TREES IN POLYHEDRAL GRAPHS

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1. Introduction. A graph is said to be *d-polyhedral* provided it is isomorphic with the graph formed by the vertices and edges of a d-dimensional bounded (convex) polyhedron (*d-polyhedron*). A *k-tree* is a connected acyclic graph in which each vertex is of valence $\leqslant k$. The principal result of this paper is the following, which solves a problem of Grünbaum and Motzkin **(7)**:

THEOREM 1. *Every 3-polyhedral graph G can be covered by a 3-tree T; that is, G admits a subgraph T such that T is a 3-tree and every vertex of G is a vertex of T.*

We shall also prove:

THEOREM 2. *For every k, there exists a 5-polyhedral graph that cannot be covered by any k-tree.*

Theorem 1 has a number of interesting consequences, some of which are summarized in the following:

THEOREM 3. *Let G be a 3-polyhedral graph with n vertices. Then:*

(i) *The vertices of G can be covered by $(n + 2)/3$ or fewer disjoint simple paths.*

(ii) *There is a simple path in G with at least $(2 \log_2 n) - 5$ vertices.*

(iii) *There is a circuit in G with at least $2\sqrt{((2 \log_2 n) - 5)}$ vertices.*

(iv) *Between any two vertices of G, there is a path with at least $\sqrt{((2 \log_2 n) - 5)}$ vertices.*

2. Proof of Theorem 1. We first observe that a graph is 3-polyhedral if and only if it is planar and 3-connected **(10; 8; 6)**. The theorem will be proved for a more general class of graphs called *circuit graphs*, which will now be defined.

Let G be a 3-polyhedral graph that is embedded in the plane Π. Let J be a simple circuit of G, and let $G(J)$ denote the graph consisting of J together with all vertices and edges of G that are interior to the region of the plane bounded by J. Then $G(J)$ is called a *circuit graph* and the edges of $G(J)$ that are not edges of J are called *interior edges*. A simple circuit of $G(J)$ bounding a connected component of $\Pi \sim G(J)$ is called a *face* of $G(J)$.

A face of $G(J)$ is said to *separate* $G(J)$ if there are vertices x and y of F that separate $G(J)$ into two components each of which contains an interior edge.

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We now proceed to prove the theorem for circuit graphs. The proof is by induction on the number of interior edges. The case in which there are no interior edges is obvious. Suppose the theorem is true for all circuit graphs with fewer than n interior edges, and consider a circuit graph $G(J)$ having n interior edges. There are two cases to be treated.

Case I. There is a vertex x of J such that x is incident to an interior edge and $G(J)$ is not separated by any face of $G(J)$ incident to x.

Let the edges of $G(J)$ incident to x be $\alpha_0, \ldots, \alpha_m$, in cyclic order, with α_0 and α_m edges of J. Let α_i have end points x and x_i. Let F_i be the face common to α_i and α_{i+1} and let Γ_i be the path from x_i to x_{i+1} along F_i which misses x. Let y be the first vertex of J encountered upon traversing Γ_{m-1} from x_{m-1} to x_m and let Γ' be the portion of Γ_{m-1} that connects x_{m-1} and y. Let P be the path $\alpha_1 \cup \Gamma_i \cup \ldots \cup \Gamma_{m-2} \cup \Gamma'$, and let P′ be the path along J from x to y that misses α_m. (See Figure 1.)

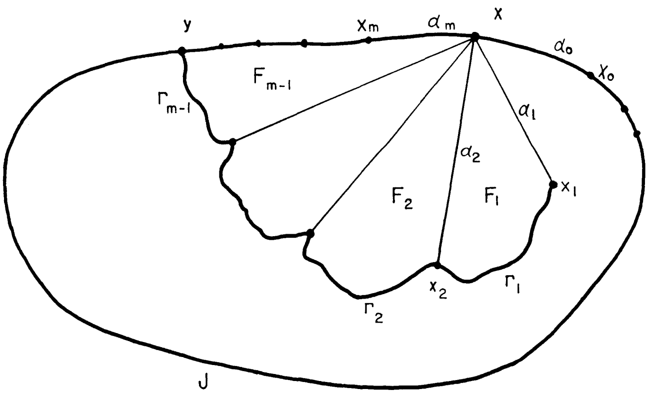

FIGURE 1.

