

## ON CERTAIN APPLICATIONS OF THE KHUKHRO–MAKARENKO THEOREM

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**Abstract.** Some recent results of Khukhro and Makarenko on the existence of characteristic  $\mathfrak{X}$ -subgroups of finite index in a group  $G$ , for certain varieties  $\mathfrak{X}$ , are used to obtain generalisations of some well-known results in the literature pertaining to groups  $G$ , in which all proper subgroups satisfy some condition or other related to the property ‘soluble-by-finite’. In addition, a partial generalisation is obtained for the aforementioned results on the existence of characteristic subgroups.

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**1. Introduction.** Let  $F$  be a free group of countable rank with basis  $\{x_1, x_2, \dots\}$ . Then an outer commutator word of weight 1 is  $x_1$ , and an outer commutator word  $\omega$  of weight  $t > 1$  is a word of the form

$$\omega(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)],$$

where  $u, v$  are outer commutator words of weight  $r, t - r$  respectively. Let  $\omega$  be an outer commutator word of weight  $t$ . We denote by  $\mathfrak{X}_\omega$  the class of groups  $G$  satisfying  $\omega(g_1, \dots, g_t) = 1$  for all  $g_1, \dots, g_t \in G$ , i.e.  $\omega(G) = 1$ .

Some recent results of Khukhro and Makarenko (see especially Lemma 2.1) establish that, for certain group-theoretic properties  $\mathfrak{Y}$ , the existence of an  $\mathfrak{Y}$ -subgroup  $H$  of finite index in a group  $G$  ensures that there is a *characteristic*  $\mathfrak{Y}$ -subgroup  $C$  of finite index in  $G$ . In the present paper we shall use these results to obtain generalisations of some well-known results on groups  $G$ , in which all proper subgroups satisfy certain conditions, in several cases the condition in question being either ‘almost in the variety  $\mathfrak{X}_\omega$ ’ for some outer commutator word  $\omega$  (see for example Theorem 2.4) or ‘ $\mathfrak{X}_\omega$ -by-Chernikov’ (see for example Theorem 2.5). We shall also obtain a generalisation of a result on barely transitive  $p$ -groups (see Theorem 2.3). Recall that a group

of permutations  $G$  of an infinite set  $\Omega$  is called a *barely transitive* group if  $G$  acts transitively on  $\Omega$  and every orbit of every proper subgroup is finite. Equivalently,  $G$  is barely transitive if  $G$  has a subgroup  $H$  such that  $|G : H|$  is infinite,  $\bigcap_{g \in G} H^g = 1$  and  $|K : K \cap H|$  is finite for every proper subgroup  $K$  of  $G$ , where the subgroup  $H$  is called a *point stabiliser*. Finally, in Section 4 of the paper, we obtain some partial generalisations of the Khukhro–Makarenko results.

We shall use the following notation for the given classes of groups.

- $\mathfrak{A}$ : Abelian groups,
- $\mathfrak{N}$ : Nilpotent groups,
- $\mathfrak{S}$ : Soluble groups,
- $\mathfrak{S}_d$ : Soluble groups of derived length at most  $d$ ,
- $\mathfrak{C}$ : Chernikov groups,
- $\mathfrak{R}$ : Groups of finite (Prüfer) rank,
- $\mathfrak{F}$ : Finite groups,
- $\mathfrak{D}$ : Divisible (radicable) groups,
- $\mathfrak{T}$ : Periodic groups,
- $\mathfrak{L}$ :  $(\mathfrak{T} \cap \mathfrak{D} \cap \mathfrak{A})$ -groups.

We also denote the class of all  $\mathfrak{X}$ -by- $\mathfrak{Y}$ -groups by  $\mathfrak{X}\mathfrak{Y}$ , and  $\mathfrak{X}\mathfrak{X}$ -groups by  $\mathfrak{X}^2$ .

**2.  $\mathfrak{X}_\omega\mathfrak{C}$ -groups.** We will use the following very useful result, referred to here as the Khukhro–Makarenko theorem.

LEMMA 2.1 ([9, Theorem 1], [11, Theorem 1] or [13]). *If a group  $G$  has a subgroup  $H$  of finite index  $n$  satisfying the identity  $\chi(H) = 1$ , where  $\chi$  is an outer commutator word of weight  $w$ , then  $G$  has also a characteristic subgroup  $C$  of finite  $(n, w)$ -bounded index satisfying the same identity  $\chi(C) = 1$ .*

Before we give an application of Lemma 2.1, we prove the following lemma.

