COMPLEMENTED SUBSPACES AND THE HAHN–BANACH EXTENSION PROPERTY IN l_p $(0<p<1)$

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In this article, we study some questions related to the complementation and the Hahn–Banach property for subspaces of l_p, for $0<p<1$. Some results which are stated here have appeared in the work of W. J. Stiles [**4, 5**] and N. Popa [**3**], but our proofs are simpler. We solve a problem raised by Popa [**3**], concerning complemented copies of l_p contained in l_p.

We fix first some terminology. A p-Banach space $(0<p\leqq 1)$ is a real vector space X, endowed with a p-norm $\|.\|$, in the sense of G. Köthe [**1**], and complete with respect to the metric defined by the p-norm. In particular, we are concerned with l_p, which is a p-Banach space with p-norm:

$$\|(\alpha_n)_n\|_p = \sum_{n=1}^{\infty} |\alpha_n|^p$$

If $(X, \|.\|_X)$ is a p-Banach space, $(Y, \|.\|_Y)$ a q-Banach space, $0<p, q\leqq 1$, and $T:X\to Y$ a continuous linear operator, the "norm" of T is defined by:

$$\|T\| = \sup_{\|x\|_X\leqq 1} \|Tx\|_Y^{1/q},$$

and is equal to the smallest $C>0$ for which:

$$\|Tx\|_Y^{1/q} \leqq C\,\|x\|_X^{1/p} \quad \text{for all } x \text{ in } X.$$

As we have said before, our techniques are simple. Two tools are used, essentially. First, a result concerning basic sequences in p-Banach spaces, whose proof is identical to the one known for the Banach case [**2**, 1.a.9]:

1. LEMMA. *Let $(x_n)_{n=1}^{\infty}$ be a monotone normalized basic sequence in a p-Banach space X, and suppose that there is a projection P of X onto the closed linear span of $(x_n)_{n=1}^{\infty}$. If $(y_n)_{n=1}^{\infty}$ is a sequence in X satisfying:*

$$\sum_{n=1}^{\infty} \|x_n - y_n\| < \frac{1}{8\,\|P\|^p},$$

then $(y_n)_{n=1}^{\infty}$ is a basic sequence, equivalent to $(x_n)_{n=1}^{\infty}$, whose closed linear span is complemented in X.

The second tool which will be used is the Mackey topology of a p-Banach space. If $(X, \|.\|)$ is a p-Banach space whose topological dual separates the points of X, the convex hulls of the balls of $(X, \|.\|)$ form a basis of zero neighbourhoods of a locally convex topology, which is the finest locally convex topology on X whose dual is X^*, i.e. the

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Mackey topology of the dual pair $\langle X, X^* \rangle$. This topology is usually called the Mackey topology of $(X, \|.\|)$, and can be defined by the norm induced by the bidual $(X^{**}, \|.\|^{**})$. In the case $X = l_p$, $X^* = l_\infty$, and it is easy to see that the Mackey topology is induced by the l_1-norm.

If Y is a closed subspace of $(X, \|.\|)$, $(Y, \|.\|)$ is a p-Banach space, for which the bidual norm $\|.\|_Y^{**}$ defines the Mackey topology. In general, the Mackey topology of Y is finer than the topology induced by the Mackey topology of X. It is not difficult to see, using duality arguments, that both topologies coincide if and only if Y has the Hahn–Banach extension property (HBEP): every $y^* \in Y^*$ is the restriction to Y of some $x^* \in X^*$. Every complemented subspace of X has the HBEP in X. More precisely, if $P: X \to X$ is a continuous linear projection onto Y:

$$\|x\|_X^{**} \leqq \|x\|_Y^{**} \leqq \|P\| \, \|x\|_X^{**} \quad \text{for every } x \text{ in } Y.$$

In general, there are closed subspaces with HBEP which are not complemented. For l_p, we have a classical example:

