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ABSOLUTE TAUBERIAN CONSTANTS FOR HAUSDORFF TRANSFORMATIONS

SORAYA SHERIF

1. Introduction. Let $\{\mu_n\}_{n=0}^{\infty}$ be a fixed sequence of real or complex numbers. The Hausdorff transform $\{t_n\}$ of a sequence $\{s_n\}$ by means of the fixed sequence $\{\mu_n\}_{n=0}^{\infty}$ (or, in short, the (H, μ_n) transform) is given by

$$t_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k}\mu_k) S_k, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where, for $r, q \geqq 0$,

$$\Delta^0 \mu_q = \mu_q, \quad \Delta\mu_q = \mu_q - \mu_{q+1}, \quad \Delta^{r+1}\mu_q = \Delta(\Delta^r \mu_q). \tag{1.2}$$

K. Knopp and G. G. Lorentz [**6**] have shown (a simpler proof was given by Jakimovski [**5**, Equation (3.1)]) that if (1.1) and (1.2) hold and if

$$t_n = b_0 + b_1 + \ldots\ldots\ldots + b_n, \qquad s_k = a_0 + a_1 + \ldots + a_k, \tag{1.3}$$

then the series-to-series Hausdorff transform b_n of $\sum a_n$ (unless otherwise indicated, the symbol $\sum$ stands for $\sum_0^\infty$) is such that

$$b_0 = \mu_0 a_0$$

$$b_n = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} k (\Delta^{n-k}\mu_k) a_k, \quad n = 1, 2, 3, \ldots \tag{1.4}$$

If (1.1)-(1.4) hold, then we say that the sequence $\{s_n\}$ is *absolutely* summable (H, μ_n) or summable $|H, \mu_n|$, if the sequence $\{t_n\}$ is of bounded variation or equivalently if

$$\sum |b_n| < \infty.$$

(For the definition of absolute summability, see [**2**; **4**; **8**; **10**].)

In sections 2 and 3 of this paper, we shall prove the following two inequalities:

$$\sum |b_n - a_n| \leqq K \sum |\Delta(na_n)|, \tag{1.5}$$

$$\sum |b_n - a_n| \leqq A \sum \left|\Delta\left(\frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu\right)\right|, \tag{1.6}$$

where K and A are *absolute* Tauberian constants.

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Estimates of this form have been shown in Sherif [**9**] for the absolute Cesàro means. Since, as is well-known (cf. [**3**, p. 251]), when

$$\mu_n = 1\Big/\binom{n+k}{k}, \tag{1.8}$$

the (H, μ_n) method reduces to the Cesàro method (C, k) the results of this paper include those of [**9**]. This will be verified in Remarks 2.1 and 3.1.

The estimates (1.5) and (1.6) are analogous to results obtained for other summability methods by various authors. For a discussion of these analogous estimates, see Sherif [**9**].

I have much pleasure in expressing my gratitude to Professor B. Kuttner for his criticisms and suggestions for improvements to present this paper.

2. THEOREM 2.1. *Let* $\{\mu_n\}$ *be a moment sequence generated by the real function of bounded variation* χ *on* $0 \leqq t \leqq 1$ *so that*

$$\mu_n = \int_0^1 t^n \, d\chi(t), \tag{2.1}$$

where

$$\chi(0+) = \chi(0) = 0, \qquad \chi(1) = 1, \tag{2.2}$$

and

$$\int_0^1 \frac{|\chi(t)|}{t} \, dt < \infty. \tag{2.3}$$

Then (1.5) *holds with*

$$K = \int_0^1 \frac{|\chi(t)|}{t} \, dt. \tag{2.4}$$

If

$$\chi(t) \geqq 0, \tag{2.5}$$

the constant given in (2.4) *is the best possible in the sense that* (1.5) *becomes false if K is replaced by any smaller constant.*

Remark 2.1. In the case of summability (C, k) $(k > 0)$, we have $\chi(t) = 1 - (1-t)^k$ $(k > 0)$, and equation (2.4) becomes

$$K = \int_0^1 \frac{1-(1-t)^k}{t} \, dt = \int_0^1 \frac{1-u^k}{1-u} \, du = \frac{\Gamma'(k+1)}{\Gamma(k+1)} + \gamma, \tag{2.6}$$

(γ is Euler's constant) by Bateman [**1**, p. 16].

Thus, Theorem 2.1 for $\chi(t) = 1 - (1-t)^k$ $(k > 0)$, is Theorem 2.1 of Sherif [**9**].

For the proof of Theorem 2.1, we require the following lemma.

