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Weinberg and DMO sum rules

As mentioned earlier in Subsection 2.2.7 of Part I, Weinberg and DMO sum rules are prototypes of QSSR, whilst their derivation is based on the asymptotic realization of chiral and flavour symmetries, or alternatively, in the world with massless quarks and without any interactions with external gluon fields. The convergence of these sum rules has been tested in QCD when the quark masses and non-perturbative power corrections are switched on [28,29,31,32]. The analysis has been reviewed in details in [30,3,34], where the QCD corrections to the WSR have been given explicitly.

We shall follow the notations and conventions in Subsection 2.2.7 of Part I. We shall be concerned here with the two-point correlator:

$$\begin{aligned}\Pi^{\mu\nu}_{LR}(q) &\equiv i\int d^4x\, e^{iqx}\langle 0|\mathcal{T} J^{\mu}_L(x)\left(J^{\nu}_R(0)\right)^{\dagger}|0\rangle \\ &= -(g^{\mu\nu}q^2-q^{\mu}q^{\nu})\Pi^{(1)}_{LR}(q^2)+q^{\mu}q^{\nu}\Pi^{(0)}_{LR}(q^2)\,,\end{aligned} \tag{50.1}$$

built from the left- and right-handed components of the local weak current:

$$J^{\mu}_L=\bar{u}\gamma^{\mu}(1-\gamma_5)d, \qquad J^{\mu}_R=\bar{u}\gamma^{\mu}(1+\gamma_5)d\,, \tag{50.2}$$

and/or using isospin rotation relating the neutral and charged weak currents. The indices (1) and (0) corresponds to the spins of the hadrons entering into the spectral function. In the chiral limit, the longitudinal part $\Pi^{(0)}_{LR}(q^2)$ of the two-point correlator vanishes, once the pion pole has been subtracted. The spectral function is normalized as:

$$\frac{1}{2\pi}\mathrm{Im}\Pi^{(1)}_{LR}\equiv\frac{1}{2\pi}\mathrm{Im}\Pi_{LR}\equiv\frac{1}{4\pi^2}(v-a)\,, \tag{50.3}$$

where the last term is the notation in [193,199].

50.1 Sacrosanct Weinberg sum rules (WSR) in the chiral limit

Here, we shall follow closely the discussions in [34].

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50.1.1 The sum rules

The 'sacrosanct' Weinberg sum rules read in the chiral limit:

$$
\begin{aligned}
I_0 &\equiv \int_0^\infty ds\, \frac{1}{2\pi} \mathrm{Im}\Pi_{LR} = f_\pi^2 \,, \\
I_1 &\equiv \int_0^\infty ds\, s\, \frac{1}{2\pi} \mathrm{Im}\Pi_{LR} = 0 \,, \\
I_{-1} &\equiv \int_0^\infty \frac{ds}{s}\, \frac{1}{2\pi} \mathrm{Im}\Pi_{LR} = -4L_{10} \,, \\
I_{\rm em} &\equiv \int_0^\infty ds\, \left(s\, \log \frac{s}{\mu^2} \right) \frac{1}{2\pi} \mathrm{Im}\Pi_{LR} = -\frac{4\pi}{3\alpha} f_\pi^2 \left(m_{\pi^\pm}^2 - m_{\pi^0}^2 \right) \,,
\end{aligned} \tag{50.4}
$$

where $f_\pi|_{\rm exp} = (92.4 \pm 0.26)$ MeV is the experimental pion decay constant that should be used here as we shall use data from τ-decays involving physical pions; $m_{\pi^\pm} - m_{\pi^0}|_{\rm exp} \simeq 4.5936(5)$ MeV; $L_{10} \equiv f_\pi^2 \langle r_\pi^2 \rangle/3 - F_A$ [where $\langle r_\pi^2 \rangle = (0.439 \pm 0.008) fm^2$ is the mean pion radius and $F_A = 0.0058 \pm 0.0008$ is the axial-vector pion form factor for $\pi \to e\nu\gamma$] is one of the low-energy constants of the effective chiral Lagrangian [498–502]. The last sum rule $I_{\rm em}$ is often called DMO sum rule and it governs the electro-magnetic mass shift of the pion.

It has been shown that in the case of massless quarks, the $SU(n)_L \times SU(n)_R$ chiral symmetry is not spontaneously broken by perturbative QCD radiative corrections in QCD to all orders of perturbation theory, in the framework where the Dirac matrix γ_5 anti-commutes with the remaining ones [118]. Therefore the WSR remains valid in this case.