2. Proposition. *There is a closed subspace of l_p $(0<p<1)$ which is not complemented but has HBEP in l_p.*

Proof. One can do the same construction that appears in **[1]**, of a non-complemented closed subspace of l_p $(1 \leqq p < \infty)$ with minor modifications. For each positive integer ν, a square matrix U_ν is considered, of order $n = 2^\nu$, which defines a linear involution on $\mathbb{R}^n$, with invariant subspace H_ν. We consider on H_ν the p-norm $\|.\|_p$, and define $X = l_p((H_\nu, \|.\|_p)_{\nu=1}^\infty)$, which is a closed subspace of $l_p((l_p^{2^\nu})_{\nu=1}^\infty)$. The proof of the fact that X is not complemented in $l_p((l_p^{2^\nu})_{\nu=1}^\infty)$ (isometric to l_p) is analogous to the one of the case $1 \leqq p < \infty$, with minor modifications. To prove that X has HBEP, it suffices to check that its Mackey topology coincides with the topology induced by the norm of $l_1((l_1^{2^\nu})_{\nu=1}^\infty)$. But this fact follows directly from the calculation of the dual:

$$X^* = l_\infty((H_\nu, \|.\|_\infty)_{\nu=1}^\infty). \quad \blacksquare$$

We will see below that under certain assumptions, every closed subspace with HBEP is complemented in l_p. Stiles has proved in **[5]** that an infinite-dimensional complemented subspace of l_p is isomorphic to l_p, so we are going to restrict ourselves to isomorphic copies of l_p. Popa **[3]** has proved that, if X is a subspace of l_p $(0<p<1)$, and there is an isometry $T: l_p \to X$, then X is complemented in l_p if and only if $\inf_n \|Te_n\|_1 > 0$, where $(e_n)_{n=1}^\infty$ is the unit basis of l_p. In the same paper, he asks if this condition is also necessary and sufficient when T is assumed to be an isomorphism. We will see here that the answer is negative, but the condition is necessary. First, we give a criterion for an isomorphic copy of l_p to have HBEP in l_p, in terms of the l_1-norm.

3. Proposition. *Let X be a subspace of l_p $(0<p<1)$, and suppose that there is an isomorphism $T: l_p \to X$. Then the following are equivalent.*

(i) *X has HBEP.*

(ii) *There exists* $C>0$ *such that, if* $\alpha_1, \ldots, \alpha_n$ *are scalars:*

$$\left\|\sum_{i=1}^{n} \alpha_i Te_i\right\|_1 \geqq C \sum_{i=1}^{n} |\alpha_i|.$$

Proof. X has HBEP if and only if the l_1-norm defines the Mackey topology of $(X, \|.\|_p)$. But, T being an isomorphism, this topology can also be defined by the norm

$$\sum_{n=1}^{\infty} \alpha_i Te_i \to \sum_{n=1}^{\infty} |\alpha_n|. \quad \blacksquare$$

The condition (ii) obviously implies $\inf_n \|Te_n\|_1 > 0$, showing that the condition proposed by Popa is necessary when T is an isomorphism. Let us see now that it is not sufficient through a counterexample inspired in [**5**].

4. Proposition. *Let* $(x_n)_{n=1}^{\infty}$ *be the sequence defined in* l_p $(0<p<1)$ *by:*

$$x_n = e_n - \tfrac{1}{2}e_{2n} - \tfrac{1}{2}e_{2n+1}.$$

Then:

(i) $\|x_n\|_p = 1+2^{1-p}$, $\|x_n\|_1 = 2$ *for every* n,

(ii) $(x_n)_{n=1}^{\infty}$ *is a basic sequence, equivalent to the* l_p*-basis,*

(iii) $\left\|\sum_{k=1}^{n} k^{-1}x_k\right\|_1 = 2$ *for every* n.

Proof. (i) is obvious.

