A GENERALIZATION OF A THEOREM OF JACOBI ON SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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JACOBI proved a curious theorem regarding the solutions of the system of equations

$$\frac{dx^1}{\lambda^1} = \frac{dx^2}{\lambda^2} = \ldots = \frac{dx^n}{\lambda^n},$$

for functions $\lambda^a(x^1, \ldots, x^n)$ satisfying

$$\frac{\partial\lambda^1}{\partial x^1} + \frac{\partial\lambda^2}{\partial x^2} + \ldots + \frac{\partial\lambda^n}{\partial x^n} = 0,$$

showing that the knowledge of $n-2$ independent integrals of the system leads, with this condition, to an exact differential equation for the last integral of the system. When the coordinates are Euclidean the left member is called the divergence of the vector λ^a. If the divergence of λ^a is non-vanishing there exists a factor M such that the divergence of $M\lambda_a$ vanishes. Jacobi's "theorem of the last multiplier"[1] states that the determination of this factor is tantamount to finding the last integral of the linear system.

Here a theorem is proved regarding a special system of k vectors, which we choose to call a Jacobian system of vectors. For $k = 1$ this theorem reduces to Jacobi's theorem of the last multiplier.

1. Conventions. The symbols $\lambda^a{}_{i|}(a = 1, \ldots, n; i = 1, \ldots, k)$ will represent functions of n independent variables $x = [x^1, \ldots, x^n]$. The ordered set of functions associated with a fixed i ($a = 1, \ldots, n$) will be called a *vector*, k linearly independent vectors, a *basis*. A vector $a^r\lambda^a{}_{r|}$, the a's dependent on the x's, will be said to belong to the basis. The totality of vectors belonging to the basis constitutes a k-uple. Repeated Latin letters indicate a summation from 1 to k, repeated Greek, from 1 to n. All functions will be assumed to have such character as to satisfy the existence theorems that are applied. Only a finite number of derivatives need be assumed to exist in any case.

A coordinate transformation will be indicated formally by the equations

$$(1.1) \qquad \bar{x}^a = \bar{x}^a(x) \qquad q \equiv \left|\frac{\partial\bar{x}}{\partial x}\right| \not\equiv 0,$$

and the inverse by

Received August 18, 1949. Presented to the American Mathematical Society at the British Columbia meeting, June 19, 1948, under similar title.

[1]Goursat-Hedrick, *Mathematical Analysis*, Vol. II, Part II, Article 32.

$$x^\alpha = x^\alpha(\bar{x}) \qquad p \equiv \left|\frac{\partial x}{\partial \bar{x}}\right|, \ pq = 1. \tag{1.2}$$

We shall have occasion to use the equations

$$\lambda^\alpha{}_{i|} \frac{\partial u}{\partial x^\alpha} = 0, \tag{1.3}$$

$$\frac{\partial x^1}{\lambda^1{}_{i|}} = \frac{\partial x^2}{\lambda^2{}_{i|}} = \dots = \frac{dx^n}{\lambda^n{}_{i|}}, \tag{1.4}$$

$$\frac{\partial M \lambda^\alpha{}_{i|}}{\partial x^\alpha} = 0, \tag{1.5}$$

defining quantities u, and M under suitable conditions. When these exist they will be defined in a new coordinate system by the following conventions. The function $u(x)$ with the x's replaced from equations (1.2) will determine a function

$$\bar{u}(\bar{x}) = u(x), \tag{1.6}$$

which will represent the scalar u in the new coordinates x. The product $M(x)p$ with x replaced by (1.2) determines a representative $\overline{M}(\bar{x})$ in the new coordinate system

$$\overline{M}(\bar{x}) = M(x)p. \tag{1.7}$$

In this case M is said to be a *relative invariant of weight* 1.