We now show that $J^* = P \cup P'$ is a simple circuit. Since Γ_i is a simple path, we see that if P intersects itself, then there are paths Γ_i and Γ_j that intersect at some vertex z. We may assume that it is not the case that $\Gamma_i = \Gamma_{j+1}$ and z is the common end point. The faces F_i and F_j meet at both x and z. Let v be a vertex between the first and second occurrence of z on P. Then v is a vertex of some face that is incident to x; thus two appropriate paths from x to z along F_i and F_j can be chosen so that v is interior to the circuit formed by these paths. But this implies that v can be separated from all vertices outside this circuit by removing x and z, which contradicts the 3-connectedness of G. Thus P has no self-intersections and it is clear that P intersects P' only at their common end points, since no face incident to x separates $G(J)$. It then follows that J^* is a simple circuit.

The circuit graph $G(J^*)$ has fewer interior edges than $G(J)$. By the inductive hypothesis, it can therefore be covered by a 3-tree T^*. The only vertices of $G(J)$ not in $G(J^*)$ are those on the path from x to y along J that misses x_0.

Since the valence of x in T is at most two, we can construct a 3-tree T that covers $G(J)$ by adding a portion of this path to T^*.

Case II. Each vertex of J that is incident to an interior edge is also incident to a face that separates $G(J)$.

For each face F that separates $G(J)$, there is a path $\Gamma(F)$ that is a subgraph of J and whose only vertices in common with F are its end points. Let A be the set of all $\Gamma(F)$ such that F separates $G(J)$. For $\Gamma(F_1)$, $\Gamma(F_2)$ in A, write $\Gamma(F_1) \leqslant \Gamma(F_2)$ provided $\Gamma(F_1)$ is a subgraph of $\Gamma(F_2)$. Let $\Gamma(F)$ be a minimal element of A, and let its end points be x and y.

We now show that (x, y) is an edge of $G(J)$. Let P_1 and P_2 be the two paths along F from x to y. Assume that P_1 is interior to the circuit $P_2 \cup \Gamma(F)$. If there is a vertex on P_1 other than x or y, then this vertex can be separated from J by removing x and y, for no internal vertex of $\Gamma(F)$ can be incident to an interior edge of $G(J)$. This contradicts the 3-connectedness of G. Thus P_1 is actually the edge (x, y).

Let H be the graph obtained from $G(J)$ by removing the edge (x, y). We shall show that H is a circuit graph.

Let K be the graph consisting of H together with a vertex w outside J and the edges (w, z) for each vertex z in J. Clearly K is planar; so to show that H is a circuit graph it suffices to show that K is 3-connected and therefore 3-polyhedral.

Suppose that K can be separated by removing two vertices. First we note that no subgraph of $G(J)$ can be separated from J by removing two vertices, for this would contradict the 3-connectedness of G. The vertex w cannot be separated from J by removing two vertices, because J has at least three vertices. Thus no subgraph of K can be separated from J, and so each component must contain at least one vertex of J. Apparently the two vertices that separate K into these components are vertices of J since it requires two vertices to separate J. But these components are connected by edges incident to w.

By the induction hypothesis H can be covered by a 3-tree, and this same tree will also cover $G(J)$. Observing that a 3-polyhedral graph is a circuit graph, we have proved Theorem 1.

3. Proof of Theorem 2.

LEMMA. *A k-tree T with n vertices can be covered by $n(k - 1)/k$ or fewer disjoint simple paths.*

Proof. The proof is by induction on the number of branch points (vertices of valence $\geqslant 3$) of T. The case where T has only one branch point is easily verified.