LEMMA 2.2. *Let  $G$  be a group and let  $\omega$  be an outer commutator word of weight  $t \geq 2$ ; then  $G^{(t-1)} \leq \omega(G)$ . In particular,*

- (i) *if  $\omega(G) = 1$ , then  $G$  is in  $\mathfrak{S}_{t-1}$ , i.e.  $\mathfrak{X}_\omega \leq \mathfrak{S}_{t-1}$ ,*
- (ii) *if  $G$  is a perfect group, then  $\omega(G) = G$ .*

*Proof.* We proceed by induction on  $t$ . If  $t = 2$ , then  $G^{(t-1)} = G^{(1)} = G' = \omega(G)$ . Now assume that  $t \geq 3$ ; then there exist outer commutator words  $\sigma, \tau$  of weight  $1 \leq t_1, t_2 < t$ , respectively, such that  $t = t_1 + t_2$  and  $\omega = [\sigma, \tau]$ , and then  $\omega(G) = [\sigma(G), \tau(G)]$ . By induction hypothesis, we have  $G^{(t_1-1)} \leq \sigma(G)$  and  $G^{(t_2-1)} \leq \tau(G)$ . Put  $m = \max\{t_1, t_2\}$ , then

$$G^{(m)} = [G^{(m-1)}, G^{(m-1)}] \leq [G^{(t_1-1)}, G^{(t_2-1)}] \leq [\sigma(G), \tau(G)] = \omega(G).$$

Clearly  $t_1 + t_2 \geq m + 1$  and thus  $t - 1 \geq m$ . So  $G^{(t-1)} \leq G^{(m)} \leq \omega(G)$  and the induction is complete.

- (i) If  $\omega(G) = 1$ , then  $G^{(t-1)} = 1$ . So  $G$  is in  $\mathfrak{S}_{t-1}$ .
- (ii) Assume that  $G$  is a perfect group. Since  $G^{(t-1)} \leq \omega(G)$ , we have  $G^{(t-1)} = G$ , and hence  $G = \omega(G)$ , as desired. □

As an application of the Khukhro–Makarenko theorem we present the following result.

**THEOREM 2.3.** *Let  $G$  be a locally finite barely transitive  $p$ -group with a point stabiliser  $H$  and let  $\omega$  be an outer commutator word of weight  $t$ . If  $H \in \mathfrak{X}_\omega$ , then  $G' \neq G$  and  $G' \in \mathfrak{X}_\omega$ .*

*Proof.* Let  $N$  be a proper normal subgroup of  $G$ ; then  $N \cap H \in \mathfrak{X}_\omega$ . Since  $|N : N \cap H|$  is finite, by Lemma 2.1,  $N$  has a characteristic subgroup  $K \in \mathfrak{X}_\omega$  such that  $N/K \in \mathfrak{F}$ . It is well known that  $G$  has no proper subgroup of finite index, so  $N/K \leq Z(G/K)$ . It follows that  $N' \leq K$  and that  $N' \in \mathfrak{X}_\omega$ . Since there exists a chain  $\{N_i : i \in I\}$  of proper normal subgroups of  $G$  such that  $G = \bigcup_{i \in I} N_i$ , it follows that

$$G' = \bigcup_{i \in I} N'_i.$$

Consequently, we have  $G' \in \mathfrak{X}_\omega$  and  $G \neq G'$  by Lemma 2.2(ii). □

Theorem 2.3 generalises [1] and [2, Theorem 2], and by using Lemma 2.2(i) we can obtain the same results as those in [1] and [2, Theorem 2]. The structure of imperfect locally finite barely transitive groups is described in [7].

Let  $v(x_1, \dots, x_s)$  and  $u(x_1, \dots, x_t)$  be two words. Then the composite of  $v$  and  $u$ ,  $v \circ u$  is defined as follows:

$$v \circ u = v(u(x_1, \dots, x_t), \dots, u(x_{(s-1)t+1}, \dots, x_{st})).$$

If  $v$  is an outer commutator word and  $u$  is a word, then it is well known that  $v \circ u(G) = v(u(G))$  for any group  $G$  (see for example [16, Lemma 2.5]).

We will use this definition to describe the structure of certain groups.

Let  $\mathfrak{V}$  be a class of groups. Recall that a group  $G$  is called a minimal non- $\mathfrak{V}$ -group if every proper subgroup of  $G$  is a  $\mathfrak{V}$ -group, but  $G$  itself is not. The minimal non- $\mathfrak{V}$ -groups are denoted by  $MN\mathfrak{V}$ .

Now define the word  $\theta$  as  $\theta(x, y) = [x, y]$ , which will be used in the sequel.

**THEOREM 2.4.** *Let  $G$  be an  $MN\mathfrak{X}_\omega\mathfrak{F}$ -group, where  $\omega$  is an outer commutator word of weight  $t > 1$ . If  $G$  has no infinite simple images, then the following properties hold.*

- (i)  $G$  has no proper subgroup of finite index and no simple images.
- (ii)  $N' \in \mathfrak{X}_\omega$  for every proper normal subgroup  $N$  of  $G$ .
- (iii)  $G$  is not perfect,  $G \in \mathfrak{X}_\omega(\mathfrak{L} \cap \mathfrak{C})$  and  $G' \in \mathfrak{X}_\omega$ . In particular,  $G \in \mathfrak{S}_t$ .
- (iv)  $(\omega \circ \theta)(G) = 1$ , i.e.  $G \in \mathfrak{X}_{\omega \circ \theta}$ .

