THE FIVE AND SIX DIMENSIONAL MAGIC HYPERCUBES OF ORDER 3

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1. Introduction. A recent publication entitled "Magic Squares and Cubes", by W. S. Andrews, published by the Dover Publications, New York, is an excellent authoritative book on magic squares, and magic cubes. In fact, chapter XIV entitled "Magic Octrahedroids", has shown examples of the extension into four-dimensional space. Andrews' method seems to be that of extending symmetrical considerations, which he calls reversions, in forming higher dimensional forms of magic squares. Neither he, nor any other author, to the best of my knowledge, has extended magic squares to higher than four dimensions.

La Hire must be given full credit for his method of breaking down magic squares into component squares, and conversely constructing magic squares from component squares. The approach of this author is to extend La Hire's method for n-dimensional space by means of a relation:

$$(A_{x_1 x_2 \ldots x_n}, 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \{a^{[j]}_{x_1 x_2 \ldots x_n}\} \cdot \mathrm{col}\{m^j\} \qquad \ldots (1)$$

where:

a) $A_{x_1 x_2 \ldots x_n}$ refers to the magic number to be assigned to the coordinate position $(x_1, x_2, \ldots, x_n)$. The magic numbers

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are of the set 1, 2, ..., m^n, where m is the order of the space, and n is the dimension.

b) $a^{[j]}_{x_1 x_2 \ldots x_n}$ refers to the component number assigned to the coordinate position $(x_1, x_2, \ldots, x_n)$, in the j'th component hypercube. The component numbers are taken from m^{n-1} sets of 0, 1, ..., m-1. The value j ranges from n-1, n-2, ..., 0.

c) $\{m^j\}$ refers to vector $(m^{n-1}, m^{n-2}, \ldots, m, 1)$.

d) $\{a^{[j]}_{x_1 x_2 \ldots x_n}\}$ refers to vector:

$$(a^{[n-1]}_{x_1 x_2 \ldots x_n}, a^{[n-2]}_{x_1 x_2 \ldots x_n}, \ldots, a^{[0]}_{x_1 x_2 \ldots x_n}) .$$

Knowing that it would be cumbersome to have n different component hypercubes for any n-dimensional hypercube, a successful attempt was made in choosing the [0] component hypercube as a basis, and relating all other n-1 [j] component hypercubes to it. This was accomplished by a simple rotation of the basic [0] component hypercube about its main n-agonal* n-1 times, and using these n hypercubes as components. Simply this reduces to:

$$a^{[j]}_{x_1 x_2 \ldots x_n} = a^{[0]}_{x_{1+j} x_{2+j} \ldots x_j} = \sigma^j . a^{[0]}_{x_1 x_2 \ldots x_n}$$

where σ indicates the operation of one rotation, σ^j indicates j rotations, (including σ^0 meaning no rotations). A note must be mentioned here, that in the case of 2 dimensions, this so-called rotation is a pure reflection across the main diagonal.

* Use 2-agonal for square, 3-agonal for cube, n-agonal for hypercube.

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Further, in the case of a 3x3 magic square, this relation between the two components does not hold. However, some higher order squares, and all examples shown here (except figure 1) bear this relationship. Hence relation (1) reduces to

$$(A_{x_1 x_2 \ldots x_n}, 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \{\sigma^j . a^{[0]}_{x_1 x_2 \ldots x_n}\} . \mathrm{col}\{m^j\} \quad \ldots (2)$$

The construction of a component hypercube is relatively easy, ensuring that each i-row* and each n-agonal of the component hypercube total m(m-1)/2. For the case of a 6x6 magic square, Tarey has shown that no Euler square of order 6 exists, which shows that another type of component square other than an Euler square must be chosen.

The magic hypercube of dimension n, and order m will contain:

a) The numbers 1, 2, ..., m^n arranged in an ordered array,

b) $n.m^{n-1}$ rows, that is m^{n-1} i-rows, each with a total of $S = \dfrac{m(m^n + 1)}{2}$, the magic sum,

c) 2^{n-1} n-agonals, running from corner to opposite corner through the center of the hypercube, each with a total S.

2. Representation. Before exhibiting the 5 and 6-dimensional examples, it would be worthwhile to show the 2 and 3 dimensional examples, and a new representation, other than W. S. Andrews' way, of the four dimensional example.

* Define an i-row to be a row parallel to the x_i axis.

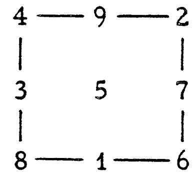

Figure 1. The 2-dimensional magic hypercube of order 3. S = 15. Broken lines have been used only on the outline of the square.

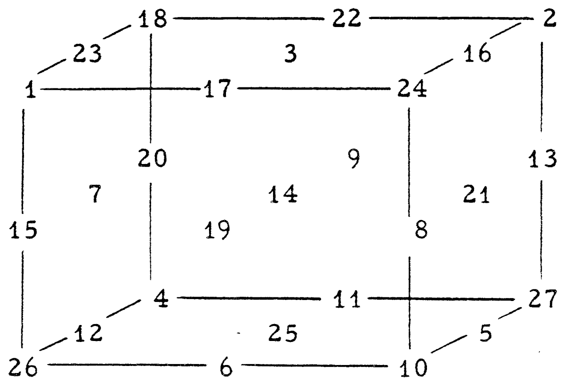

Figure 2. The 3-dimensional magic hypercube of order 3. S = 42. Broken lines have been used on the outline of the cube.

In the example of the 3-dimensional magic hypercube of order 3, (magic cube). All 1-rows may be considered parallel to a set such as 1, 17 and 24; all 2-rows may be considered parallel to a set such as 1, 23, 18; all 3-rows may be considered parallel to a set such as 1, 15, and 26. There are four 3-agonals defined by the sets |1, 14, 27| , |18, 14, 10| , | 2, 14, 26| , and |24, 14, 4| .

In the example of the 4-dimensional magic hypercube of order 3, (magic tessaract), all 1-rows may be considered parallel to a set such as 1, 80, 42; all 2-rows may be considered parallel to a set such as 1, 72, 50; all 3-rows may be considered parallel to a set such as 1, 54, 68; all 4-rows may be considered parallel to a set such as

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1, 78, 44. There are eight 4-agonals defined by the sets |1, 41, 81|, |44, 41, 38|, |57, 41, 25|, |50, 41, 32|, |61, 41, 21|, |14, 41, 68|, |73, 41, 9| and |42, 41, 40|. It is interesting to note that this example of the 4-dimensional magic hypercube of order 3 is different than the example given by W. S. Andrews.