Recent measurement of the difference between the vector and axial-vector spectral function has been performed by ALEPH/OPAL using hadronic τ-decay data [33] as shown in Fig. 25.7. This has permitted us to have a detailed analysis of the spectral part of the WSR. In order to exploit these sum rules using the ALEPH/OPAL data, we shall work with their finite energy sum rule (FESR) versions (see e.g. [28,325] for such a derivation). In the chiral limit ($m_q = 0$ and $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$), this is equivalent to truncate the LHS at t_c until which the data are available, while the RHS of the integral remains valid to leading order in the $1/t_c$ expansion in the chiral limit, because, in this limit the breaking of these sum rules by higher dimension $D = 6$ condensates, which is of the order of $1/t_c^3$, is numerically negligible [29]. The analysis of these different sum rules using the τ decay data is shown in Fig. 50.1.

50.1.2 Matching between the low- and high-energy regions

In order to fix the t_c values which separate the low and high energy parts of the spectral functions, we require that the second Weinberg sum rule (WSR) I_1 should be satisfied by the present data. As shown in Fig. 50.1, this is obtained for two values of t_c:

$$
t_c \simeq (1.4 \sim 1.5)\ \mathrm{GeV}^2 \quad \text{and} \quad (2.4 \sim 2.6)\ \mathrm{GeV}^2 \,. \tag{50.5}
$$

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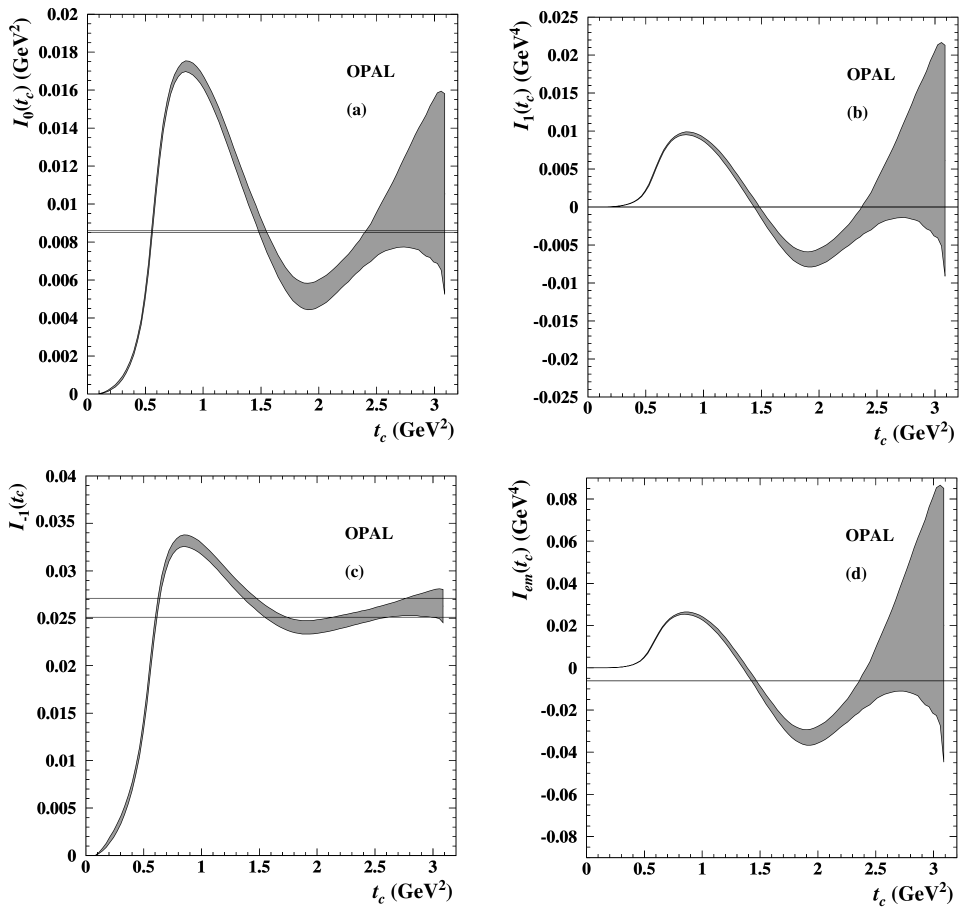

Fig. 50.1. Measurements of the different WSR until an energy cut t_c from τ-decay data by OPAL [33]. A similar result has been obtained by ALEPH. The RHS of the sum rules is given by the straight line ($\pm 1\sigma$) when two lines are present.

Although the second value is interesting from the point of view of the QCD perturbative calculations (better convergence of the QCD series), its exact value is strongly affected by the inaccuracy of the data near the τ-mass (with the low values of the ALEPH/OPAL data points, the second Weinberg sum rule is only satisfied at the former value of t_c).