*Proof.* We first assume that  $G$  has a proper subgroup  $K$  of finite index. Since  $K \in \mathfrak{X}_\omega\mathfrak{F}$ ,  $K$  has a normal subgroup  $L \in \mathfrak{X}_\omega$  such that  $K/L \in \mathfrak{F}$ . Hence,  $\text{core}_G L \in \mathfrak{X}_\omega$  and has finite index in  $G$ , i.e.  $G \in \mathfrak{X}_\omega\mathfrak{F}$ . But this is a contradiction. So  $G$  has no proper subgroup of finite index and it has no simple images. Thus (i) holds.

Now let  $N$  be a proper normal subgroup of  $G$ . Since  $N \in \mathfrak{X}_\omega\mathfrak{F}$ ,  $N$  has a characteristic subgroup  $S \in \mathfrak{X}_\omega$  of finite index in  $N$  by Lemma 2.1. Put  $\bar{G} := G/S$ , then  $\bar{G} = C_{\bar{G}}(\bar{N})$ , since  $G/S$  has no proper subgroup of finite index and so we have  $[G, N] \leq S$ . Since  $\mathfrak{X}_\omega$  is subgroup-closed,  $N' \in \mathfrak{X}_\omega$ , and thus (ii) holds.

Now assume that  $G$  is perfect. Since  $G$  has no simple images, it is a union of a chain of proper normal subgroups. If  $N$  is a proper normal subgroup of  $G$ , then  $N' \in \mathfrak{X}_\omega$  by (ii) and so  $G = G'$  is a union of  $\mathfrak{X}_\omega$ -groups. So  $G \in \mathfrak{X}_\omega$ , a contradiction.

Thus,  $G$  is not perfect and  $G/G'$  has a proper subgroup  $R/G'$  such that  $G/R \in \mathfrak{L} \cap \mathfrak{C}$ . Now by Lemma 2.1,  $R$  has a characteristic subgroup  $W \in \mathfrak{X}_\omega$  such that  $G/W \in \mathfrak{C}$ . Since  $G/W$  has no proper subgroup of finite index, we have  $G/W \in \mathfrak{L} \cap \mathfrak{C}$ .

Consequently,  $G \in \mathfrak{X}_\omega(\mathfrak{L} \cap \mathfrak{C})$ . In particular,  $G' \leq W$  and hence  $G' \in \mathfrak{X}_\omega$ . In particular,  $G \in \mathfrak{S}_t$  by Lemma 2.2(i). So (iii) holds.

Finally, since  $G' \in \mathfrak{X}_\omega$ , we have  $(\omega \circ \theta)(G) = \omega(G') = 1$ , and (iv) holds. □

The following is the  $\mathfrak{X}_\omega\mathfrak{C}$  version of Theorem 2.4.

**THEOREM 2.5.** *Let  $G$  be an  $MN\mathfrak{X}_\omega\mathfrak{C}$ -group. If  $G$  has no infinite simple images, then the following are satisfied.*

- (i)  $G$  has no proper subgroup of finite index and no simple images.
- (ii)  $N' \in \mathfrak{X}_{\omega \circ \theta}$  for every proper normal subgroup  $N$  of  $G$ , i.e.  $N \in \mathfrak{X}_{\omega \circ \theta^2}$ .
- (iii)  $G$  is not perfect and  $G \in \mathfrak{X}_{\omega \circ \theta}(\mathfrak{L} \cap \mathfrak{C})$ . In particular,  $G' \in \mathfrak{X}_{\omega \circ \theta}$  and  $G \in \mathfrak{S}_{t+1}$ .

*Proof.* By a similar argument to that used in the proof of Theorem 2.4,  $G$  has no proper subgroup of finite index. So (i) holds.

Now let  $N$  be a proper normal subgroup of  $G$ , then it has a normal subgroup  $S \in \mathfrak{X}_\omega$  such that  $N/S \in \mathfrak{C}$ . So  $N/S$  has a normal subgroup  $R/S \in \mathfrak{L} \cap \mathfrak{C}$  such that  $N/R \in \mathfrak{F}$ . Since  $R/S$  is in  $\mathfrak{A}$ ,  $R \in \mathfrak{X}_{\omega \circ \theta}$ . By Lemma 2.1  $N$  has a characteristic subgroup  $M \in \mathfrak{X}_{\omega \circ \theta}$  such that  $N/M \in \mathfrak{F}$  and hence  $N' \leq M$ , i.e.  $N' \in \mathfrak{X}_{\omega \circ \theta}$ . So (ii) holds.

Suppose next that  $G$  has a non-trivial  $\mathfrak{C}$ -image  $G/N$ . Then  $N$  has a normal subgroup  $S \in \mathfrak{X}_\omega$  such that  $N/S \in \mathfrak{C}$  and  $N/S$  has a normal subgroup  $M/S \in \mathfrak{L} \cap \mathfrak{C}$  such that  $N/M \in \mathfrak{F}$ . So  $N \in \mathfrak{X}_{\omega \circ \theta}\mathfrak{F}$ . By Lemma 2.1  $N$  has a characteristic subgroup  $T \in \mathfrak{X}_{\omega \circ \theta}$  with  $N/T \in \mathfrak{F}$ . This implies that  $G/T \in \mathfrak{C}$ , and hence  $G/T \in \mathfrak{L} \cap \mathfrak{C}$  by (i) and  $G \in \mathfrak{X}_{\omega \circ \theta}(\mathfrak{L} \cap \mathfrak{C})$  in this case.