Vectors $\bar{\lambda}^\alpha{}_{i|}$, representatives of $\lambda^\alpha{}_{i|}$, will be defined in a new coordinate system by the law of transformation of contravariant tensors: $\bar{\lambda}^\alpha{}_{i|} = \lambda^\rho{}_{i|} \dfrac{\partial \bar{x}^\alpha}{\partial x^\rho}$. With these conventions, the left members of (1.3) are invariant, and the left members of (1.5) are relative invariants of weight 1. The equations (1.3), (1.4) and (1.5) will imply like equations in new coordinates. If u and M are solutions of (1.3) and (1.5), $\bar{u}$ and $\overline{M}$ will be solutions of their representatives in the new coordinates. If $u = c$ is an integral of (1.4), $\bar{u} = c$ will be a representative integral in the new coordinates.

2. Complete basis. From the two contravariant vectors $\lambda^\alpha{}_{i|}$ and $\lambda^\alpha{}_{j|}$ an *associated contravariant* vector is defined by the equations

$$T^\beta{}_{ij|} \equiv \lambda^\alpha{}_{i|} \frac{\partial \lambda^\beta{}_{j|}}{\partial x^\alpha} - \lambda^\alpha{}_{j|} \frac{\partial \lambda^\beta{}_{i|}}{\partial x^\alpha}. \tag{2.1}$$

When the associate vectors of all pairs in a basis belong to the k-uple, the basis will be said to be *complete*. This is in agreement with the classical terminology that the system (1.3) is complete when the equations

$$T^\beta{}_{ij|} \frac{\partial u}{\partial x^\beta} = 0, \tag{2.2}$$

https://doi.org/10.4153/CJM-1950-040-6 Published online by Cambridge University Press

are dependent on (1.3). Similarly, when the associate vectors are null vectors the basis is said to be Jacobian. Some theorems in the theory of the linear systems of partial differential equations (1.3) will be restated in terms of these definitions:[2]

(2.3) *A system of k linearly independent vectors is always complete if $k = n$. If $k < n$ and the system is not complete, vectors $T^{\beta}{}_{ij|}$ may be adjoined to the system to form a set of $k' > k$ independent vectors. When the new system is not complete the process may be repeated until a complete system is obtained.* Completeness is a property of the k-uple.

(2.4) *A complete k-uple has bases that are Jacobian.* This is a property of a basis.

(2.5) *These properties are invariant under coordinate transformations.*

3. Normal form for a complete basis. The equation of (1.3) with $i = 1$ has $n - 1$ independent solutions $\phi^{A}(x)(A = 2, \ldots, n)$. Adjoin to these a function $\phi^1(x)$ such that the n functions are functionally independent. In new coordinates $\bar{x}^{\alpha} = \phi^{\alpha}(x)$ $(\alpha = 1, \ldots, n)$; this equation has solutions $\bar{x}^{A}$. Hence $\bar{\lambda}^{A}{}_{1|} \equiv 0$. Since $\lambda^{\alpha}{}_{1|}$ is a non-null vector $\bar{\lambda}^{\alpha}{}_{1|}$ is non-null and $\bar{\lambda}^{1}{}_{1|} \not\equiv 0$. Consequently there is no loss in generality in taking $\lambda^{1}{}_{1|}, 0, \ldots, 0$ as the components of the first vector in the original coordinates. By a subsequent transformation of coordinates $\bar{\lambda}^{\alpha}{}_{1|} = \lambda^{\beta}{}_{1|} \dfrac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} = \lambda^{1}{}_{1|} \dfrac{\partial \bar{x}^{\alpha}}{\partial x^{1}}$. By choosing $\bar{x}^{A}$ independent of x^1 and $\bar{x}^1 = \int dx^1/\lambda^{1}{}_{1|}$, the vector transforms to $1, 0, \ldots, 0$, and the corresponding equation takes the form $\dfrac{\partial u}{\partial x^1} = 0$.