The Magic Tessaract (3rd order)

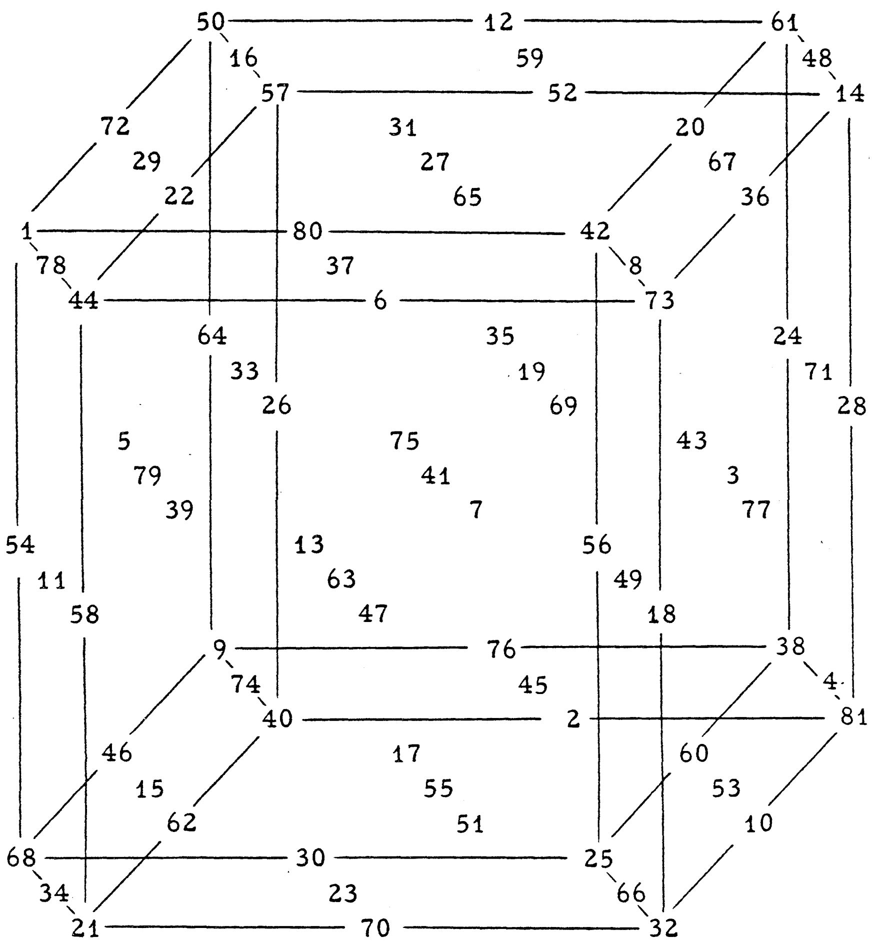

Figure 3. The 4-dimensional magic hypercube of order 3. S = 123. Broken lines have been used on the outline of the tessaract.

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Although W. S. Andrews in his book states on page 351, "for rows of numbers can be arranged side by side to represent a visible square, squares can be piled one upon another to make a visible cube, but cubes cannot be so combined in drawing as to picture to the eye their higher relations", figure 3, in the opinion of the author, seems to "picture to the eye the higher relation".

In considering figure 4, consider figures 4a, 4b and 4c as one diagram. A 3-dimensional model could be made by placing 4a, 4b, and 4c one upon another, and joining the numbers with vertical broken lines.

In the example of the 5-dimensional magic hypercube of order 3, all 1-rows may be considered parallel to a set such as 1, 212, 153; all 2-rows may be considered parallel to a set such as 1, 206, 159; all 3-rows may be considered parallel to a set such as 1, 132, 233; all 4-rows may be considered parallel to a set such as 1, 150, 215; and all 5-rows may be considered parallel to a set such as 1, 152, 213. The 16 5-agonals may be defined by the sets $|1, 122, 243|$, $|215, 122, 29|$, $|10, 122, 234|$, $|159, 122, 85|$, $|153, 122, 91|$, $|4, 122, 240|$, $|162, 122, 82|$, $|191, 122, 53|$, $|99, 122, 145|$, $|49, 122, 195|$, $|227, 122, 17|$, $|31, 122, 213|$, $|233, 122, 11|$, $|87, 122, 157|$, $|242, 122, 2|$ and $|28, 122, 216|$.

On the pages that follow is represented figure 5 in totality. To gain a greater appreciation of figure 5, figures 5(1, 1) to figures 5(3, 3) should be laid out in the following pattern:

figure 5 (1, 1)	figure 5 (1, 2)	figure 5 (1, 3)
figure 5 (2, 1)	figure 5 (2, 2)	figure 5 (2, 3)
figure 5 (3, 1)	figure 5 (3, 2)	figure 5 (3, 3)

Schematic diagram for illustrating figure 5.

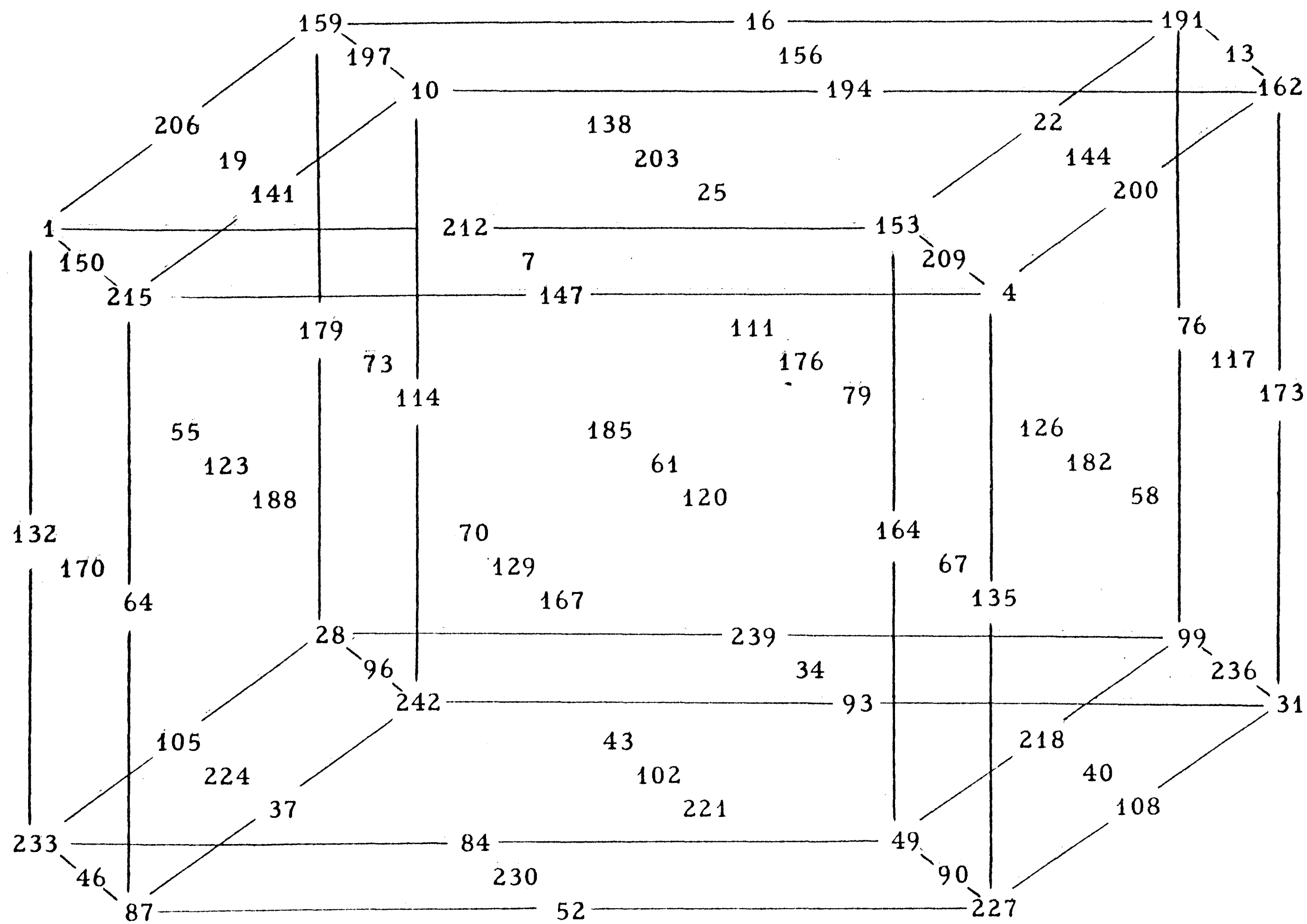

Figure 4 a. The first of three figures showing the 5-dimensional hypercube of order 3. S = 366. Broken lines have been used to outline each tessaract and the 5th direction is taken by choosing the corresponding elements of figures 4a, 4b and 4c.