Now if  $G$  is perfect, then as in the proof of Theorem 2.4,  $G$  is a union of proper normal subgroups and so we have  $G \in \mathfrak{X}_{\omega \circ \theta^2}$ , and hence  $\omega(G) = 1$ , a contradiction. So  $G$  is not perfect and  $G/G'$  has a proper normal subgroup  $R/G'$  such that  $G/R \in \mathfrak{L} \cap \mathfrak{C}$ . By the previous argument  $G \in \mathfrak{X}_{\omega \circ \theta}(\mathfrak{L} \cap \mathfrak{C})$  and so  $G' \in \mathfrak{X}_{\omega \circ \theta}$  and  $G \in \mathfrak{S}_{t+1}$  by Lemma 2.2(i). Thus, (iii) holds. □

**3. Applications to  $MN\mathfrak{S}_n\mathfrak{C}$  and  $MN\mathfrak{S}_n\mathfrak{F}$ -groups.** Since a group is in  $\mathfrak{S}\mathfrak{C}$  if and only if it is in  $\mathfrak{S}\mathfrak{F}$ , we see that a group is in  $MN\mathfrak{S}\mathfrak{C}$  if and only if it is in  $MN\mathfrak{S}\mathfrak{F}$ .

We know that the celebrated example of Heineken and Mohamed (see [15, Theorem 6.2.1]) is an  $MN\mathfrak{A}\mathfrak{F}$ -group which is in  $\mathfrak{A}\mathfrak{C}$ . So an  $MN\mathfrak{S}_n\mathfrak{F}$ -group (for a positive integer  $n$ ) is not in general an  $MN\mathfrak{S}_n\mathfrak{C}$ -group.

The locally graded groups with all proper subgroups in  $\mathfrak{S}\mathfrak{F}$  are classified by [6, Theorem C], as follows

**THEOREM 3.1.** *Let  $G$  be a locally graded group with all proper subgroups in  $\mathfrak{S}\mathfrak{F}$ . Then either*

- (i)  $G$  is locally soluble, or
- (ii)  $G \in \mathfrak{S}\mathfrak{F}$ , or
- (iii)  $G$  is  $\mathfrak{S}$ -by- $PSL(2, \mathfrak{F})$ , or
- (iv)  $G$  is  $\mathfrak{S}$ -by- $Sz(\mathfrak{F})$ ,

where  $F$  is an infinite locally finite field with no infinite proper subfields.

By the remark above, we see that Theorem 3.1 also gives a classification of the locally graded groups with all proper subgroups in  $\mathfrak{S}\mathfrak{C}$ .

If  $G$  is a countable locally graded simple group with all subgroups in  $\mathfrak{S}\mathfrak{F}$  (or in  $\mathfrak{S}\mathfrak{C}$ ), then a super-inert subgroup  $R$  (see [6] for the definition) of  $G$  either has non-trivial Hirsch–Plokin radical or is in  $\mathfrak{F}$ , hence  $G$  is locally finite [6, Theorem 2]. So by [12]

$G$  is isomorphic either to  $PSL(2, \mathbb{F})$  or to  $Sz(\mathbb{F})$  for some infinite locally finite field  $\mathbb{F}$  containing no infinite proper subfield.

**THEOREM 3.2.** *There are infinite locally finite simple  $MN\mathfrak{S}_2\mathfrak{F}$  and  $MN\mathfrak{S}_3\mathfrak{F}$ -groups.*

*Proof.* Let  $G := PSL(2, \mathbb{F})$  or  $G := Sz(\mathbb{F})$  for some infinite locally finite field  $\mathbb{F}$  containing no infinite proper subfield. In the first case every proper subgroup is either in  $\mathfrak{A}^2$  or in  $\mathfrak{F}$  and so in  $\mathfrak{S}_2\mathfrak{F}$  by [4, Example 3]. Clearly  $G \notin \mathfrak{S}_2\mathfrak{F}$ . So  $G \in MN\mathfrak{S}_2\mathfrak{F}$ .

In the second case every proper subgroup of  $G$  is in  $\mathfrak{F}$  or is  $\mathfrak{N}_2$ -by-locally cyclic (i.e. in  $\mathfrak{S}_3\mathfrak{F}$ ) by the proof of [5, Lemma 2]. Consequently,  $G$  is in  $MN\mathfrak{S}_3\mathfrak{F}$ .  $\square$

Let  $G$  be a group,  $H$  a subgroup of  $G$ ; then the *isolator*  $I_G(H)$  of  $H$  in  $G$  is defined as

$$I_G(H) = \{x \in G \mid \text{there is a non-zero integer } n \text{ such that } x^n \in H\}.$$

We prove the following general lemma.