LEMMA [**7**, p. 167, Theorem 5]. *Let*

$$A_n = \sum_\nu \alpha_{n,\nu} f_\nu. \tag{2.7}$$

Suppose that

$$\sum_n |\alpha_{n,\nu}| \text{ is bounded.} \tag{2.8}$$

Let

$$K = \sup_\nu \sum |\alpha_{n,\nu}|. \tag{2.9}$$

Then

$$\sum |A_n| \leqq K \sum |f_\nu|, \tag{2.10}$$

and this constant is the best possible in the sense that (2.11) *becomes false if K is replaced by any smaller constant.*

Proof of Theorem 2.1. Since

$$a_n = \frac{1}{n} \cdot na_n = -\frac{1}{n} \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu), \tag{2.11}$$

it follows from (1.4) and (2.11) that for $n \geqq 1$,

$$\begin{aligned} b_n &= -\frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} (\Delta^{n-k}\mu_k) \sum_{\nu=0}^{k-1} \Delta(\nu a_\nu) \\ &= -\frac{1}{n} \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left\{ \sum_{k=\nu+1}^{n} \binom{n}{k} (\Delta^{n-k}\mu_k) \right\}. \end{aligned} \tag{2.12}$$

It thus follows from (2.11) and (2.12) that

$$\begin{aligned} b_n - a_n &= \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left[\frac{1}{n} \left\{ 1 - \sum_{k=\nu+1}^{n} \binom{n}{k} (\Delta^{n-k}\mu_k \right\} \right] \\ &= \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left[\frac{1}{\mathrm{n}} \sum_{k=0}^{\nu} \binom{n}{k} (\Delta^{n-k}\mu_k) \right] \quad (\textit{cf.}\ [\mathbf{3}, \textit{Formula}\ (11.5.5)]) \\ &= \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left[\frac{1}{n} \sum_{k=0}^{\nu} \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} d\chi(t) \right] \\ &= \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left[\frac{1}{n} \int_0^1 d\chi(t) \sum_{k=0}^{\nu} \binom{n}{k} t^k (1-t)^{n-k} \right]. \end{aligned} \tag{2.13}$$

Now,

$$\begin{aligned} \frac{d}{dt} \left\{ \sum_{k=0}^{\nu} \binom{n}{k} t^k (1-t)^{n-k} \right\} &= \sum_{k=1}^{\infty} k \binom{n}{k} t^{k-1} (1-t)^{n-k} \\ &\quad - \sum_{k=0}^{\nu} (n-k) \binom{n}{k} t^k (1-t)^{n-1-k}, \end{aligned} \tag{2.14}$$

where the first term on the right hand side of (2.14) is to be taken to be 0 if $\nu = 0$. Since

$$(n-k)\binom{n}{k} = (k+1)\binom{n}{k+1} \tag{2.15}$$

we see on replacing k by $k + 1$ in the first sum that the expression (2.14) reduces to

$$-(\nu+1)\binom{n}{\nu+1}t^{\nu}(1-t)^{n-1-\nu}. \tag{2.16}$$

Integrating by parts in (2.13) and using (2.16) we obtain

$$b_n - a_n = \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu)\left[\binom{n-1}{\nu}\int_0^1 t^{\nu}(1-t)^{n-1-\nu}\chi(t)dt\right], \text{ where } \binom{0}{0} = 1. \tag{2.17}$$

Now, (2.17) is a transformation of the type considered in the Lemma, and for $n \geqq 1$,

$$\alpha_{n,\nu} = \begin{cases} 0, & (\nu \geqq n) \\ \binom{n-1}{\nu}\int_0^1 t^{\nu}(1-t)^{n-1-\nu}\chi(t)dt, & (0 \leqq \nu \leqq n-1). \end{cases} \tag{2.18}$$

Thus, the conditions of the Lemma are satisfied with

$$K = \sup_{\nu} S_\nu, \tag{2.19}$$

where

$$S_\nu = \sum_{n=\nu+1}^{\infty}\binom{n-1}{\nu}\left|\int_0^1 t^{\nu}(1-t)^{n-1-\nu}\chi(t)dt\right|$$

$$\leqq \sum_{n=\nu+1}^{\infty}\binom{n-1}{\nu}\int_0^1 |\chi(t)|t^{\nu}(1-t)^{n-1-\nu}dt \tag{2.20}$$

$$= \int_0^1 |\chi(t)|\left\{\sum_{n=\nu+1}^{\infty}\binom{n-1}{\nu}t\,(1-t)^{n-1-\nu}\right\}dt$$

$$= \int_0^1 \frac{|\chi(t)|}{t}\,dt. \tag{2.21}$$

Hence (1.5) holds with K given by (2.4), as claimed. Further if (2.5) holds, then there is equality in (2.20) and the final conclusion follows from the Lemma.