Because of the hypothesis that the vectors form a complete basis the equations (1.3) have $n - k$ solutions ϕ^{A} $(A = k + 1, \ldots, n)$ that are now independent of x^1. Adjoining functions $\phi^1 \equiv x^1$, ϕ^{B} $(B = 2, \ldots, k)$ independent of x^1, so that $\phi^{\alpha}(\alpha = 1, \ldots, n)$ are independent, a transformation of coordinates may be defined by $\bar{x}^{\alpha} = \phi^{\alpha}(x)$. In the new coordinates the equations (1.3) are satisfied by $\bar{x}^{A}$, which implies that $\bar{\lambda}^{A}{}_{i|} = 0$. The components of the vector $1, 0, \ldots, 0$, are unchanged by this transformation. Hence:

(3.1) A *complete k-basis can be transformed to*

$$\lambda^{\alpha}{}_{1|} \equiv \delta^{\alpha}{}_{1} \ (\alpha = 1, \ldots, n), \qquad \lambda^{\alpha}{}_{i|} \equiv 0 \ (\alpha > k;\, i = 2, \ldots, k).$$

4. Normal form for a Jacobian system. It will be proved that:

(4.1) *A coordinate system exists in which a Jacobian system takes the normal form*

$$\lambda^{\alpha}{}_{i|} \equiv \delta^{\alpha}{}_{i} \qquad (i = 1, \ldots, k;\, \alpha = 1, \ldots, n).$$

[2]Goursat-Hedrick, *op. cit.*, Section 89, p. 267.

To construct a proof by induction let $h - 1 < k$ of the vectors be assumed to be in the form of the theorem. The Jacobian condition $T^{\beta}{}_{ij|} = 0$, implies on some remaining vector $\lambda^{\alpha}{}_{h|}$ that:

$$\frac{\partial \lambda^{\alpha}{}_{h|}}{\partial x^{i}} = 0 \qquad (\alpha = 1, \ldots, n;\ i = 1, \ldots, h - 1) \tag{4.2}$$

so that the components $\lambda^{\alpha}{}_{h|}$ are functions of $y \equiv [x^h, \ldots, x^n]$. The equations (1.4) $i = h$ have integrals $\phi^{\alpha} = c^{\alpha}$, $\alpha \neq h$ such that

$$\phi^{A} \equiv x^{A} - f^{A}(y) \qquad (A = 1, \ldots, h - 1),$$

$$\phi^{A} \equiv \phi^{A}(y) \qquad (A = h + 1, \ldots, h).$$

Let $\phi^h(y)$ be any function such that a proper transformation of coordinates may be defined by $\bar{x}^{\alpha} = \phi^{\alpha}(x, y)$: In the new coordinates only the hth component of $\lambda^{\alpha}{}_{h|}$ is non-vanishing, and it is a function of the variables y so may be reduced to unity by a transformation on these variables. These transformations do not affect the components of the vectors $\lambda^{\alpha}{}_{i|}(i = 1, \ldots, h - 1)$. This completes the induction and the theorem follows.

5. Multipliers. A function M, satisfying an equation (1.5) has been called by Lagrange, a *multiplier* of the vector $\lambda^{\alpha}{}_{i|}$. In this case the vector $M\lambda^{\alpha}{}_{i|}$ is said to be solenoidal. To investigate the conditions that the system of k vectors admit the same multiplier, set $\mu_i \equiv -\dfrac{\partial \lambda^{\alpha}{}_{i|}}{\partial x^{\alpha}}$ and define the dependent variable M implicitly by an unknown function $Q(x, M) = 0$. The equations then take the homogeneous form

$$\lambda^{\alpha}{}_{i|} \frac{\partial Q}{\partial x^{\alpha}} + M\mu_i \frac{\partial Q}{\partial M} = 0. \tag{5.1}$$

Every solution Q of these equations that depends on M yields, with $Q = 0$, a solution M of (1.5). Every solution $M \equiv \phi(x)$ of (1.5) gives a $Q \equiv M - \phi(x)$ satisfying (5.1) for $Q = 0$. The problem of solving (1.5) for M therefore reduces to the problem of finding solutions of (5.1) that are dependent on M.

