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COINCIDENCE OF NODES FOR GENERALIZED CONVEX FUNCTIONS

BY
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In a recent paper [1] I. B. Lazarevic announced an extension of results of L. Tornheim [2; Theorems 2 & 3] concerning points of contact between two distinct members of an n-parameter family and between a member of an n-parameter family and a corresponding convex function. In the proofs of these extensions [1; Theorems 3.1 & 3.2] use is made of Tornheim's Convergence Theorem [2; Theorem 5]; however this theorem is not correctly applied in [1] since it requires distinct limiting nodes, and that hypothesis necessarily fails in the approach used in [1]. In this note proofs of results more general than those in [1] are given independent of convergence theorems. Throughout this note $F\subset C^r(I)$ for $r\geq 0$ and I is an interval of the reals. Let $\lambda(n)=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ where $\lambda_1,\ldots,\lambda_k$ and k are positive integers satisfying $\lambda_1+\lambda_2+\cdots+\lambda_k=n$. F is said to be a $\lambda(n)$-*parameter family* on I in case for every choice of k points $x_1<x_2<\cdots<x_k$ in I and every set $\{y_i^j\}$ of n real numbers there is a unique $f\in F$ satisfying $f^{(j)}(x_i)=y_i^j$, $j=0,1,\ldots,\lambda_i-1$, $i=1,2,\ldots,k$. A function g is said to be $\lambda(n)$-*convex* with respect to F on I in case for every choice of k points $x_1<x_2<\cdots<x_k$ from I and every f in F satisfying $f^{(j)}(x_i)=g^{(j)}(x_i)$ for $j=0,1,\ldots,\lambda_i-1$, $i=1,2,\ldots,k$, we have

$$(1)\qquad (-1)^{M(i)}(g(x)-f(x))\geq 0 \quad \text{when} \quad x_{i-1}<x<x_i$$

for $i=2,3,\ldots,k$ where $M(i)=n+\lambda_1+\cdots+\lambda_{i-1}$.

In the case that $\lambda_1=\lambda_2=\cdots=\lambda_k=1$ we call a $\lambda(n)$-parameter family an n-parameter family and a $\lambda(n)$-convex function is called an n-convex function. Replacing $\geq$ by $\leq$ in (1) replaces convex by concave in the definition.

Definition. Functions f and g defined on an interval I are said to graze (or have *a point of contact*) at an interior point z of I if $f(z)=g(z)$ and there is a $d>0$ so that $f(x)-g(x)$ is of constant sign for $0<|x-z|<d$.

Theorem 1. *Let g be convex and h be concave with respect to the n-parameter family F on an interval I of the real numbers. If g and h graze at k points and $g-h$ changes sign at m points in I, then $2k+m\leq n$ unless g and h are identical on a subinterval of I. Moreover, if $h, g\in F$, then $2k+m<n$.*

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The proof of this theorem consists of showing it first for g and h both in F, then for only one in F, and finally for neither in F. Because of the similarity of these arguments, only the first case will be considered here.

We remark that in the case where $2k+m=n$ we may conclude that $g-h>0$ to the right of the last zero of $g-h$ in I and $(-1)^n(g-h)>0$ to the left of the first zero of $g-h$ in I. Hence the sign of $g-h$ between its zeros is dependent only on the relative positions of these zeros and whether or not they are points of contact for g and h.

Proof. The case $k=1$ is Theorem 3 in [2], and this same theorem resolves the cases $n=2$ and $n=3$. We assume the theorem is true for $n-1$ in place of n, and show that it is then true for n. Let f_1 and f_2 graze at points $z_1<z_2<\cdots<z_k$ in I, and let f_1-f_2 change sign at $x_1<x_2<\cdots<x_m$ in I. Suppose that $m+2k>n$. First observe that $z_1<x_1$. For if not, we could consider the $n-1$-parameter family G consisting of all $f\in F$ with $f(x_1)=f_1(x_1)$ restricted to the interval $I\cap(x_1,\infty)$. Then f_1 and f_2 are in G and graze at k points, while f_1-f_2 changes sign at $m-1$ points. Hence $2k+m-1\geq n-1$ contracting our induction assumption. Similarly for $z_k>x_m$ and for $m+2k>n$. Thus we can and do assume that $m+2k=n$.

Let f_1-f_2 have zeros $z_1=y_1<y_2<\cdots<y_j=z_k$ where $j=k+m$. Pick $u<y_1$ and $v>y_j$. Pick $f\in F$ so that $f(y_i)=f_2(y_i)$ for $i=1,2,\ldots,j$, $f(u)=f_1(u)$, and $f(v)=f_2(v)$. In addition if f_1 and f_2 graze at y_i for $1<i<j$, pick u_i between y_i and y_{i+1} and let $f(u_i)=f_2(u_i)$. Then f is specified at $j+2+k-2=m+2k=n$ points and so is uniquely determined. Also $f-f_2$ has $n-1$ zeros, and so it changes sign at each of these zeros. We shall without loss of generality assume that $f_2(v)<f_1(v)$. There are two cases to consider:

CASE 1. $f(x)<f_2(x)$ for $z_k<x<v$.