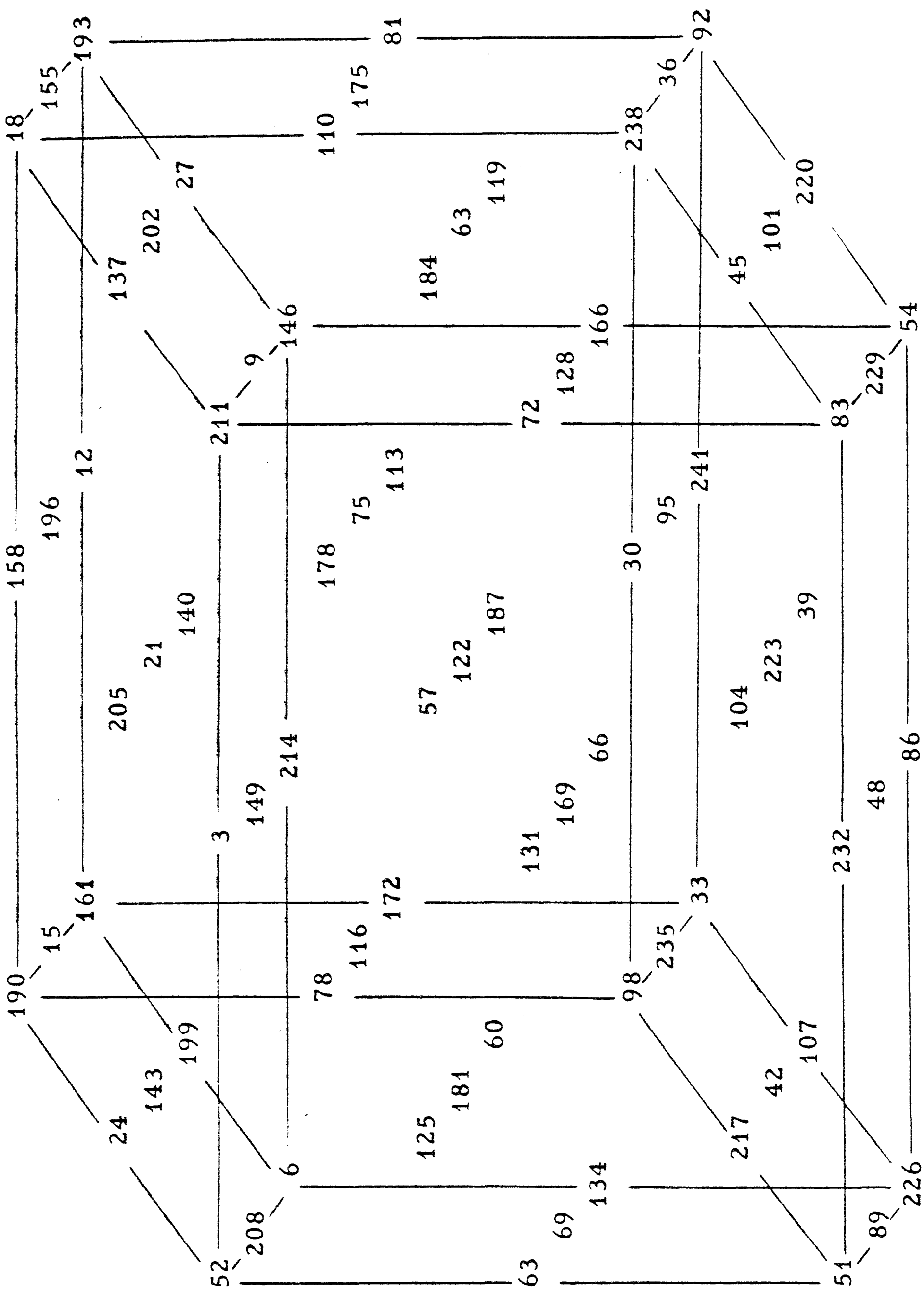

Figure 4 b.

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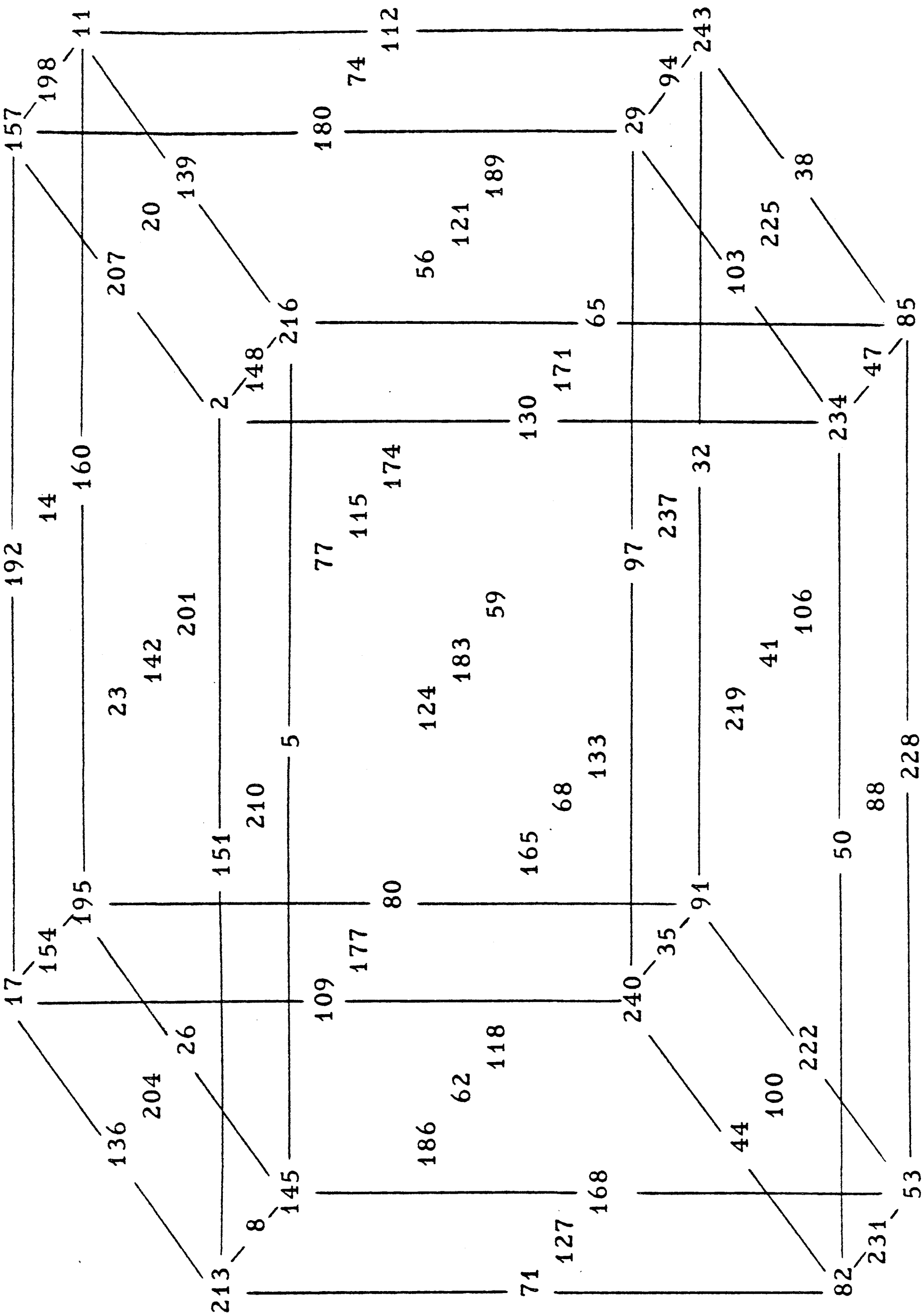

Figure 4 c.

