \varepsilon.$$

Thus we have

$$\||f| - f^*\| \leq \int_0^\infty \|e\big(A(u)\big) - e\big((0,u)\big)\|\,d\mu(u) \leq 2\varepsilon + \delta_4(\varepsilon).$$

So

$$\|f - f^*\| \leq \|f - |f|\| + \|f^* - |f|\| \leq 2\delta_6(\varepsilon) + 2\varepsilon + \delta_4(\varepsilon) = \delta_5(\varepsilon),$$

and $\delta_5(\varepsilon) \to 0$ as $\varepsilon \to 0$.

3. Main results

The following lemma is taken from [5].

LEMMA 3.1. *Let f be a nonnegative function on $(0,\infty)$. Given $\varepsilon > 0$, there exists a positive surjective isometry T of $L_{w,1}$, which is also a weak-star automorphism, such that $T(f)^* = f^*$ and $\|T(f) - f^*\| < \varepsilon$.*

THEOREM 3.2 (Main Theorem). *The Lorentz space $L_{w,1}(0,\infty)$ has the weak-star uniform Kadec-Klee property if and only if w belongs to both C_1 and C_2.*

PROOF. We first prove the sufficiency. Let $\varepsilon > 0$ be given. Suppose that $\|f_n\| = 1$ for all n, that $\|f_n - f_m\| \geq \varepsilon$, $(m \neq n)$, and that (f_n) converges weak-star to f. We may assume that $\|f\| = 1 - \delta$, and we shall show that $\delta \geq \delta(\varepsilon) > 0$. The quantities δ_1, δ_2, etc. which arise in the proof depend only on δ and all approach zero as δ approaches zero.

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1°. We may assume that $f \geq 0$ and by Lemma 3.1 that $\|f - f^*\| < \delta$.

2°.

$$\int_0^\infty f(t)\,d\phi(t) = \int_0^\infty f^*(t)\,d\phi(t) - \int_0^\infty \big(f^*(t) - f(t)\big)\,d\phi(t)$$
$$\geq \|f\| - \|f - f^*\| \geq 1 - 2\delta.$$

3°. Choose $0 < m, M < \infty$ such that $\int_m^M f(t)\,d\phi(t) \geq 1 - 3\delta$. Recall that $f_n \to f$ weak-star simply means that $\int_0^\infty f_n(t)g(t)\,dt \to \int_0^\infty f(t)g(t)\,dt$, for all g belonging to the predual of $L_{w,1}$. In particular, $\int_0^\infty f_n I\big((m, M)\big)(t)\,d\phi(t) \to \int_0^\infty f I\big((m, M)\big)(t)\,d\phi(t)$. Therefore, by passing to a subsequence, we may assume that $\int_m^M f_n(t)\,d\phi(t) \geq 1 - 4\delta$ for all n.

Since $\|f_n\| = 1$ and since (by Proposition 2.6) property P holds, we have

$$\|f_n I\big((0, m)\big) + f_n I\big((M, \infty)\big)\| \leq \delta_1.$$

Thus

$$\int_0^\infty f_n(t)\,d\phi(t) \geq 1 - 4\delta - \delta_1 \quad \text{for all } n.$$

4°. By Proposition 2.10, we have $\|f_n - f_n^*\| \leq \delta_2$.

5°. By Helly's Selection Theorem we may assume, by passing to a subsequence, that $f_n^* \to g$ pointwise and, in particular, that $f_n^* \to g$ weak-star. Since $f_n^* - f_n \to g - f$ weak-star, we have $\|g - f\| \leq \liminf_{n\to\infty} \|f_n^* - f_n\| \leq \delta_2$. Hence $\|g\| \geq \|f\| - \delta_2 = 1 - \delta_3$.

6°. Select $0 < m_1, M_1 < \infty$ such that $\|gI\big((m_1, M_1)\big)\| \geq 1 - 2\delta_3$. By Egorov's theorem,

$$\|f_n^* I\big((m_1, M_1)\big) - gI\big((m_1, M_1)\big)\| \to 0.$$

So by passing to a subsequence we may assume that

$$\|f_n^* I\big((m_1, M_1)\big) - gI\big((m_1, M_1)\big)\| \leq \delta_3 \quad \text{for all } n.$$

In particular, we get that $\|f_n^* I\big((m_1, M_1)\big)\| \geq 1 - 3\delta_3$.

7°. Since $\|f_n\| = 1$ and $\|g\| \leq 1$ it follows from property P and from step 6° that $\|f_n^* - f_n^* I\big((m_1, M_1)\big)\| \leq \delta_4$ and $\|g - gI\big((m_1, M_1)\big)\| \leq \delta_4$. Consequently,

$$\|f_n^* - g\| \leq \|f_n^* - f_n^* I\big((m_1, M_1)\big)\| + \|f_n^* I\big((m_1, M_1)\big) - gI\big((m_1, M_1)\big)\|$$
$$+ \|gI\big((m_1, M_1)\big) - g\| \leq \delta_4 + \delta_3 + \delta_4 = \delta_5.$$

8°. $\|f_n^* - f_m^*\| \leq \|f_n^* - g\| + \|g - f_m^*\| \leq 2\delta_5$.