3. THEOREM 3.1. *Let $\chi(t)$ be a real valued function defined for $0 \leqq t \leqq 1$, satisfying* (2.2), *and with $\chi(t)/t$ of bounded variation there. Let $\{\mu_n\}$ be the moment sequence generated by χ so that* (2.1) *holds. Then* (1.6) *holds with*

$$A = \sup_{\nu}\left\{(\nu+1)\int_0^1 t^{\nu-1}|\chi(t)|dt + \int_0^1 t(1-t^{\nu})\left|d\left(\frac{\chi(t)}{t}\right)\right|\right\}. \tag{3.1}$$

If (2.5) *holds, and if* $\chi(t)/t$ *is monotonic* (in the case in which $\chi(t)/t$ is non-increasing, this hypothesis implies that (2.5) holds (since $\chi(1) = 1$) and the assumption (2.5) may therefore be omitted), *then* (3.1) *is the best possible result in the sense that* (1.6) *becomes false if* A *is replaced by any smaller constant.*

Further (3.1) *can be simplified; if* $\chi(t)/t$ *is non-increasing, we have*

$$A = \int_0^1 \frac{\chi(t)}{t}\,dt; \tag{3.2}$$

if (2.5) *holds and* $\chi(t)/t$ *is increasing, we have*

$$A = 2(1 - l) - \int_0^1 \frac{\chi(t)}{t}\,dt, \tag{3.3}$$

where

$$l = \lim_{n\to\infty} \mu_n = \chi(1) - \chi(1-). \tag{3.4}$$

Remark 3.1. When $\chi(t) = 1 - (1 - t)^k$, then condition (2.5) is satisfied in any case. If $k > 1$, $\chi(t)/t$ is non-increasing, while if $0 < k < 1$, $\chi(t)/t$ is non-decreasing. Hence, it is easily seen using (2.6) that Theorem 3.1 for $\chi(t) = 1 - (1 - t)^k$ is Theorem 3.1 of Sherif [**9**].

Proof of Theorem 3.1. Let

$$u_n = \begin{cases} 0, & n = 0. \\ \dfrac{1}{n+1} \displaystyle\sum_{\nu=1}^{n} \nu a_\nu, & n \geqq 1. \end{cases}$$

Write

$$\phi_n = -\Delta u_{n-1} = u_n - u_{n-1} \quad \text{for } n \geqq 1.$$

Then

$$\begin{aligned} n a_n &= (n+1)u_n - n u_{n-1}, \\ &= u_n + n\phi_n, \\ &= \sum_{\nu=1}^{n} \phi_\nu + n\phi_n. \end{aligned}$$

Thus

$$a_n = \frac{1}{n} \sum_{\nu=1}^{n} \phi_\nu + \phi_n, \quad n \geqq 1. \tag{3.5}$$

Substituting with (3.5) in (1.4), we find that for $n \geqq 1$,

$$\begin{aligned} b_n &= \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} k(\Delta^{n-k}\mu_k)\left\{\frac{1}{k} \sum_{\nu=1}^{k} \phi_n + \phi_k\right\} \\ &= B + D \quad \text{(say)}. \end{aligned} \tag{3.6}$$

But

$$B = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(\Delta^{n-k}\mu_k)\sum_{\nu=1}^{k}\phi_\nu$$

$$= \frac{1}{n}\sum_{\nu=1}^{n}\phi_\nu\sum_{k=\nu}^{n}\binom{n}{k}(\Delta^{n-k}\mu_k)$$

$$= \frac{1}{n}\sum_{\nu=1}\phi_\nu\left[1 - \sum_{k=0}^{\nu-1}\binom{n}{k}(\Delta^{n-k}\mu_k)\right]$$

$$= \frac{1}{n}\sum_{\nu=1}^{n}\phi_\nu\left[1 - \int_0^1 d\chi(t)\sum_{k=0}^{\nu-1}\binom{n}{k}t^k(1-t)^{n-k}\right].$$

Using an argument similar to that used in deducing (2.16), and integration by parts, we obtain

$$B = \frac{1}{n}\sum_{\nu=1}^{n}\phi_\nu\left[1 - \nu\binom{n}{\nu}\int_0^1 \chi(t)t^{\nu-1}(1-t)^{n-\nu}dt\right]. \tag{3.7}$$

Also

$$D = \frac{1}{\mathrm{n}}\sum_{\nu=1}^{n}\binom{n}{\nu}\nu\phi_\nu\int_0^1 d\chi(t)t\ (1-t)^{n-\nu}. \tag{3.8}$$

Integrating the integral in (3.8) by parts, and using (2.15), it follows that

$$D = \frac{1}{n}\sum_{\nu=1}^{n}\nu\phi_\nu\int_0^1 \chi(t)\left\{(\nu+1)\binom{n}{\nu+1}t^\nu(1-t)^{n-1-\nu} - \nu\binom{n}{\nu}t^{\nu-1}(1-t)^{n-\nu}\right\}dt + \phi_n. \tag{3.9}$$