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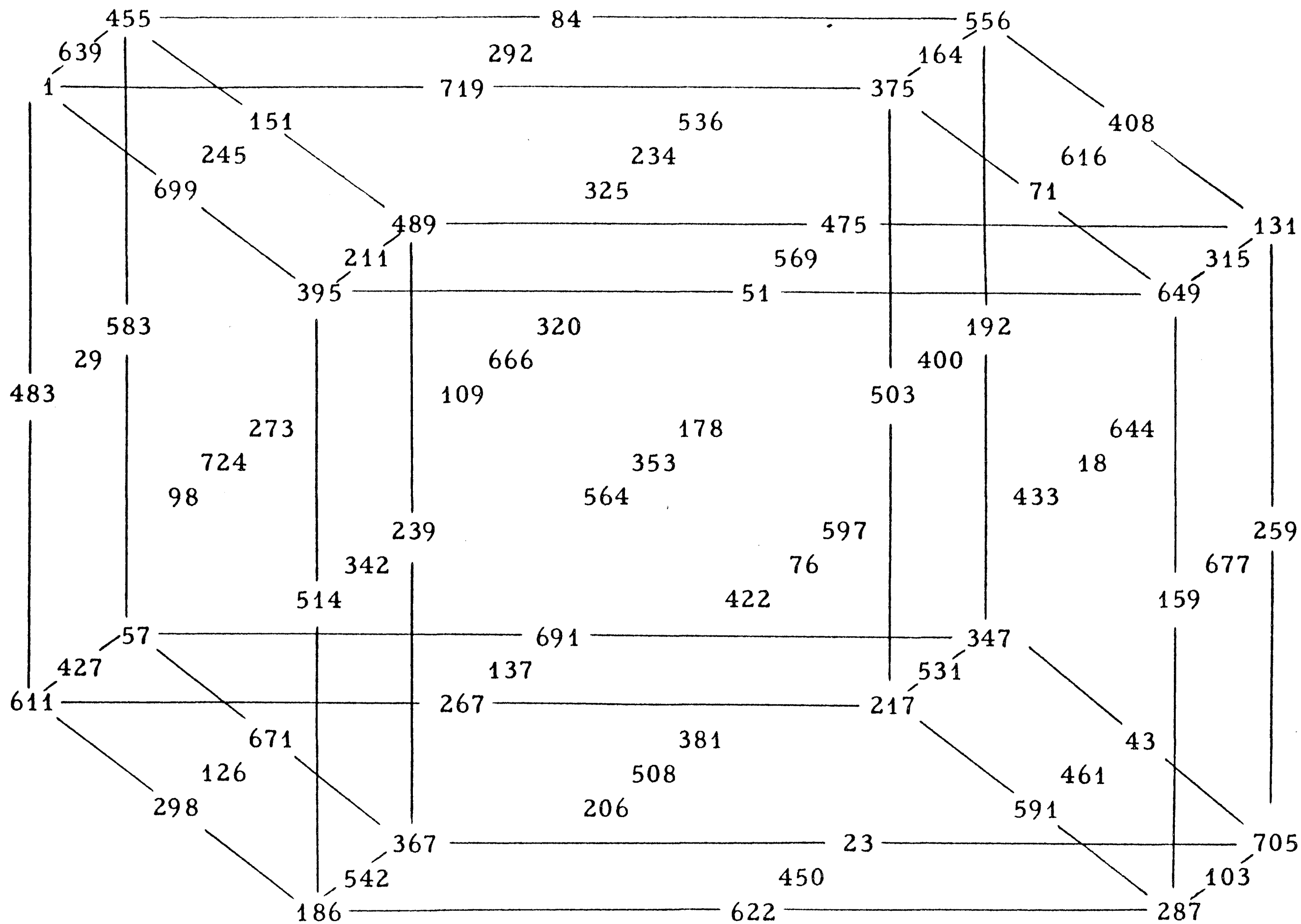

Figure 5(1, 1). The first of nine figures showing the 6-dimensional hypercube of order 3. S = 1095. Broken lines have been used to outline each tessaract. The 5th direction is taken by choosing the corresponding elements of figures 5(x, 1), (x, 2), (x, 3) and the 6th direction is taken by choosing the corresponding elements of figures 5(1, x), (2, x), (3, x) where x is a constant.

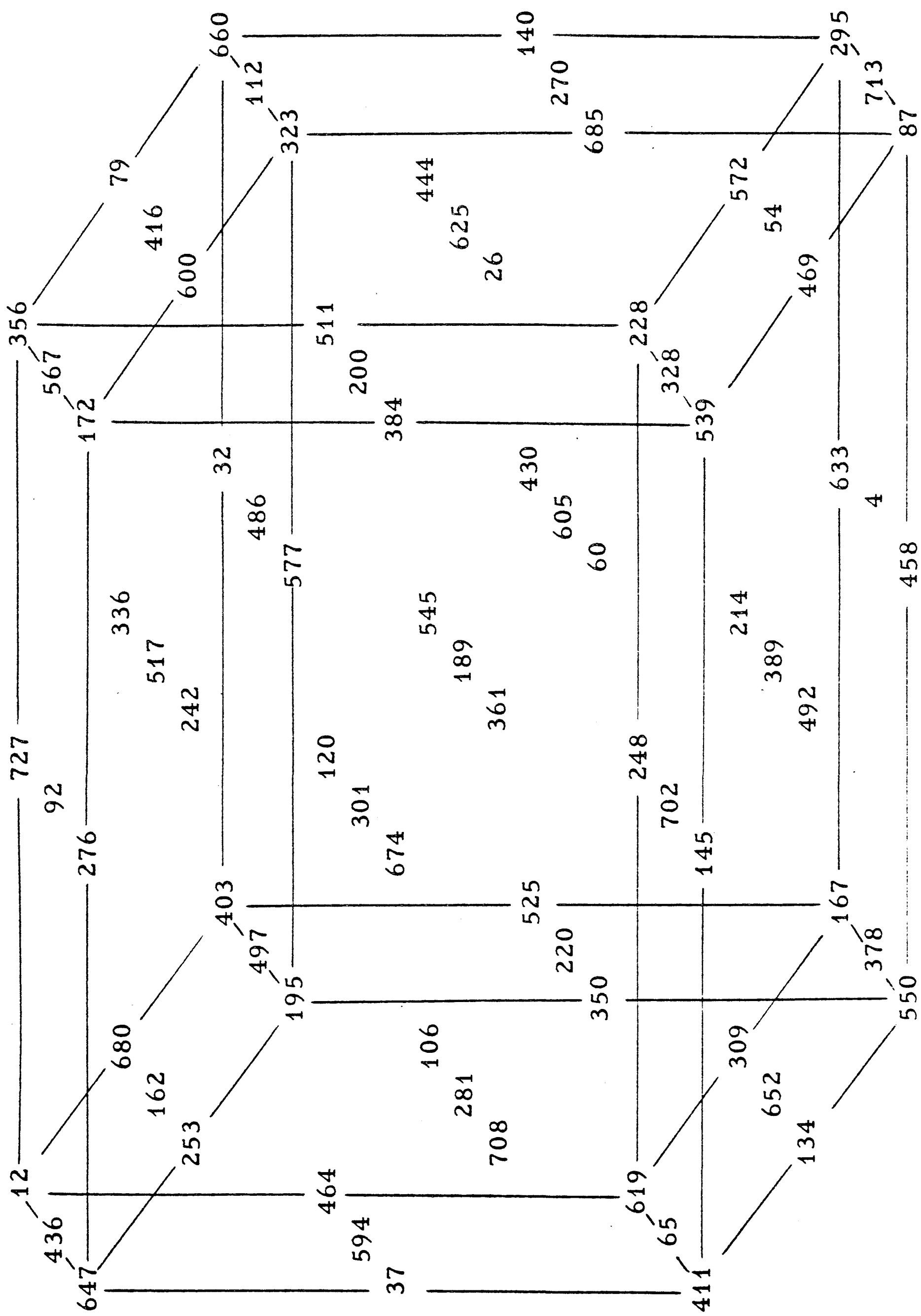
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Figure 5(1,2).