(ii) $(x_n)_{n=1}^{\infty}$ is bounded in l_p, and so $\sum_{n=1}^{\infty} \alpha_k x_k$ converges for every sequence of scalars $(\alpha_n)_{n=1}^{\infty}$ belonging to l_p. Conversely, for every finite sequence of scalars whose number of terms is odd, $\alpha_1, \ldots, \alpha_{2n+1}$:

$$\begin{aligned}\left\|\sum_{k=1}^{2n+1} \alpha_k x_k\right\|_p &= |\alpha_1|^p + \sum_{k=1}^{n} \left|\alpha_{2k} - \frac{\alpha_k}{2}\right|^p + \sum_{k=1}^{n} \left|\alpha_{2k+1} - \frac{\alpha_k}{2}\right|^p + 2\sum_{k=n+1}^{2n+1} \left|\frac{\alpha_k}{2}\right|^p \\ &\geqq \sum_{k=1}^{n} \left[\left|\frac{\alpha_k}{2}\right|^p - |\alpha_{2k}|^p\right] + \sum_{k=1}^{n} \left[\left|\frac{\alpha_k}{2}\right|^p - |\alpha_{2k+1}|^p\right] + 2\sum_{k=n+1}^{2n+1} \left|\frac{\alpha_k}{2}\right|^p \\ &= (2^{1-p}-1)\sum_{k=1}^{2n+1} |\alpha_k|^p.\end{aligned}$$

(iii) Let $u_n = \sum_{k=1}^{n} k^{-1}x_k$. We know $\|u_1\|_1 = 2$, and hence it is enough to see $\|u_n\|_1 = \|u_{n+1}\|_1$ for every n. For instance, for $n=2k$:

$$u_{2k+1} = u_{2k} + \frac{1}{2k+1}e_{2k+1} - \frac{1}{2(2k+1)}e_{4k+2} - \frac{1}{2(2k+1)}e_{4k+3},$$

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and hence:

$$\|u_{2k+1}\|_1 = \|u_{2k}\|_1 - \frac{1}{2k} + \left|\frac{1}{2k+1} - \frac{1}{2k}\right| + \frac{1}{2k+1} = \|u_{2k}\|_1.$$

The case $n = 2k+1$ can be checked in the same way. ■

The preceding proposition shows that the closed linear hull of $(x_n)_{n=1}^{\infty}$ is isomorphic to l_p and does not have HBEP, but $\inf_n \|x_n\| > 0$, providing the announced counterexample. The failure of HBEP, which follows here from Proposition 2, can be obtained using some results of J. Lindenstrauss on l_1, as in **[5]**, but our proof is straightforward.

In certain situations, the condition $\inf_n \|Te_n\|_1 > 0$ is sufficient:

5. Proposition. *Let X be a subspace of l_p $(0<p<1)$, and $T: l_p \to X$ an isomorphism such that*:

(a) $C = \inf_n \|Te_n\|_1 > 0$,

(b) $\operatorname{supp}(Te_i) \cap \operatorname{supp}(Te_j) = \varnothing$ *for* $i \neq j$.

Then there exists a projection $P: l_p \to X$ *with* $\|P\| \leq \frac{1}{C}$.

Proof. Put $x_i = Te_i$, $\Delta_i = \operatorname{supp}(x_i)$. For every i, consider $x_i^* \in l_\infty = l_p^*$ such that:

$$\operatorname{supp}(x_i^*) \subset \Delta_i, \qquad x_i^*(x_i) = 1 \quad \text{and} \quad \|x_i^*\|_\infty = \frac{1}{\|x_i\|_1}$$

(remark that every x_i can be considered as a point in $l_p(\Delta_i)$). Then

$$\left|x_i^*\left(\sum_{n\in\Delta_i} \alpha_n e_n\right)\right|^p \leq (\|x_i^*\|_1)^{-p} \cdot \sum_{n\in\Delta_i} |\alpha_n|^p,$$

and for $u = \sum_{n=1}^{\infty} \alpha_n e_n$ in l_p, one can define:

$$Pu = \sum_{i=1}^{\infty} x_i^*\left(\sum_{n\in\Delta_i} \alpha_n e_n\right) x_i,$$

and P is a projection onto X, satisfying:

$$\|Pu\|_p \leq \sum_{i=1}^{\infty} (\|x_i\|_1)^{-p} \cdot \sum_{n\in\Delta_i} |\alpha_n|^p \leq \frac{1}{C^p} \|u\|_p. \quad ■$$

We obtain thus easily the mentioned result of Popa:

6. Corollary. *Let X be a subspace of l_p $(0<p<1)$, and $T: l_p \to X$ an isometry. The following are equivalent:*

(i) *X has HBEP,*

(ii) *X is complemented in l_p,*

(iii) $\inf_n \|Te_n\|_1 > 0$.