(The extra term ϕ_n occurs in (3.9) since in the integration by parts of the integral in (3.8), the term $(1-t)^{n-\nu}$ does not vanish at $t = 1$ in the case $\nu = n$.) It thus follows from (3.5), (3.6), (3.7) and (3.9) that

$$b_n - a_n = \frac{1}{n}\sum_{\nu=1}^{n}\phi_\nu\left[\int_0^1 \chi(t)\left\{\nu(\nu+1)\binom{n}{\nu+1}t\ (1-t)^{n-1-\nu} - \nu\ (\nu+1)\binom{n}{\nu}t^{\nu-1}(1-t)^{n-\nu}\right\}dt\right].$$

Thus

$$b_n - a_n = \sum_{\nu=1}^{n}\binom{n-1}{\nu-1}\phi_\nu\int_0^1 \frac{\chi(t)}{}\left\{(n-\nu)t^{\nu+1}(1-t)^{n-1-\nu} - (\nu+1)t^\nu(1-t)^{n-\nu}\right\}dt. \tag{3.10}$$

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Now (3.10) is a transformation of the type considered in the Lemma, and for $n \geqq 1$,

$$(3.11)\quad \alpha_{n,\nu} = \begin{cases} 0, \quad (\nu > n) \\ \binom{n-1}{\nu-1} \int_0^1 \frac{\chi(t)}{t} \\ \times \{(n-\nu)t^{\nu-1}(1-t)^{n-1-\nu} - (\nu+1)t^{\nu}(1-t)^{n-\nu}\}dt, \ (\nu \leqq n-1), \\ -(\nu+1)\int_0^1 t^{\nu-1}\chi(t)dt, \quad (\nu = n). \end{cases}$$

It follows by integration by parts that for $1 \leqq \nu \leqq n-1$,

$$(3.12)\qquad \alpha_{n,\nu} = \binom{n-1}{\nu-1} \int_0^1 (1-t)^{n-\nu} t^{\nu+1} a\left(\frac{\chi(t)}{t}\right).$$

Thus, the conditions of the Lemma are satisfied with

$$(3.13)\qquad A = \sup_{\nu} \psi_{\nu}.$$

where

$$(3.14)\qquad \psi_{\nu} = \sum_{n=\nu}^{\infty} |\alpha_{n,\nu}|.$$

Now, we can deduce at once from (3.11), (3.12) and (3.14) that

$$(3.15)\qquad \begin{aligned} \psi_{\nu} &\leqq (\nu+1)\int_0^1 t^{\nu-1}|\chi(t)|dt \\ &\quad + \int_0^1 \left\{\sum_{n=\nu+1}^{\infty} \binom{n-1}{\nu-1} t^{\nu+1}(1-t)^{n-\nu}\right\} \left|d\left(\frac{\chi(t)}{t}\right)\right| \\ &= (\nu+1)\int_0^1 t^{\nu-1}|\chi(t)|dt + \int_0^1 t(1-t^{\nu})\left|d\left(\frac{\chi(t)}{t}\right)\right| \end{aligned}$$

with equality in the case in which (2.5) holds and $\chi(t)/t$ monotonic.

This completes the proof except for the simplification when (2.5) holds and $\chi(t)/t$ is monotonic.

If $\chi(t)/t$ is non-increasing, we may omit the modulus signs in (3.15) provided we alter the sign of the second integral. On integrating the second integral by parts we get

$$(3.16)\qquad \psi_{\nu} = \int_0^1 \frac{\chi(t)}{t}\, dt.$$

Thus, equation (3.2) follows from (3.13) and (3.16).

If $\chi(t)/t$ is non-decreasing, we omit the modulus signs in (3.15). Again integrating the second integral by parts we get

$$(3.16)\qquad \psi_{\nu} = 2(\nu+1)\int_0^1 t^{\nu-1}\chi(t)dt - \int_0^1 \frac{\chi(t)}{t}\, dt.$$

Now, since $\chi(t)/t$ is non-decreasing, it follows from (3.4) that for $0 \leqq t < 1$,

$$\chi(t)/t \leqq \chi(1-) = 1 - l.$$

Hence, the first integral in (3.16) does not exceed

$$2(\nu + 1)(1 - l) \int_0^1 t^\nu dt = 2(1 - l). \tag{3.17}$$

On the other hand, it tends to $2(1 - l)$ as $\nu \to \infty$, so that its supremum equals $2(1 - l)$. This can be seen most easily by integrating it by parts, when we find that it is equal to

$$\frac{2(\nu + 1)}{\nu}(1 - \mu_\nu). \tag{3.18}$$

Combining (3.4), (3.13) and (3.16)–(3.18), equation (3.3) clearly follows. This completes the proof of Theorem 3.1.

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Education College for Women,
Riyadh, Saudi Arabia