CASE 2. $f(x)>f_2(x)$ for $z_k<x<v$. We consider Case 1 first. $f(x)-f_2(x)$ and $f_1(x)-f_2(x)$ have opposite signs for $z_k<x<v$. $f_1(x)-f_2(x)$ changes sign m times for $u<x<v$, and $f(x)-f_2(x)$ changes signs $m+2(k-2)$ times for $u<x<v$. So $f_1(x)-f_2(x)$ and $f(x)-f_2(x)$ have opposite signs for $u<x<z_1$. This contradicts $f_1(u)=f(u)$. Next for Case 2 $f(x)-f_2(x)$ and $f_1(x)-f_2(x)$ have the same sign for $z_k<x<v$. Either $f(x)-f_1(x)$ has a zero for $z_k<x<v$ or else f and f_1 graze at z_k. $f-f_2$ changes sign at z_k and f_1-f_2 does not. Thus $f-f_2$ and f_1-f_2 have opposite signs in a small interval with right endpoint z_k. Next pick the largest $i<j$ for which f_1 and f_2 graze at y_i. If $i=j-1$, $f-f_2$ changes sign at u_i implies that $f(x)-f_1(x)$ has a zero for $y_i<x<u_i$ or else f and f_1 graze at y_i. Also y_i is the right endpoint of an interval on which $f-f_2$ and f_1-f_2 have the same sign. If $i<j-1$, $f-f_2$ and f_1-f_2 both change sign at y_q for $i<q<j$, and hence $f-f_2$ and f_1-f_2 have the same sign in an interval with right endpoint y_q. The above argument can be applied to show that for each y_i for $i>1$ at which f_1 and f_2

graze either f and f_1 graze at y_i or else $f(x)-f_1(x)$ has a zero for $y_i<x<u_i$. Thus twice the number of points of contact of f and f_1 plus the number of zeros of $f-f_1$ (not counting u) which are not points of contact is at least as large as $m+2k-1=n-1$. $f-f_1$ has a zero at u, and this is impossible as pointed out previously when we observed that $z_1<x_1$. Hence the theorem is established.

In the case of n-convex functions, if (1) holds for some fixed i, $1\le i\le k+1$, then it holds for every i in that range. In the case of coincidence of nodes, i.e., $\lambda_i>1$ for at least one i, a similar result holds if

$$\lambda_i \le 2 \quad \text{for all} \quad i=1,2,\ldots,k. \tag{6}$$

By $[\lambda(n)]$ we shall mean $\lambda(n)$ together with the set of all ordered partitions $\mu(n)$ obtained from $\lambda(n)$ by replacing a 2 in $\lambda(n)$ by two 1's. See [3, page 39].

THEOREM 2. *Let F be a $\mu(n)$-parameter family for all $\mu(n)$ in $[\lambda(n)]$. Assume* (6). *If* (1) *holds for a fixed i between* 1 *and* $k+1$ *inclusive, then it holds for every i in this range. Moreover there is no distinction between $\lambda(n)$- and $\lambda(n)^*$-convexity in this case. See* [3, *page* 37].

This theorem is an immediate consequence of the lemma that follows.

LEMMA. *Let F be a* $(1,1)$*-parameter and a* (2)*-parameter family on the open interval I. Let g be a differentiable real valued function defined on I having the property that if $g(x_0)=f(x_0)$ and $g'(x_0)=f'(x_0)$ for some f in F and some x_0 in I, then*

$$g(x)\ge f(x) \tag{1}$$

whenever x is in I and

$$x\ge x_0. \tag{2}$$

Then g is convex with respect to F on I and, for f as above, (1) *holds for all x in I.*

Proof. Suppose g is not convex. Then there are points $x_1<x_2$ in I and a function f in F so that $f(x_1)=g(x_1)$, $f(x_2)=g(x_2)$, and $f(x)<g(x)$ for $x_1<x<x_2$. Consider the cases (i) $f'(x_1)<g(x_1)$, and (ii) $f'(x_1)=g'(x_1)$. In case (i) pick $h\in F$ so that $h(x_1)=g(x_1)$ and $h'(x_1)=g'(x_1)$. Then $h(x)\le g(x)$ for $x\ge x_1$, and since $h(x)>f(x)$ for x near and $>x_1$, f and h must intersect in (x_1,x_2). This contradiction shows that the case (i) is impossible. In case (ii) pick a point u between x_1 and x_2. We get an immediate contradiction by considering $h\in F$ satisfying $h(x_1)=g(x_1)$, $h(u)=g(u)>f(u)$. This shows that g is convex. Suppose that $f(u)>g(u)$ for some point u of I with $u<x_0$. Then $f(x)>g(x)$ for all $x<x_0$. The function h in F satisfying $h(x_0)=g(x_0)$ and $h(u)=g(u)$ must satisfy $h'(x_0)=f'(x_0)$ which is not possible.

Clearly if (2) is replaced by

$$x\le x_0 \tag{2$'$}$$

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then the same result follows. Also if (1) is replaced by $g(x) \leq f(x)$, then g is concave with repsect to F whether or not (2) is replaced by (2)′.

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