After having these t_c solutions, we can improve the constraints by requiring that the first Weinberg sum rule I_0 reproduces the experimental value of f_π[1] within an accuracy that is twice the experimental error. This condition allows us to fix t_c in a very narrow margin due

[1] Although we are working here in the chiral limit, the data are obtained for physical pions, such that the corresponding value of f_π should also correspond to the experimental one.

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to the sensitivity of the result on the changes of t_c values:[2]

$$t_c = (1.475 \pm 0.015)\ \mathrm{GeV}^2\ . \tag{50.6}$$

50.2 $L_{10}, m_{\pi^\pm} - m_{\pi^0}$ and f_π in the chiral limit

Using the previous value of t_c into the I_{-1} sum rule, we deduce:

$$L_{10} \simeq -(6.26 \pm 0.04) \times 10^{-3}\ , \tag{50.7}$$

which agrees quite well with more involved analysis including chiral symmetry breakings [651,33], and with the one using a lowest meson dominance (LMD) of the spectral integral [500].

Analogously, one obtains from the $I_{em}(t_c)$ FESR:

$$\Delta m_\pi \equiv m_{\pi^\pm} - m_{\pi^0} \simeq (4.84 \pm 0.21)\ \mathrm{MeV}\ . \tag{50.8}$$

This result is 1σ higher than the data 4.5936(5) MeV, but agrees within the errors with the more detailed analysis from τ-decays [30,33] and with the LMD result of about 5 MeV [500]. We have checked that moving the subtraction point μ from 2 to 4 GeV slightly decreases the value of Δm_π by 3.7%, which is relatively weak as expected. Indeed, in the chiral limit, the μ dependence does not appear in the RHS of the I_{em} sum rule, and so it looks natural to choose:

$$\mu^2 = t_c\ , \tag{50.9}$$

because t_c is the only external scale in the analysis. At this scale the result increases slightly by 2.5%. One can also notice that the prediction for Δm is more stable when one changes the value of $t_c = \mu^2$. Therefore, the final predictions from the value of t_c in Eq. (50.6) fixed from the first and second Weinberg sum rules are:

$$\begin{aligned} \Delta m &\simeq (4.96 \pm 0.22)\ \mathrm{MeV}\ , \\ L_{10} &\simeq -(6.42 \pm 0.04) \times 10^{-3}\ , \end{aligned} \tag{50.10}$$

which we consider as our 'best' predictions.

For some more conservative results, we also give the predictions obtained from the second t_c–value given in Eq. (50.5). In this way, one obtains:

$$\begin{aligned} f_\pi &= (87 \pm 4)\ \mathrm{MeV}\ , \\ \Delta m &\simeq (3.4 \pm 0.3)\ \mathrm{MeV}\ , \\ L_{10} &\simeq -(5.91 \pm 0.08) \times 10^{-3}\ , \end{aligned} \tag{50.11}$$

where one can notice that the results are systematically lower than those obtained in Eq. (50.10) from the first t_c value given previously, which may disfavour a posteriori the second choice of t_c values, although, in principle, we do not have a strong argument favouring one

[2] For the second set of t_c-values in Eq. (50.5), one obtains a slightly lower value: $f_\pi = (84.1 \pm 4.4)$ MeV.

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with respect to the other. However, approach based on $1/N_c$ expansion and a saturation of the spectral function by the lowest state within a narrow width approximation (NWA) as discussed in Section 43.2 favours the former value of t_c given in Eq. (50.6) [500]. A similar value of t_c is also obtained from the FESR constraint using the naïve duality ansatz of the vector spectral function. Taking as a conservative value the largest range spanned by the two sets of results, one obtains:

$$\begin{aligned} f_\pi &= (86.8 \pm 7.1)\ \text{MeV}\ , \\ \Delta m &\simeq (4.1 \pm 0.9)\ \text{MeV}\ , \\ L_{10} &\simeq -(5.8 \pm 0.2) \times 10^{-3}\ , \end{aligned} \tag{50.12}$$

which we found to be quite satisfactory in the chiral limit. The previous tests are very useful, as they will allow us to gauge the confidence level of the sum rule predictions in the following chapters.

50.3 Masses and power corrections to the Weinberg sum rules

It has been shown [28,29,31,32] that:

- The $SU(n)_L \times SU(n)_R$ chiral symmetry is broken by massive quarks. The first WSR is broken to order $\alpha_s\, m_u m_d$ but is still convergent, whereas the second WSR is not convergent in its mathematical sense. However, this non-convergence does not affect the success of the A_1 mass prediction from the second WSR, as phenomenologically the light running quark mass effects are small.
- The $SU(n)_L \times SU(n)_R$ chiral symmetry is broken spontaneously by the dimension-six four-quark condensate, which affects the WSR. However, the effect is relatively small and vanishes as $1/q^4$, where q^2 is the typical scale of the sum rule.