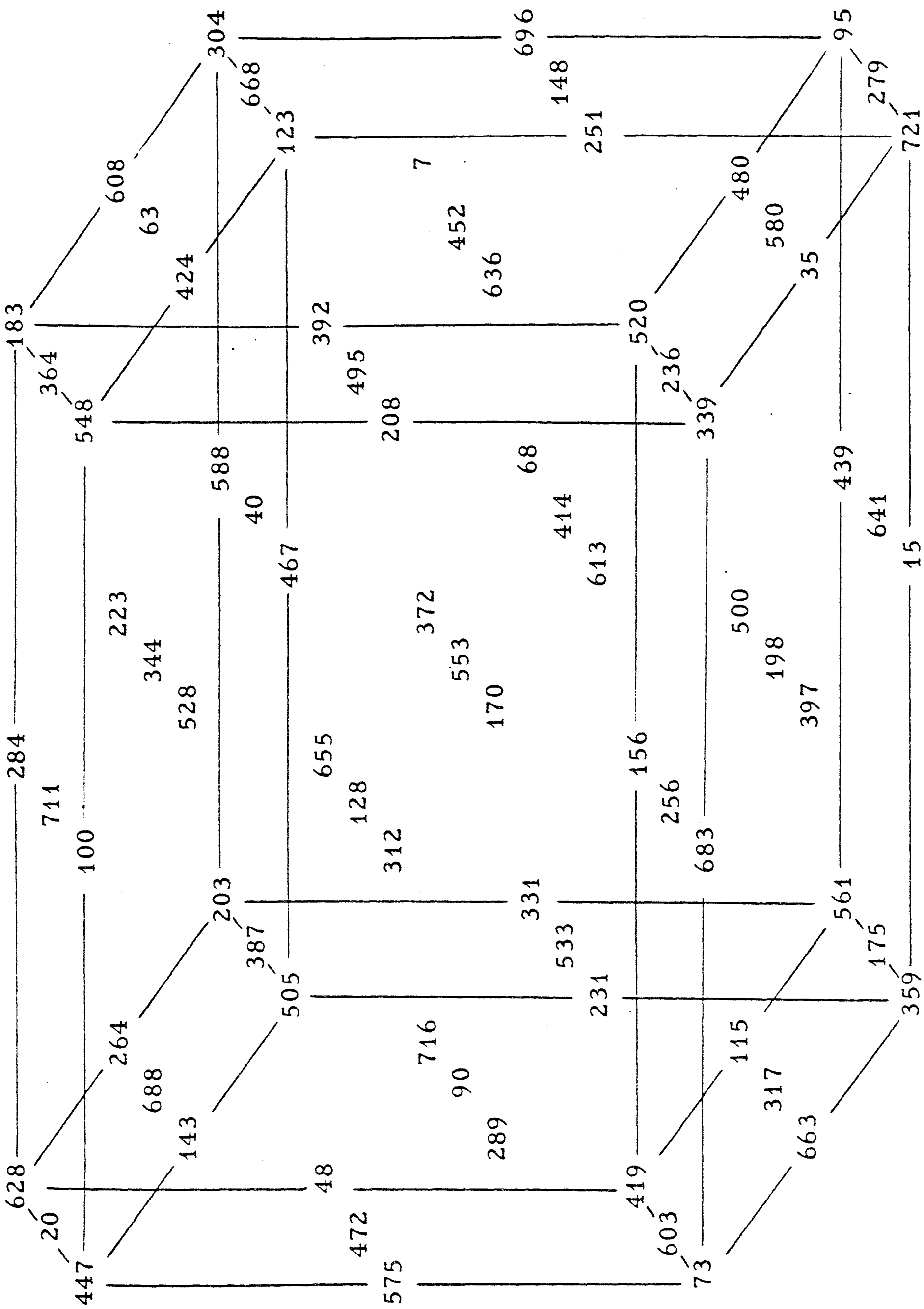

Figure 5(1, 3).

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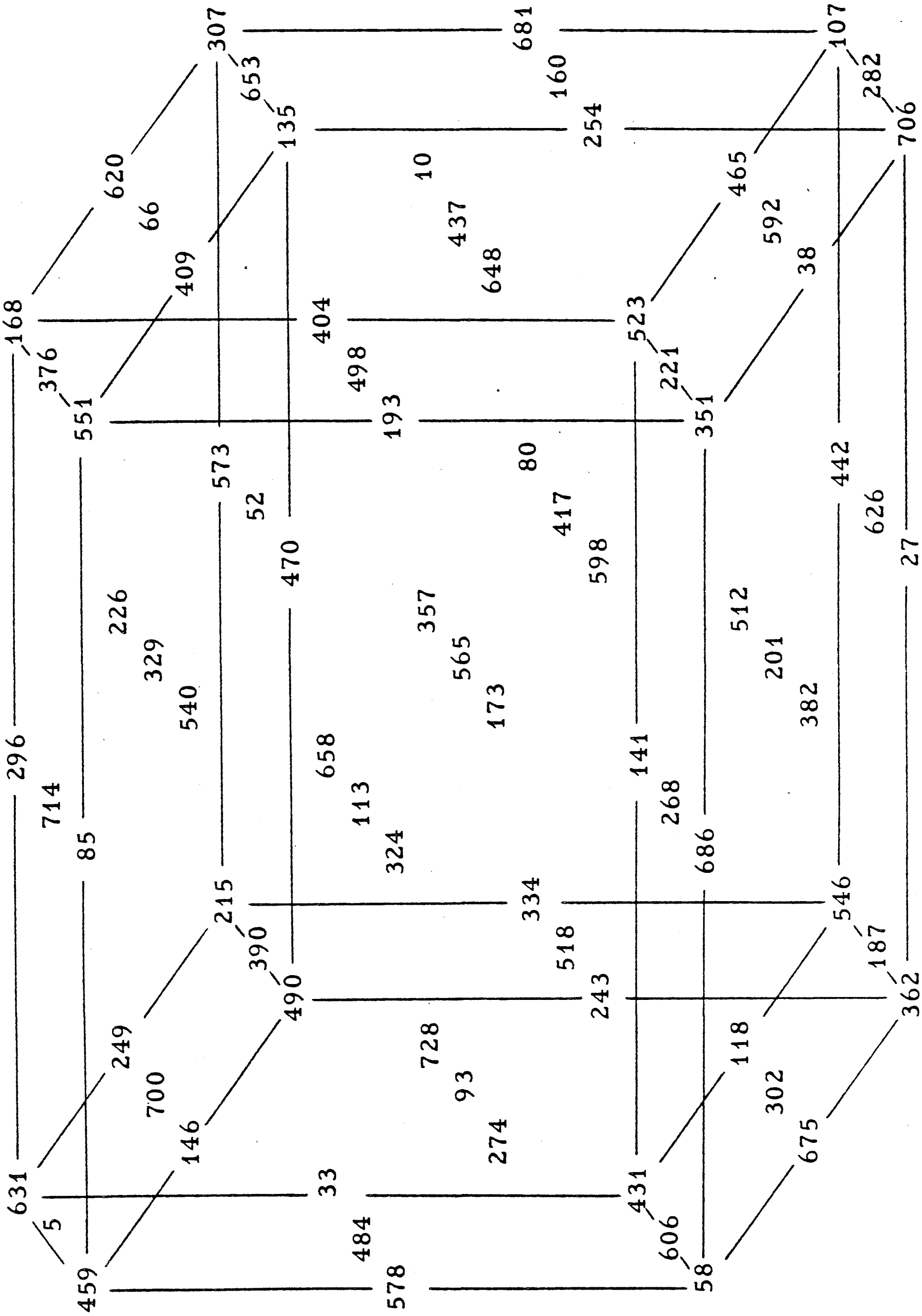