9°. Finally, combining steps 4° and 8° and the hypothesis $\|f_n - f_m\| \geq \varepsilon, m \neq n$, we have

$$\varepsilon \leq \|f_n - f_m\| \leq \|f_n - f_n^*\| + \|f_n^* - f_m^*\| + \|f_m^* - f_m\| \leq 2\delta_2 + 2\delta_5.$$

Since $2\delta_2 + 2\delta_5 \to 0$ as $\delta \to 0$, it follows that $\delta \geq \delta(\varepsilon)$ as required. This proves that the conditions are sufficient.

We now prove that the conditions are necessary. First suppose that $w \notin C_1$, that is, that there exists $\alpha \in (0, 1)$ such that $\sup_{t>0}(\phi(\alpha t)/\phi(t)) = 1$. Therefore there is a sequence $\langle t_k \rangle$ such that $(\phi(\alpha t_k)/\phi(t_k)) \to 1$. Consider the Rademacher functions

$$r_n(t) = \operatorname{sign} \sin(2^n \pi t), \quad n = 1, 2, \ldots, t \in [0, 1].$$

Let

$$y_{k,n} = \begin{cases} r_n((t - \alpha t_k)/(t_k - \alpha t_k)), & t \in [\alpha t_k, t_k], \\ 0, & t \notin [\alpha t_k, t_k]. \end{cases}$$

Clearly, $y_{k,n} \underset{n}{\to} 0$ weak-star. Let

$$x_k(t) = \begin{cases} 2, & t \in [0, \alpha t_k], \\ 1, & t \in (\alpha t_k, t_k), \\ 0, & t \in (t_k, \infty). \end{cases}$$

Then

$$(x_k + y_{k,n})^* = \begin{cases} 2, & t \in [0, (\alpha t_k + t_k)/2], \\ 0, & t \in ((\alpha t_k + t_k)/2, \infty). \end{cases}$$

Hence

$$\|x_k\| = \phi(\alpha t_k) + \phi(t_k),$$

and

$$\|x_k + y_{k,n}\| = 2\phi\left(\frac{\alpha t_k + t_k}{2}\right) \equiv a_k,$$

say. Thus,

$$\left\|\frac{x_k + y_{k,n}}{a_k}\right\| = 1, \quad \frac{x_k + y_{k,n}}{a_k} \underset{n}{\to} \frac{x_k}{a_k} \quad \text{weak-star},$$

and

$$\left\|\frac{x_k}{a_k}\right\| = \frac{\phi(\alpha t_k) + \phi(t_k)}{2\phi((\alpha t_k + t_k)/2)},$$

and finally (by concavity of ϕ)

$$\frac{\|y_{k,n} - y_{k,m}\|}{a_k} = \frac{2\phi\big((1-\alpha)t_k/2\big)}{2\phi\big((1+\alpha)t_k/2\big)} > \frac{1-\alpha}{1+\alpha}$$

for $n \neq m$.

Since $\phi(\alpha t_k)/\phi(t_k) \to 1$, and since ϕ is an increasing function, it follows that $\|x_k/a_k\| \to 1$ as $k \to \infty$. Hence, for $\varepsilon = (1-\alpha)/(1+\alpha)$, we can find sequences $\langle x_k + y_{k,n}\rangle_n$ lying on the unit sphere of $L_{w,1}$ such that $(x_k+y_{k,n})/a_k \underset{n}{\to} x_k/a_k$ weak-star for all k, and such that $\|y_{k,n} - y_{k,m}\|/a_k \geq \varepsilon$ $(m \neq n)$. Since $\|x_k/a_k\| \to 1$, there cannot exist $\delta(\varepsilon) > 0$ for which $\|x_k/a_k\| \leq 1 - \delta(\varepsilon)$ for all k. Therefore $L_{w,1}$ does not have the weak-star uniform Kadec-Klee property.

Now suppose that $w \notin C_2$, that is, that there exists $\alpha \in (0,1)$ such that $\inf_{t>0}(w(\alpha t)/w(t)) = 1$. Thus there is a sequence $\langle t_k\rangle$ such that $w(\alpha t_k)/w(t_k) \to 1$. Observe that this implies that

$$\frac{\int_{\alpha t_k}^{t_k} w(t)\,dt - \int_{\alpha t_k}^{(1+\alpha)t_k/2} 2w(t)\,dt}{\int_0^{\alpha t_k} w(t)\,dt} \to 0$$

as $k \to \infty$.

Let x_k, $y_{k,n}$ be defined as above. It follows from the above observation that, for each n, we have

$$\lim_{k\to\infty} \frac{\|x_k\|}{\|x_k + y_{k,n}\|} = \lim_{k\to\infty} \frac{\int_0^{\alpha t_k} 2w(t)\,dt + \int_{\alpha t_k}^{t_k} w(t)\,dt}{\int_0^{\alpha t_k} 2w(t)\,dt + \int_{\alpha t_k}^{(1+\alpha)t_k/2} 2w(t)\,dt} = 1.$$

So $L_{w,1}$ does not have the weak-star uniform Kadec-Klee property, which completes the proof.

Let K be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : K \to K$ is said to be *non-expansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in K, and K is said to have the *fixed point property* if every non-expansive mapping on K has a fixed point. By van Dulst and Sims [7], who utilized Kirk's important concept of *normal structure* [11], we have the following corollary.

COROLLARY 3.3. *If $w \in C_1 \bigcap C_2$ then all weak-star compact convex subsets of $L_{w,1}$ have the fixed point property. In particular, if $w \in C_1 \bigcap C_2$, then the closed unit ball of $L_{w,1}$ has the fixed point property.*

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