Suppose that T has m branch points and that the lemma is true for every k-tree with fewer than m branch points. Let x be an arbitrary vertex of T. For any vertex y of T, let $d(x, y)$ be the number of vertices on the path from

x to y. Choose a branch point y so that $d(x, y)$ is maximal. Let T' be the largest subtree of T that contains every branch point other than y. The remaining vertices belong to a tree T'' which can be covered by $l - 1$ disjoint simple paths, where l is the valence of y. If T has n vertices and T' has n_1 vertices, then T can be covered by

$$(n_1 k - n_1 + kl - k)/k$$

or fewer disjoint simple paths. But

$$n(n-1)/k \geqslant (n_1 k - n_1 + kl - l)/k \geqslant (n_1 k - n_1 + kl - k)/k.$$

To prove Theorem 2, we shall construct a 5-polytope with n vertices that cannot be covered by $n(k-1)/k$ or fewer disjoint simple paths. The construction involves the use of cyclic polytopes, and of the following theorem of Gale **(5)**:

If a cyclic d-polytope P has v vertices, then the number of $(d-1)$-dimensional faces of P is equal to

$$2\binom{v-u}{u-1} \qquad \textit{when } d = 2u - 1.$$

Let P be a cyclic 5-polytope with v vertices, and let P' be the polytope obtained by adding pyramidal caps to each 4-dimensional face of P. We call the vertices added in this way *new vertices* and the other vertices of P' *old vertices*.

Between any two new vertices on a simple path on P', there is an old vertex. Thus each simple path has at most one more new vertex than old. If n is the number of new vertices, then at least $n - v$ disjoint simple paths are required to cover the vertices of P'. Then $m = n + v$ and $n = v^2 - 7v + 12$; hence at least $m - 2\sqrt{(m-3)} - 6$ disjoint simple paths are required to cover P. To complete the proof, we simply choose v large enough so that

$$m - 2\sqrt{(m-3)} - 6 < m(k-1)/k.$$

4. Consequences of Theorem 1. The *path number*, $m(G)$, of a graph G is the number of simple disjoint paths necessary to cover the vertices of G. Let

$$m(n) = \max\{m(G)\colon G \text{ is 3-polyhedral and } G \text{ has } n \text{ vertices}\}.$$

Brown **(1)** has shown that $m(n) \geqslant (n-10)/3$. We shall show in Theorem 3 that $m(n) \leqslant (n+2)/3$.

The *path length* $p(G)$ of a graph G is the maximum number of vertices contained in a simple path in G. Let

$$p(n) = \min\{p(G)\colon G \text{ is 3-polyhedral and } G \text{ has } n \text{ vertices}\}.$$

Moon and Moser **(9)** have shown that

$$(\log_2 n/\log_2 \log_2 n) - 1 \leqslant p(n) \leqslant 9((n-2)/2)^{\log 2/\log 3} - 1.$$

We shall show that $(2 \log_2 n) - 5 \leqslant p(n)$.

Proof of Theorem 3. An argument similar to that of the lemma for Theorem 2 will show that a 3-tree can be covered by $(n + 2)/3$ or fewer disjoint simple paths, and this implies (i).

Let T be a 3-tree with n vertices that covers G. Let P be a longest path in T, let its end points be y and z, and let l be the number of vertices in P. Choose a vertex x of P such that $\min(d(x, y), d(x, z))$ is maximal, where $d(x, y)$ is the number of edges on the path from x to y.

The number of vertices of T of distance one from x is $\leqslant 3$. The number of distance two is $\leqslant 3 \times 2$. The number of distance i is $\leqslant 3 \times 2^{i-1}$. Thus

$$n \leqslant 1 + 3 \sum_{i=1}^{[(l+1)/2]} 2^{i-1}$$

where [ ] denotes the least integer function. This shows that

$$(2 \log_2 n) - 5 \leqslant l.$$

To prove (iii), we use the following theorem by Dirac **(3)**:

If G is a 2-connected graph with a simple path of length l, then G has a simple circuit of length at least $2\sqrt{l}$.