Figure 5(2,1).

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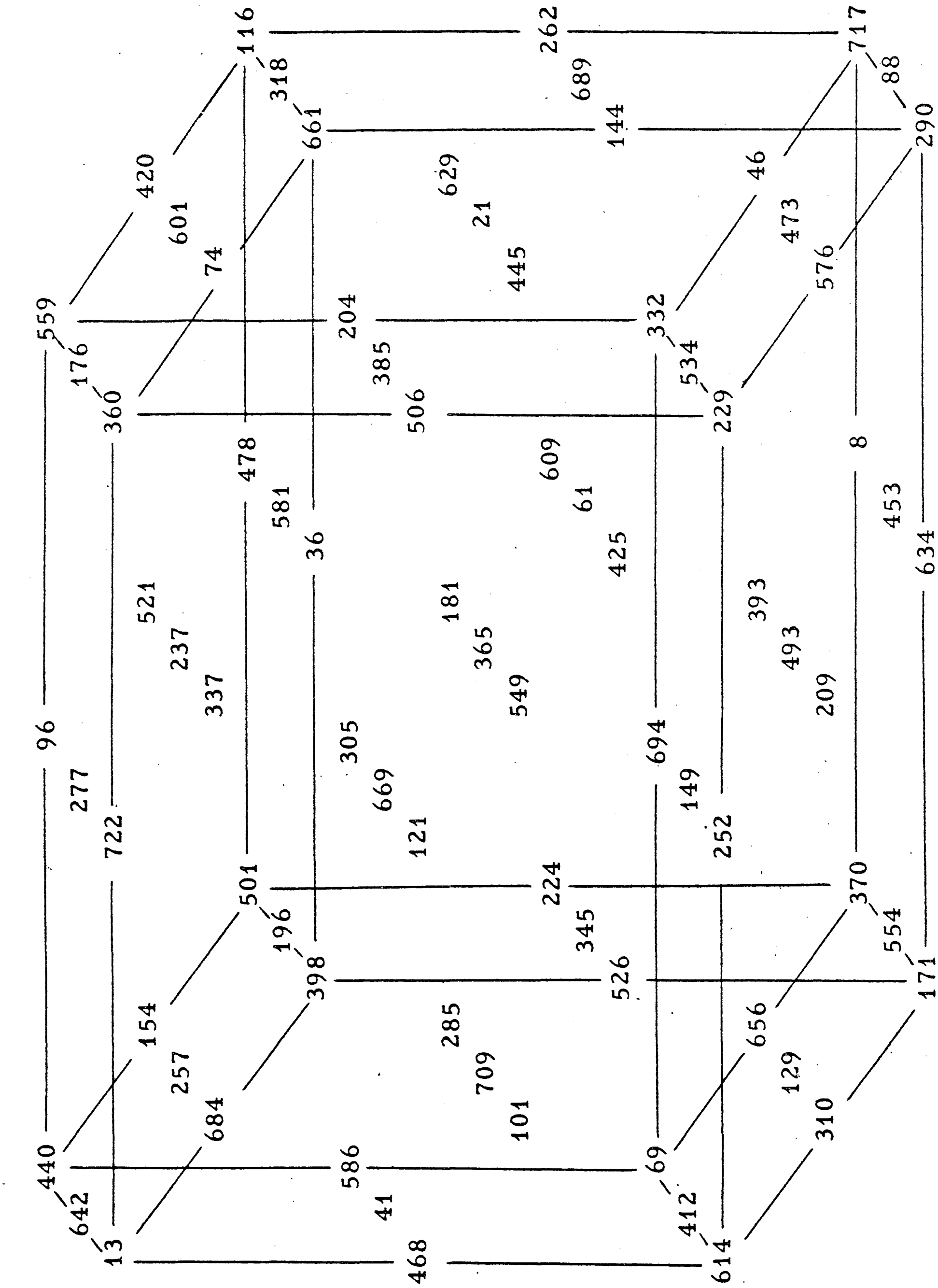
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Figure 5(2,2).