Proof. It suffices to remark that T satisfies the condition (b) of the preceding proposition. This fact can be shown as in the case $p=1$. ■

7. COROLLARY. *Let $(x_n)_{n=1}^{\infty}$ be a normalized block of the l_p-basis $(0<p<1)$. Then the closed linear hull of $(x_n)_{n=1}^{\infty}$ is isometric to l_p, and it is complemented in l_p if and only if* $\inf_n \|x_n\|_1 > 0$.

Proof. The first assertion can be proved as in the case $p=1$, and the other follows from the first one and the preceding corollary. ■

8. PROPOSITION. *Let X be an infinite dimensional closed subspace of l_p $(0<p<1)$. The following are equivalent.*

(i) *There is a subspace Y of X, isometric to l_p, and complemented in l_p.*

(ii) *The unit ball of X is not relatively compact in l_1.*

(iii) *There is a bounded sequence $(x_n)_{n=1}^{\infty}$ in X, such that:*

(a) $\inf_n \|x_n\|_1 > 0$,

(b) *$(x_n)_{n=1}^{\infty}$ converges to zero coordinatewise.*

Proof. i⇒ii. If the unit ball of $(X, \|.\|_p)$ is relatively compact in l_1, the same is true for any subspace Y of X. If Y has HBEP, the l_1-norm defines the Mackey topology of $(Y, \|.\|_p)$, and Y must be finite dimensional.

ii⇒iii. If the unit ball of X is not relatively compact in l_1, there is a bounded sequence $(z_n)_{n=1}^{\infty}$ in X which does not have any subsequence converging in l_1. Replacing $(z_n)_{n=1}^{\infty}$ by a subsequence if it is needed, we can suppose that $(z_n)_{n=1}^{\infty}$ converges coordinatewise to some $z \in l_p$, and assume $\inf_n \|z_n - z_{n+1}\|_1 > 0$. Then $x_n = z_n - z_{n+1}$ gives the sequence we are looking for.

iii⇒i. We can suppose $(x_n)_{n=1}^{\infty}$ normalized in l_p. It is easy to see that a subsequence $(x_{n_k})_{k=1}^{\infty}$ and a block sequence $(u_k)_{k=1}^{\infty}$ of the l_p-basis can be constructed inductively, such that:

$$\|x_{n_k} - u_k\|_p < \frac{C}{2^{4+k}} \quad \text{for every } k.$$

By virtue of (a), we can apply Corollary 7 to the sequence $(z_k)_{k=1}^{\infty}$ defined by

$$z_k = \|u_k\|_p^{-1/p} \,.\, u_k,$$

and the closed linear hull of $(u_k)_{k=1}^{\infty}$ is complemented in l_p. By the choice of the u_k's, the sequence $(x_{n_k})_{k=1}^{\infty}$ generates a closed subspace of X, isomorphic to l_p, and complemented in l_p (direct application of Lemma 1). The details are easy. ■

It is easy to see that we can assume in (i) only that Y has HBEP, and the proposition is still true. Thus:

9. COROLLARY. *Every closed, infinite dimensional subspace of l_p $(0<p<1)$ which has HBEP contains an isomorphic copy of l_p, complemented in l_p.*

10. COROLLARY. *Let X be a subspace of l_p $(0<p<1)$, and $T: l_p \to X$ an isomorphism, with $\lim_n \|Te_n\|_1 = 0$. Then X does not contain any infinite dimensional subspace complemented in l_p.*

Proof. It is easy to see that, in this situation, the unit ball of X is relatively compact in l_1. ■

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