To prove (iv), we use another theorem by Dirac **(4)**:

If A and B are two disjoint sets of vertices in a 2-connected graph, then there are two independent simple paths from A to B.

Let C be a simple circuit of G with at least $2\sqrt{((2 \log n) - 5)}$ vertices. Let A be the set of vertices of C, let x and y be two arbitrary vertices of G, and let B be the set of vertices in $\{x, y\}$ that are not vertices of C. By using Dirac's theorem, it is now easy to construct a simple path connecting x and y that has at least half as many vertices as C.

Note that if *G is a 3-polyhedral graph in which each vertex is of valence* 3, *the bounds in* (ii), (iii), *and* (iv) *can be replaced by* $(2 \log_2 n) - 6$, $(3 \log_2 n) - 10$, *and* $(\log_2 n) - 3$, *respectively.*

Proof. Since G is 3-connected, there are independent paths P_1, P_2, and P_3 connecting any two vertices x and y. An argument similar to that of **(3**, Theorem 5**)** shows that each P_i must have at least $(\log_2 n) - 3$ vertices. Thus any two of these paths form the required circuit for (ii). If we remove an edge of P_1 incident to x and an edge of P_2 incident to y, then the remaining paths together with P_3 form the required path for (iii). The proof of (iv) is similar to the proof where G is not necessarily simple.

5. Remarks.

1. It should be possible to find smaller bounds for the path number of 3-valent 3-polyhedral graphs. Some results have been obtained by Brown **(2)**, but so far no good upper bounds have been found.

2. Theorem 1 does not hold for unbounded polyhedra. For each k, one can construct a 3-polyhedron P whose graph is a k-tree. (By the graph of P, we mean the graph consisting of vertices and bounded edges of P.)

3. What is the largest n such that every 3-valent 3-polyhedral graph can be covered by a disjoint collection of simple paths of length $\geqslant n$?

4. How does the number of disjoint simple circuits necessary to cover a 3-polyhedral graph depend on the number of vertices of the graph?

5. The following is an unpublished result of Grünbaum:

For every k there is a 4-polyhedral graph that cannot be covered by a k-tree.

Proof. Let $G(v)$ be the polytope obtained by adding pyramidal caps to each face of a cyclic 4-polytope with v vertices. In $G(v)$ there are $v(v-3)/2$ new vertices. Thus if T is a k-tree that covers $G(v)$, then T has at least $v(v-3)/2$ edges incident to the new vertices. But the other end points of these edges are old vertices. Therefore we must have $kv \geqslant v(v-3)/2$, and $v \leqslant 2k+3$. Taking v greater than this yields a $G(v)$ not coverable by any k-tree.

References

1. T. A. Brown, *Simple paths on convex polyhedra*, Pacific J. Math., *11* (1961), 1211–1214.

2. ——— *Hamiltonian paths on convex polyhedra* (unpublished note P-2069, The Rand Corporation, Santa Monica, Calif., 1960).

3. G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc., (3), *2* (1952), 69–81.

4. ——— *Connectivity theorems for graphs*, Quart. J. Math. Oxford Ser. (2), *3* (1952), 171–174.

5. D. Gale, *Neighborly and cyclic polytopes*, Proc. Symp. Pure Math., *7*, Convexity (Providence, R.I., 1963), 225–232.

6. B. Grünbaum, *Steinitz's characterization of convex polyhedra in E^3* (hectographed notes, University of Washington, 1963).

7. B. Grünbaum and T. S. Motzkin, *Longest simple paths in polyhedral graphs*, J. London Math. Soc., *37* (1962), 152–160.

8. ——— *On polyhedral graphs*, Proc. Symp. Pure Math., *7*, Convexity (Providence, R.I., 1963), 285–290.

9. J. W. Moon and L. Moser, *Simple paths on polyhedra*, Pacific J. Math., *13* (1963), 629–631.

10. E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, (Berlin, 1934).

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