The completeness conditions of (5.1), the analogues of (2.2) are

$$T^{\beta}{}_{ij|} \frac{\partial Q}{\partial x^{\beta}} + Mt_{ij} \frac{\partial Q}{\partial M} = 0, \tag{5.2}$$

$$t_{ij} \equiv \lambda^{\alpha}{}_{i|} \frac{\partial \mu_j}{\partial x^{\alpha}} - \lambda^{\alpha}{}_{j|} \frac{\partial \mu_i}{\partial x^{\alpha}} \equiv -\frac{\partial}{\partial x^{\alpha}} T^{\alpha}{}_{ij|}.$$

The coefficients of $\dfrac{\partial Q}{\partial M}$ in (5.2) and in (5.1) are the same functions of the remaining differential coefficients, hence (5.2) may be assumed to be included in (5.1) which consequently, when integrable, may be assumed to be complete.

https://doi.org/10.4153/CJM-1950-040-6 Published online by Cambridge University Press

This requires that the basis $\lambda^{\alpha}{}_{i|}$ is also complete. The converse is not true. But when the basis is complete and (5.1) is not complete an equation $\dfrac{\partial Q}{\partial M} = 0$ may be deduced as an essential condition on a solution of (5.1). These facts may be summarized in the theorem:

(5.3) *Sufficient conditions that a basis admit a multiplier are that the basis is complete, that is, that functions $a^{r}{}_{ij}$ exist such that*

$$T^{\alpha}{}_{ij} \equiv a^{r}{}_{ij}\lambda^{\alpha}{}_{r|},$$

and that these functions also reduce the equations

$$\frac{\partial T^{\beta}{}_{ij}}{\partial x^{\beta}} = a^{r}{}_{ij} \frac{\partial \lambda^{\beta}{}_{r|}}{\partial x^{\beta}},$$

to identities. These conditions are necessary when the basis has been completed.

These conditions are satisfied for Jacobian bases. The $a^{r}{}_{ij}$ being identically equal to zero, hence:

(5.4) *Each Jacobian basis of a complete k-uple admits a common multiple M such that the contravariant vectors of weight* 1, $M\lambda^{\alpha}{}_{i|}$ *are solenoidal.*

6. Vector product. The vector product (non-metric) of $n - 1$ vectors may be defined by the covariant vector of weight -1 :

(6.1) $$\lambda_{\alpha} \equiv \epsilon_{\alpha\sigma_1 \ldots \sigma_n}\lambda^{\sigma_1}{}_{1|} \ldots \lambda^{\sigma_k}{}_{k|}, \qquad k = n - 1.$$

For a scalar μ of weight 1, $\mu\lambda_{\alpha}$ is a covariant vector and

(6.2) $$a_{\alpha\beta} \equiv \frac{\partial \mu\lambda_{\alpha}}{\partial x^{\beta}} - \frac{\partial \mu\lambda_{\beta}}{\partial x^{\alpha}},$$

is the covariant tensor known as the curl. From (6.1) it appears that

(6.3) $$\lambda^{\alpha}{}_{i|}\lambda_{\alpha} = 0 \qquad (i = 1, \ldots, k = \mu - 1).$$

Conversely these equations determine λ_{α} to within a factor of proportionality. By differentiating these the definition (6.2) leads to

(6.4) $$\mu T^{\beta}{}_{ij|}\lambda_{\beta} \equiv a_{\alpha\beta}\lambda^{\alpha}{}_{i|}\lambda^{\beta}{}_{j|}.$$

The elimination of the factor μ from these equations gives

(6.5) $$T^{\beta}{}_{ij|}\lambda_{\beta} \equiv \left(\frac{\partial \lambda_{\alpha}}{\partial x^{\beta}} - \frac{\partial \lambda_{\beta}}{\partial x^{\alpha}}\right)\lambda^{\alpha}{}_{i|}\lambda^{\beta}{}_{j|}.$$