**LEMMA 3.3.** *Let  $\omega$  be an outer commutator word of weight  $t$ , and let  $H$  be a subgroup of the locally nilpotent torsion-free group  $G$ . Then*

$$\omega(I_G(H)) \leq I_G(\omega(H)).$$

*Proof.* First let  $U$  and  $V$  be subgroups of  $G$ . Then with the notation of [14, Section 2.3] we have  $I_G(U) \sim U$  and  $I_G(V) \sim V$ . It follows that  $[I_G(U), I_G(V)] \sim [U, V]$  by [14, 2.3.5]. So we see that

$$[I_G(U), I_G(V)] \leq I_G([U, V]).$$

Now we proceed by induction on  $t$ . If  $t = 1$ , then the result is immediate. If  $t > 1$ , then  $\omega = [\varphi, \delta]$  for some outer commutator words  $\varphi$  and  $\delta$  of weights  $1 \leq t_1 < t$ ,  $1 \leq t_2 < t$  such that  $t_1 + t_2 = t$ . By induction hypothesis and the above remark, we have

$$\begin{aligned} \omega(I_G(H)) &= [\varphi(I_G(H)), \delta(I_G(H))] \leq [I_G(\varphi(H)), I_G(\delta(H))] \\ &\leq I_G([\varphi(H), \delta(H)]) = I_G(\omega(H)), \end{aligned}$$

and the proof is complete.  $\square$

**THEOREM 3.4.** *Let  $G$  be a locally nilpotent torsion-free group.*

- (i) *If all proper subgroups of  $G$  are in  $\mathfrak{X}_\omega\mathfrak{T}$ , then  $G \in \mathfrak{X}_\omega$ .*
- (ii) *If all proper subgroups of  $G$  are in  $\mathfrak{X}_\omega\mathfrak{R}$ , then  $G \in \mathfrak{X}_\omega(\mathfrak{R} \cap \mathfrak{N})$ . In particular,  $G$  is in  $\mathfrak{S}$ .*

*Proof.* (i) Let  $K$  be a proper subgroup of  $G$ . Then  $K$  has a normal subgroup  $N \in \mathfrak{X}_\omega$  such that  $K/N \in \mathfrak{T}$ , and so  $I_K(N) = K$ . By Lemma 3.3

$$\omega(K) = \omega(I_K(N)) \leq I_K(\omega(N)) = I_K(1) = 1.$$

This means that every proper subgroup  $K$  of  $G$  is in  $\mathfrak{X}_\omega$ .

If  $G$  is not finitely generated, then every finitely generated subgroup of  $G$  is in  $\mathfrak{X}_\omega$ , and thus  $G \in \mathfrak{X}_\omega$ . Otherwise,  $G$  is finitely generated, and by [18, 5.2.2.1] it has a normal

subgroup  $H \in \mathfrak{X}_\omega$  of finite index, since it is nilpotent. Hence,  $I_G(H) = G$  and as above  $\omega(G) = 1$ . So  $G \in \mathfrak{X}_\omega$ .

(ii) Assume for a contradiction that  $G$  is not in  $\mathfrak{X}_\omega\mathfrak{A}$ , and first suppose that  $G$  has a proper normal subgroup of  $N$  such that  $G/N$  is in  $\mathfrak{A}$ . Then  $N$  has a normal subgroup  $M$  such that  $M \in \mathfrak{X}_\omega$  and  $N/M \in \mathfrak{A}$ . By [10, Theorem 3], we may assume that  $M$  is characteristic in  $N$  so that  $M$  is normal in  $G$ . Clearly  $G/M$  is in  $\mathfrak{A}$ , so  $G \in \mathfrak{X}_\omega\mathfrak{A}$ , a contradiction. Therefore,  $G$  has no proper images which are in  $\mathfrak{A}$  and hence it is perfect.

Let  $H$  be any proper normal subgroup of  $G$  and let  $K$  be a characteristic subgroup of  $H$  such that  $K \in \mathfrak{X}_\omega$  and  $H/K \in \mathfrak{A}$ . If  $T/K$  denotes the torsion subgroup of  $H/K$ , then  $T \in \mathfrak{X}_\omega\mathfrak{T}$  and hence  $T \in \mathfrak{X}_\omega$  by (i). Since torsion-free locally nilpotent  $\mathfrak{A}$ -groups are in  $\mathfrak{N}$  [18, Theorem 6.36], we have  $H/T \in \mathfrak{N}$ . So  $H/T$  has a finite characteristic series whose factors are torsion-free  $\mathfrak{A} \cap \mathfrak{N}$ -groups. If  $U$  is such a factor, then  $G/C_G(U)$  is nilpotent by [17, Part 2, Lemma 6.37] and hence  $G = C_G(U)$  since  $G$  is perfect. We deduce that  $H/T$  is contained in the hypercentre of  $G/T$ , which equals the centre, as  $G$  is perfect. Thus,  $H/T \leq Z(G/T)$  and so  $H' \leq T$ . We deduce that  $H' \in \mathfrak{X}_\omega$ . As before, since there exists a chain  $\{N_i : i \in I\}$  of proper normal subgroups of  $G$  such that  $G = \bigcup_{i \in I} N_i$ , it follows that  $G = G' = \bigcup_{i \in I} N'_i$ . Consequently, we have  $G \in \mathfrak{X}_\omega$ , a contradiction. Therefore,  $G \in \mathfrak{X}_\omega\mathfrak{A}$ .