Using the QCD expressions of the vector and axial-vector two-point correlators given in Part VIII from [325], it is easy to derive the different power corrections to the Weinberg sum rules. Introducing the running coupling $\bar{\alpha}_s$ and masses $\bar{m}_i$ evaluated at Q^2, one can deduce for the spin $1+0$ combination:

$$\begin{aligned} Q^2\big[\Pi^{(1+0)}_{ij,LR}\big]^{(D=4)} &= -\frac{1}{\pi^2}\left(\frac{\bar{\alpha}_s}{\pi}\right)\bar{m}_i\bar{m}_j \\ Q^4\big[\Pi^{(1+0)}_{ij,LR}\big]^{(D=4)} &= \frac{4}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right)\langle m_j\bar{\psi}_i\psi_i + m_i\bar{\psi}_j\psi_j\rangle - \frac{8}{7\pi^2}\bar{m}_i\bar{m}_j\big[\bar{m}_i^2 + \bar{m}_j^2\big] \\ Q^6\big[\Pi^{(1+0)}_{ij,LR}\big]^{(D=6)} &= 8\pi\left(\frac{\bar{\alpha}_s}{\pi}\right)[\langle(\bar{\psi}_i\gamma_\mu T_a\psi_j)(\bar{\psi}_j\gamma_\mu T_a\psi_i)\rangle \\ &\quad -\langle(\bar{\psi}_i\gamma_\mu\gamma_5 T_a\psi_j)(\bar{\psi}_j\gamma_\mu\gamma_5 T_a\psi_i)\rangle] \\ &\simeq -\frac{64\pi}{9}\rho\alpha_s\langle\bar{u}u\rangle^2 \\ Q^8\big[\Pi^{(1+0)}_{ij,LR}\big]^{(D=8)} &\approx 8\pi\alpha_s M_0^2\langle\bar{u}u\rangle^2\ , \end{aligned} \tag{50.13}$$

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where $\rho = 1$ in the large N_c-limit; $T_a \equiv \lambda_a/2$ is the $SU(3)_c$ matrix defined in Appendix B; $M_0^2 \simeq 0.8$ GeV^2 is the scale introduced in Chapter 27 in order to parametrize the mixed condensate. The $D = 8$ contribution comes from [652]. For the spin 0 component of the two-point function, one can also deduce from [325]:

$$
\begin{aligned}
Q^2\left[\Pi^{(0)}_{ij,LR}\right]^{(D=2)} &= \frac{3}{2\pi^2} m_i m_j \left[\ln \frac{Q^2}{\nu^2} + \mathcal{O}(1)\right] \\
Q^4\left[\Pi^{(0)}_{ij,LR}\right]^{(D=4)} &= \langle (m_i - m_j)(\bar\psi_i\psi_i - \bar\psi_j\psi_j)\rangle - \langle (m_i + m_j)(\bar\psi_i\psi_i + \bar\psi_j\psi_j)\rangle \\
&\quad + \frac{1}{4\pi^2}\left[-\frac{12}{7}\left(\frac{\bar\alpha_s}{\pi}\right)^{-1} + \frac{11}{14}\right]\left\{[\bar m_i - \bar m_j]\left[\bar m_i^3 - \bar m_j^3\right]\right. \\
&\quad \left. - [\bar m_i + \bar m_j]\left[\bar m_i^3 + \bar m_j^3\right]\right\} \\
&\quad - \frac{3}{4\pi^2}\bar m_i \bar m_j [\bar m_i - \bar m_j]^2 + \frac{3}{4\pi^2}\bar m_i \bar m_j [\bar m_i + \bar m_j]^2 \,. \qquad (50.14)
\end{aligned}
$$

With these expressions, it is easy to derive the QCD expressions of the different WSR. The phenomenology of these FESR sum rules and especially their Laplace transform have been explictly discussed in [3], which the readers may also consult.

50.4 DMO sum rules in QCD

The DMO sum rule which controls the $SU(n)$ flavour symmetry has been analyzed in QCD in [31], [32] and in [354]. Phenomenologically, it has been used to extract the value of the quark masses and to predict the splittings due to $SU(3)$ breakings among the mesons. In particular, its τ-like version has been used to extract the value of the running strange quark mass, which has the advantage to be model-independent. We shall come back to this point in the chapter on light quark masses.

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