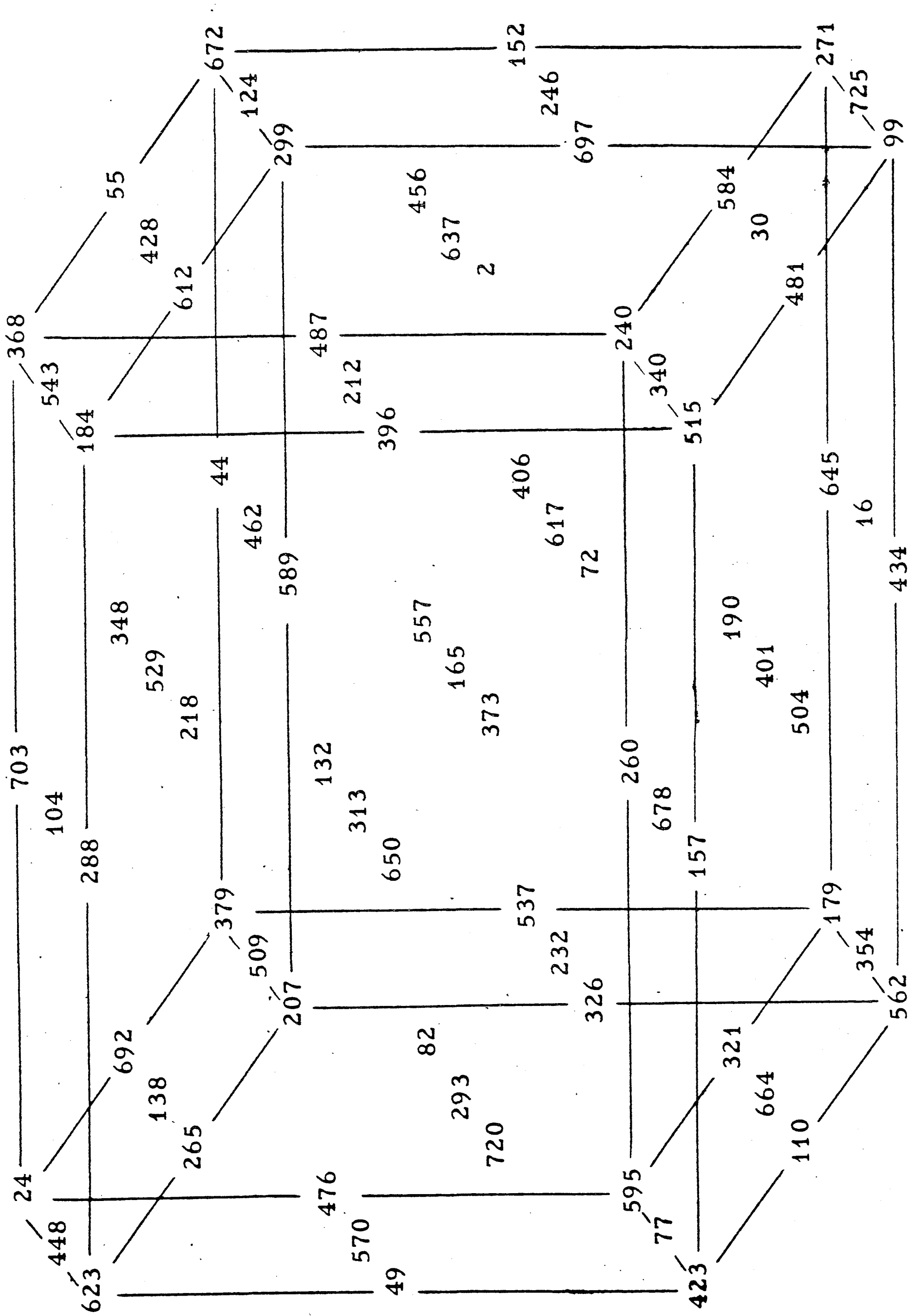

Figure 5(2, 3).

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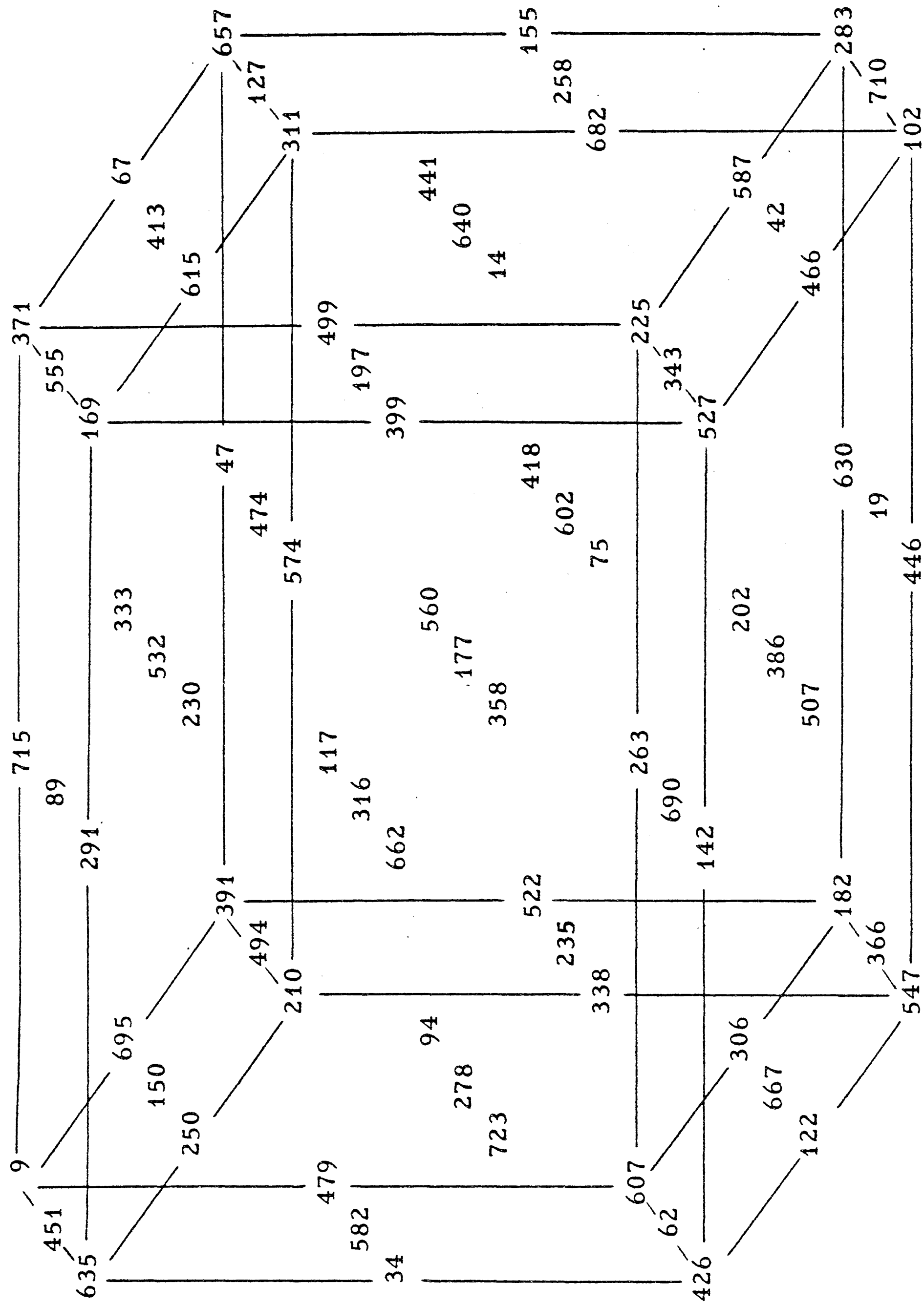

Figure 5(3,1).