Let  $N$  be a normal subgroup of  $G$  such that  $N \in \mathfrak{X}_\omega$  and  $G/N \in \mathfrak{A}$ . If  $T/N$  denotes the torsion subgroup of  $G/N$ , then again by (i)  $T \in \mathfrak{X}_\omega$ , and since  $G/T$  is a locally nilpotent torsion-free  $\mathfrak{A}$ -group, it is in  $\mathfrak{N}$ . Therefore,  $G \in \mathfrak{X}_\omega(\mathfrak{A} \cap \mathfrak{N})$ . By Lemma 2.2(i), we deduce that  $G$  is in  $\mathfrak{S}$ , as claimed. □

Let us define the outer commutator word  $\phi_j$  for every  $j \geq 0$  as follows:  
 $\phi_0(x) = x$  and for  $i \geq 1$

$$\phi_i(x_1, \dots, x_{2^i}) = [\phi_{i-1}(x_1, \dots, x_{2^{i-1}}), \phi_{i-1}(x_{2^{i-1}+1}, \dots, x_{2^i})].$$

Then  $G$  is in  $\mathfrak{S}$  if and only if there is a positive integer  $n$  such that  $\phi_n(G) = 1$ .

**THEOREM 3.5.** *Let  $G$  be a group without infinite simple images. Then the following are satisfied.*

- (i) *If every proper subgroup of  $G$  is in  $\mathfrak{S}_n\mathfrak{F}$  for some fixed positive integer  $n$ , then either  $G \in \mathfrak{S}_n\mathfrak{F}$  or  $G \in \mathfrak{S}_n(\mathfrak{L} \cap \mathfrak{C})$ . So if  $G$  is an  $MN\mathfrak{S}_n\mathfrak{F}$ -group, then  $G \in \mathfrak{S}_n\mathfrak{C} \cap \mathfrak{S}_{n+1}$ .*
- (ii) *If every proper subgroup of  $G$  is in  $\mathfrak{S}_n\mathfrak{C}$  for some fixed positive integer  $n$ , then  $G \in \mathfrak{S}_n\mathfrak{C}$  or  $G \in \mathfrak{S}_{n+1}(\mathfrak{L} \cap \mathfrak{C})$ . So if  $G$  is an  $MN\mathfrak{S}_n\mathfrak{C}$ -group, then  $G \in \mathfrak{S}_{n+1}\mathfrak{C} \cap \mathfrak{S}_{n+2}$ .*

*Proof.* (i) Take  $\omega = \phi_n$ . If  $G \notin \mathfrak{S}_n\mathfrak{F}$ , then  $G$  is an  $MN\mathfrak{S}_n\mathfrak{F}$ -group. By Theorem 2.4 (iii)  $G \in \mathfrak{X}_{\phi_n}(\mathfrak{L} \cap \mathfrak{C})$ .

(ii) Again take  $\omega = \phi_n$  so that  $\omega \circ \theta = \phi_{n+1}$ . If  $G \notin \mathfrak{S}_n\mathfrak{C}$ , then by Theorem 2.5 (iii)  $G \in \mathfrak{X}_{\phi_{n+1}}(\mathfrak{L} \cap \mathfrak{C})$ , and the proof is complete. □

The following lemma will be generalised in Section 4 (see Lemma 4.1), but since the ‘soluble’ version of the lemma is useful here, we shall prove it.

**LEMMA 3.6** (c.f. [3, Proposition 1]). *Let  $G$  be in  $\mathfrak{T} \cap \mathfrak{N}$ ,  $A \in \mathfrak{S}_n$  ( $n \geq 1$ ) a normal subgroup of  $G$  such that  $G/A \in \mathfrak{L}$ . Then also  $G \in \mathfrak{S}_n$ .*

*Proof.* We proceed by induction on  $n$ . If  $n = 1$ , then  $A$  is in  $\mathfrak{A}$  and hence  $A \leq C_G(A)$ . So  $T := G/C_G(A) \in \mathfrak{L}$  is isomorphic to a subgroup of  $\text{Aut } A$ . By [3, Lemma 1],  $A \leq Z(G)$ , and by [15, Section 5.3.5]  $G$  is in  $\mathfrak{A}$ . Now let  $n > 1$  and consider  $G/A^{(n-1)}$ .