From (6.1) it is apparent that the vanishing of the left members of either of these sets of identities implies that the $(n - 1)$-uple be complete. For μ to be an integrating factor of $\lambda_{\alpha}dx^{\alpha}$ that is, for $\mu\lambda_{\alpha}$ to be a gradient, it is necessary and sufficient that $a_{\alpha\beta} = 0$. Then by (6.4) the basis is complete. For a complete $n-1$ basis in normal form, by Theorem (3.1) $\lambda_\alpha = 0$ ($\alpha = 1, \ldots, n-1$), $\lambda_n \neq 0$. Choosing $\mu = \phi/\lambda_n$, ϕ an arbitrary function of x^n, $\mu\lambda_\alpha$ is a gradient. Hence

(6.6) *The necessary and sufficient condition that the vector product of an $n-1$ basis be proportional to a gradient is that the basis be complete.*

This theorem may be stated in the equivalent form:

(6.7) *The necessary and sufficient conditions that the vector field λ_α be lamellar is that the basis be complete.*

The vector product of $k = n-1$ gradients may be defined by the relative contravariant tensor of weight 1

$$\lambda^\alpha \equiv \epsilon^{\alpha\sigma_1 \cdots \sigma_k} \frac{\partial u_1}{\partial x^{\sigma_1}} \cdots \frac{\partial u_n}{\partial x^{\sigma_k}} \tag{6.8}$$

where $u_1, \ldots, u_k$ are $n-1$ scalars. It is interesting to compare Theorem (6.7) with the well known theorem[3] that λ^α is solenoidal, and that any solenoidal vector is the vector product of $n-1$ gradients.

7. Generalization of a theorem of Jacobi. When the Jacobian system of $k = n-1$ vectors $\lambda^\alpha{}_{i|}$ is represented in the normal form (4.1), their vector product $\lambda_\alpha \equiv \delta_{\alpha n}$ and all factors μ are given by $\mu \equiv \phi(x^n)$, ϕ being any integrable function. All multipliers of the basis are given by $M \equiv \phi(x^n)$ and therefore:

(7.1) *A Jacobian basis with $n-1$ vectors $\lambda^\alpha{}_{i|}$ has multipliers M. For all such the vectors $M\lambda^\alpha{}_{i|}$ are solenoidal and $M\lambda_\alpha$ is a gradient. Conversely all factors M such that $M\lambda_\alpha$ is a gradient imply that the vectors $M\lambda^\alpha{}_{i|}$ are solenoidal.*

A system of contravariant vectors satisfying the hypotheses of (7.1) may be obtained as follows: Let $\phi^{k+2}, \ldots, \phi^n$ be $n-k-1$ integrals of a Jacobian system (1.3). Adjoin functions so that ϕ^α are n independent functions. The transformation $x^\alpha = \phi^\alpha(x)$ reduces this system to a Jacobian system of k equations in $k+1$ independent variables. With $k+1$ playing the role of n the conditions of the theorem are satisfied.

Let $\theta = C$ be the integral of the exact equation $\mu\lambda_\alpha dx^\alpha = 0$; then $\dfrac{\partial\theta}{\partial x^\alpha} = \mu\lambda_\alpha$. It follows from (6.3) that

$$\lambda^\alpha{}_{i|} \frac{\partial\theta}{\partial x^\alpha} \equiv 0, \tag{7.2}$$

and $\theta(x)$ is the "last" solution of the system (1.3). Although the index α is assumed to run from 1 to $k+1$ in these equations, it may as well run from

[3]Goursat-Hedrick, *op. cit.*

1 to n, the remaining terms vanishing. The equations are invariant and imply the following theorem:

(7.3) *Every system of equations of the form* (1.3) *is equivalent to a complete system*

$$\mu^{\alpha}{}_{i|}(x)\frac{\partial u}{\partial x^{\alpha}} = 0 \qquad (\alpha = 1, \ldots, n, i = 1, \ldots, h < n),$$

such that the vectors $\mu^{\alpha}{}_{i|}$ *admit a common multiplier M for which*

$$\frac{\partial}{\partial x^{\alpha}}\,(M\mu^{\alpha}{}_{i|}) = 0.$$

The system has $n - h$ *independent solutions: and a knowledge of* $n - h - 1$ *independent integrals, together with such a multiplier M, leads to an exact differential equation for the last solution.*

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