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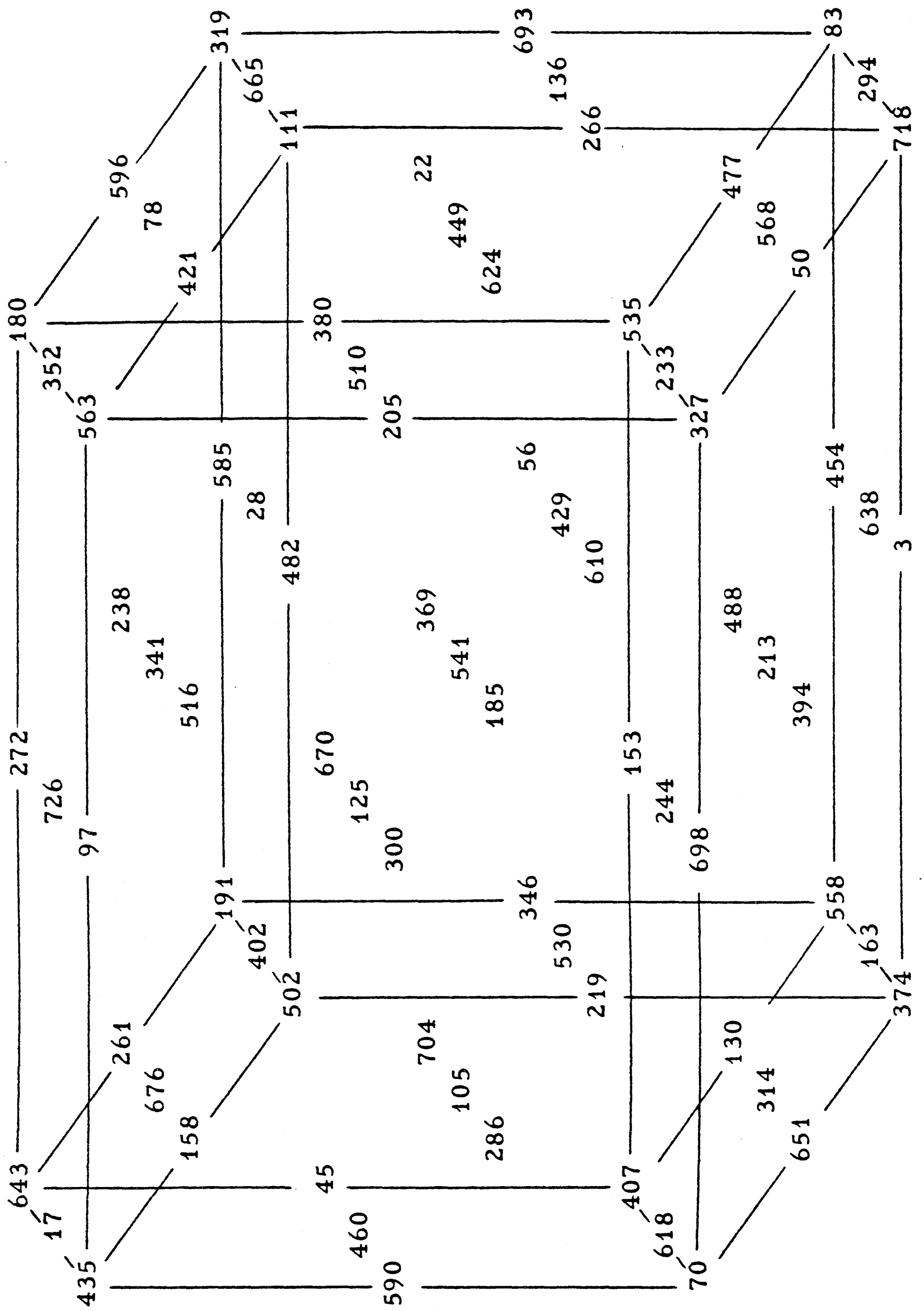

Figure 5(3,2).

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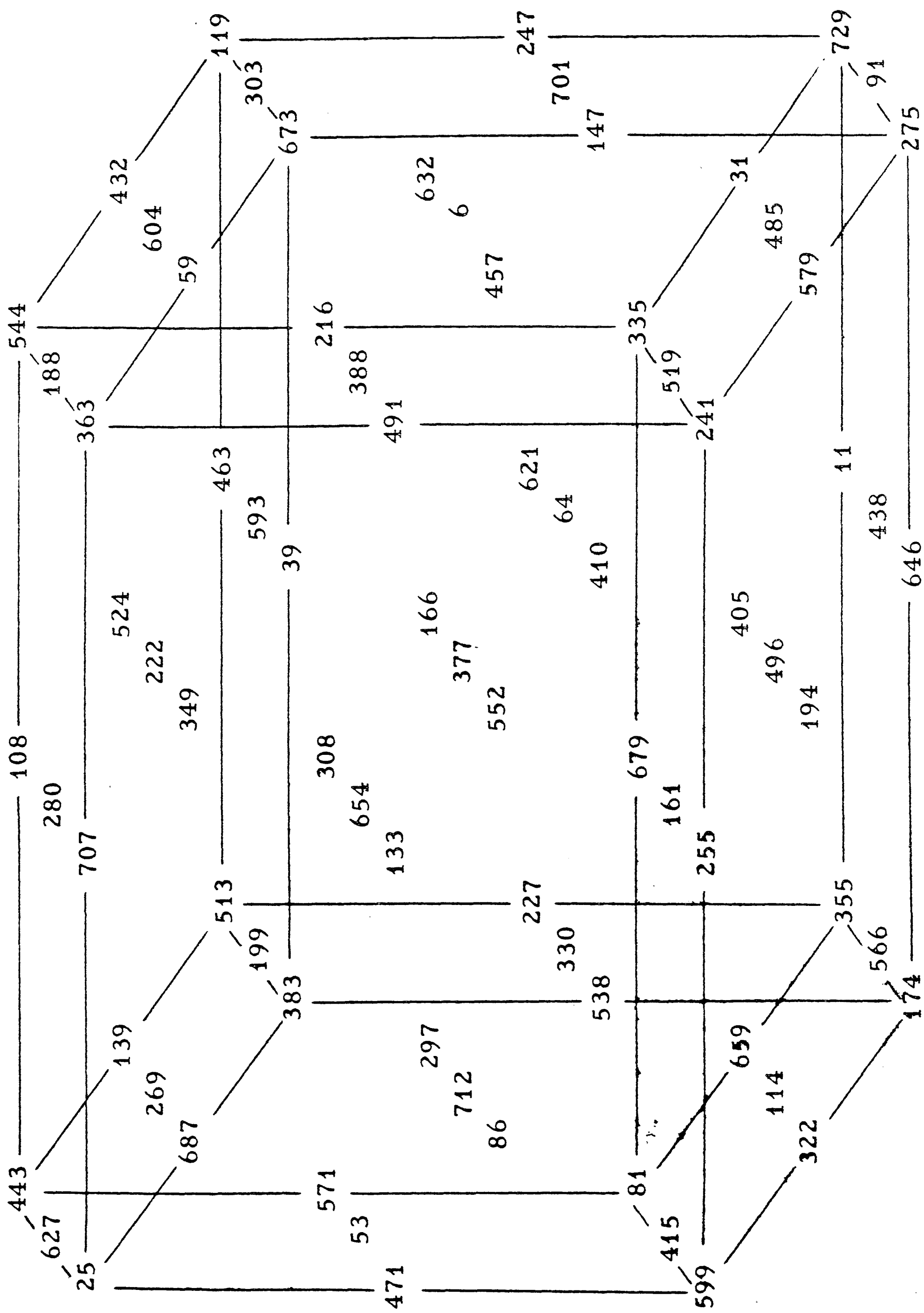

Figure 5(3, 3).

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In the example of the 6-dimensional magic hypercube of order 3, figure 5, all 1-rows may be considered parallel to a set such as 1, 719, 375; all 2-rows may be considered parallel to a set such as 1, 639, 455; all 3-rows may be considered parallel to a set such as 1, 483, 611; all 4-rows may be considered parallel to a set such as 1, 699, 395; all 5-rows may be considered parallel to a set such as 1, 647, 447; and all 6-rows may be considered parallel to a set such as 1, 459, 635.

In the 6-M3 portrayed, the 6-agonals defined by:
1,365,729|455,365,275|395,365,335|489,365,241|375,365,355|
556,365,174|649,365,81|131,365,599|611,365,119|57,365,673|
186,365,544|367,365,363|217,365,513|347,365,383|287,365,443|
705,365,25|447,365,283|628,365,102|505,365,225|203,365,527|
548,365,182|183,365,547|123,365,607|304,365,426|73,365,657|
419,365,311|359,365,371|561,365,169|339,365,391|520,365,210|
721,365,9| and 95,365,635 all add to a common total of 1095.

A 4-dimensional magic hypercube of order 4 was also constructed by this technique, but has not been shown here.