Then

$$\frac{G/A^{(n-1)}}{A/A^{(n-1)}} \in \mathfrak{L} \text{ and } A/A^{(n-1)} \in \mathfrak{S}_{n-1}.$$

By induction hypothesis  $G/A^{(n-1)} \in \mathfrak{S}_{n-1}$  and thus  $G^{(n-1)} = A^{(n-1)}$ . This implies that  $G^{(n)} = 1$  and  $G \in \mathfrak{S}_n$ , as desired. □

**THEOREM 3.7.** *Let  $G$  be a locally graded  $\mathfrak{T}$ -group and suppose that every proper subgroup of  $G$  is in  $\mathfrak{S}_n\mathfrak{C}$  for some fixed positive integer  $n$ . If  $G$  contains a normal  $\mathfrak{N}$ -subgroup  $N$  such that  $G/N \in \mathfrak{C}$ , then  $G \in \mathfrak{S}_n\mathfrak{C}$ .*

*Proof.* Assume for a contradiction that  $G$  is an  $MN\mathfrak{S}_n\mathfrak{C}$ -group. Since  $G$  has no proper subgroup of finite index, we have  $G/N \in \mathfrak{L}$  or  $G = N$ . Hence,  $G' \neq G$  and thus  $1 \neq G/G' \in \mathfrak{L}$ . If  $N = G$ , then we have the contradiction that  $G$  is in  $\mathfrak{A}$  by [18, Section 5.2.5], since  $G$  is in  $\mathfrak{T} \cap \mathfrak{N}$ . So  $N \neq G$  and hence  $N$  has a normal subgroup  $S \in \mathfrak{S}_n$  such that  $N/S \in \mathfrak{C}$ . So  $N/S$  contains a maximal  $\mathfrak{L}$ -subgroup  $R/S$  such that  $N/R \in \mathfrak{F}$ . By Lemma 3.6,  $R \in \mathfrak{S}_n$ . Therefore, we can assume by Lemma 2.1 that  $R$  is characteristic in  $N$  and hence  $R$  is normal in  $G$ . So  $G/R$  is in  $\mathfrak{C}$ . Consequently,  $G \in \mathfrak{S}_n\mathfrak{C}$ , a contradiction, and the proof is complete. □

**4.  $\mathfrak{N}\mathfrak{C}$ -groups with certain characteristic subgroups.**

**LEMMA 4.1.** *Let  $G$  be in  $\mathfrak{T} \cap \mathfrak{N}$ ,  $N$  a normal subgroup of  $G$ ,  $\omega$  an outer commutator word of weight  $t \geq 2$  such that  $\omega(N) = 1$ . If  $G/N$  is in  $\mathfrak{A} \cap \mathfrak{D}$ , then  $\omega(G) = 1$ .*

*Proof.* We proceed by induction on  $t$ . If  $t = 2$ , then  $\omega(x, y) = [x, y]$  and

$$\omega(N) = [N, N] = 1,$$

i.e.  $N$  is in  $\mathfrak{A}$ . So  $G/C_G(N)$  is in  $\mathfrak{A} \cap \mathfrak{D}$  and isomorphic to a subgroup of  $\text{Aut } N$ . By [3, Lemma 1]  $G = C_G(N)$  and thus  $N \leq Z(G)$ . Applying [18, 5.3.5] we have that  $G$  is in  $\mathfrak{A}$ . Let  $t > 2$ ; then  $\omega = [\psi, \phi]$  for some outer commutator words  $\psi, \phi$  of weight  $1 \leq t_1, t_2 < t$  such that  $t = t_1 + t_2$ . Now  $G/N$  is in  $\mathfrak{L}$  and  $\psi(N/\psi(N)) = 1$ . If  $t_1 > 1$ , then by induction hypothesis  $\psi(G/\psi(N)) = 1$ , i.e.  $\psi(G) \leq \psi(N)$ . Clearly  $\psi(N) \leq \psi(G)$  and it follows that  $\psi(G) = \psi(N)$ . If also  $t_2 > 1$ , then similarly  $\phi(G) = \phi(N)$ , and we have

$$\omega(G) = [\psi(G), \phi(G)] = [\psi(N), \phi(N)] = 1,$$

as required. So we may assume that  $t_2 = 1$  and hence  $t_1 > 1$ , since  $t > 2$ . (If  $t_1 = 1$ , then a similar argument works.) Then  $\omega(N) = [\psi(N), N] = 1$  and hence  $N \leq C_G(\psi(N))$ . We also have that  $\psi(N)$  is in  $\mathfrak{A}$ . Then  $G/C_G(\psi(N))$  is in  $\mathfrak{A} \cap \mathfrak{D}$  and isomorphic to a subgroup of  $\text{Aut } \psi(N)$ . So by [3, Lemma 1]  $\psi(N) \leq Z(G)$ ; in other words  $[\psi(N), G] = 1$ . It follows that

$$\omega(G) = [\psi(G), G] = [\psi(N), G] = 1,$$

and the proof is complete. □

**THEOREM 4.2.** *Let  $G$  be a  $\mathfrak{T}$ -group and let  $N \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$  be a normal subgroup of  $G$  such that  $G/N \in \mathfrak{C}$  for some outer commutator word  $\omega$ . Then  $G$  contains a characteristic*

(even invariant under all surjective endomorphisms) subgroup  $S \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$  such that  $G/S \in \mathfrak{C}$ .

*Proof.* Let  $W := \langle N^\alpha \mid \alpha \in \text{Aut } G \rangle$ , then  $W$  is characteristic in  $G$  and  $W/N \in \mathfrak{C}$ . By [8, Lemma 4.7]  $W$  is in  $\mathfrak{N}$ . We also have that  $W/N$  has a normal  $\mathfrak{A} \cap \mathfrak{D}$ -subgroup  $R/N \in \mathfrak{R}$  such that  $W/R$  is in  $\mathfrak{F}$ . Now by Lemma 4.1 we have  $R \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$ . By Lemma 2.1  $W$  has characteristic (even invariant under all surjective endomorphisms) subgroups  $S_1 \in \mathfrak{N}_c$  and  $S_2 \in \mathfrak{X}_\omega$  such that  $W/S_i$  is in  $\mathfrak{F}$  for  $i = 1, 2$ . Put  $S = S_1 \cap S_2$ , then  $|W : S| < \infty$  and  $S$  is contained in  $\mathfrak{N}_c \cap \mathfrak{X}_\omega$ . Since  $W$  is characteristic in  $G$ , we see that  $S$  is characteristic in  $G$ , and since  $G/W \in \mathfrak{C}$  and  $W/S$  is finite, we have  $G/S \in \mathfrak{C}$ . The proof is complete. □

If we take  $\omega = \gamma_{c+1}$ , then

$$\mathfrak{N}_c \cap \mathfrak{X}_\omega = \mathfrak{N}_c \cap \mathfrak{N}_c = \mathfrak{N}_c.$$

Hence, we obtain the following result.

**COROLLARY 4.3.** *Let  $G$  be a  $\mathfrak{T}$ -group and let  $N \in \mathfrak{N}_c$  be a normal subgroup of  $G$  such that  $G/N \in \mathfrak{C}$ . Then  $G$  contains a characteristic (even invariant under all surjective endomorphisms) subgroup  $S \in \mathfrak{N}_c$  such that  $G/S \in \mathfrak{C}$ .*

Corollary 4.3 sharpens [8, Lemma 4.7] and generalises [3, Lemma 3] and [9, Corollary 1(i)] in the periodic case.

In [8, p. 321] Hartley gives an example that shows that the ‘periodicity’ condition cannot be removed from the hypothesis of Corollary 4.3 and defined Chernikov-subnormality ( $\mathfrak{C}$ -subnormality, in short) as follows:

A subgroup  $N$  of a group  $G$  is called  $\mathfrak{C}$ -subnormal in  $G$  if there is a finite series

$$N = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_r = G$$

such that  $N_{i+1}/N_i \in \mathfrak{C}$  for  $0 \leq i \leq r - 1$ .

**COROLLARY 4.4.** *Let  $G$  be a  $\mathfrak{T}$ -group containing a  $\mathfrak{C}$ -subnormal subgroup  $N \in \mathfrak{N}_c$ . Then  $G$  contains a characteristic (even invariant under all surjective endomorphisms) subgroup  $S \in \mathfrak{N}_c$  such that  $G/S \in \mathfrak{C}$ .*

*Proof.* The result follows by Corollary 4.3 and a simple induction. □

We can give an immediate application of Corollary 4.3 by considering the following result due to Hartley.

**THEOREM 4.5 [8, Theorem B].** *If  $G$  is a locally finite group admitting an involutory automorphism  $\phi$  such that  $C_G(\phi)$  is in  $\mathfrak{C}$ , then both  $[G, \phi]'$  and  $G/[G, \phi]$  are in  $\mathfrak{C}$ .*

As Shumyatsky mentions in [19, p. 160], if we take  $N = C_{[G, \phi]}([G, \phi]')$ , then  $N \in \mathfrak{N}_2$ ,  $G/N \in \mathfrak{C}$  and  $N$  is  $\phi$ -invariant. So by Corollary 4.3,  $G$  has a characteristic subgroup  $S \in \mathfrak{N}_2$  such that  $G/S \in \mathfrak{C}$ .

We record here the following theorem, which is an immediate consequence of Lemma 2.1.

**THEOREM 4.6.** *Let  $G$  be a group and let  $N \in \mathfrak{X}_\omega$  be a normal subgroup of  $G$  for some outer commutator word  $\omega$  such that  $G/N \in \mathfrak{C}$ . Then  $G$  contains a characteristic (even invariant under all surjective endomorphisms) subgroup  $S \in \mathfrak{X}_{\omega \circ \theta}$  such that  $G/S$  is finite.*



*Proof.* Since  $G/N \in \mathfrak{C}$ , there exists a normal  $\mathfrak{A} \cap \mathfrak{D}$ -subgroup  $R/N$  of  $G/N$  such that  $G/R$  is in  $\mathfrak{F}$ . Since  $N \in \mathfrak{X}_\omega$  and  $R/N$  is in  $\mathfrak{A}$ , we have  $R \in \mathfrak{X}_{\omega\circ\theta}$ . By Lemma 2.1  $G$  has a characteristic subgroup (even invariant under all surjective endomorphisms)  $S \in \mathfrak{X}_{\omega\circ\theta}$  such that  $G/S$  is in  $\mathfrak{F}$ , and the result is established.  $\square$

Of course, if we replace the condition  $G/N \in \mathfrak{C}$  with  $G/N \in \mathfrak{A}\mathfrak{F}$  in Theorem 4.6, then the result remains true.

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