A REMARK ON THE GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR A SUMMABILITY METHOD OF MELIKOV

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Let $u = \Sigma u_k$ be a given series and let $s_n = \Sigma_0^n u_k$. Melikov [4] has defined the n-th σ-transform of u by

$$(1) \qquad \sigma_n^{(\varepsilon,\theta)}(u) = \Sigma_0^n \{1 - \frac{k-\varepsilon}{n+\theta}\} u_k ,$$

where ε and θ are assumed to be non-negative. This is easily shown to be equivalent to

$$(2) \qquad \sigma_n^{(\varepsilon,\theta)}(s_n) = \frac{1}{n+\theta} \Sigma_0^{n-1} s_k + \frac{\theta+\varepsilon}{n+\theta} s_n .$$

The method is a generalization of a method used by Kaufman [1], and of another one used by Melikov [5]. It reduces to the (n-1)th (C;1) mean when $\theta = 0$ and $\varepsilon = 0$, and to the n-th (C;1) mean when $\theta = 1$ and $\varepsilon = 0$.

The purpose of this note is to show that for every choice of ε and θ, this method fails to display the Gibbs phenomenon and that the Lebesgue constants are bounded and tend to 1 as $n \to \infty$. We first consider the Gibbs phenomenon. Let

$$\phi(x) = \Sigma \frac{\sin kx}{k} .$$

Then

$$s_n(x) = \int_0^x \frac{\sin(n+1/2)t}{2 \sin 1/2\, t}\, dt + o(1), \quad x \to 0 .$$

The σ-transform of the sequence $s_n(x)$ is given by (2), which we may put in the form

$$(3) \qquad \sigma_n^{(\varepsilon,\theta)}(s_n(x)) = \frac{n}{n+\theta}\{\frac{1}{n} \Sigma_0^{n-1} s_k(x)\} + \frac{\theta+\varepsilon}{n+\theta} s_n(x) .$$

Now $s_n(x)$ is bounded uniformly in x and n ; θ and ε being constant, it follows that the last term is $o(1)$, $n \to \infty$, uniformly in x. The term inside the braces is the (n-1)th (C;1) mean of the sequence $\{s_n(x)\}$, and it is known that this does not display the Gibbs phenomenon.

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Noting that the coefficient $n(n+\theta)^{-1}$ tends to unity as $n \to \infty$, it follows that the sequence $\{\sigma_n^{(\varepsilon,\theta)}(s_n(x)\}$ tends to the (C;1) means of $\{s_n(x)\}$ uniformly in x, $n \to \infty$. Hence it cannot display the Gibbs phenomenon.

We consider the Lebesgue constants in a similar manner. The σ-transform of the Dirichlet kernel is given by

$$(4) \qquad \sigma_n^{(\varepsilon,\theta)}(D_n(s)) = \frac{1}{n+\theta} \Sigma_0^{n-1} D_k(s) + \frac{\theta+\varepsilon}{n+\theta} D_n(s),$$

where

$$D_n(s) = \frac{\sin (n+1/2)s}{2 \sin 1/2\, s} .$$

The Lebesgue constants $L_n(\sigma)$ for this method are then given by

$$(5) \quad L_n(\sigma) = \frac{2}{\pi} \int_o^\pi | \sigma_n^{(\varepsilon,\theta)}(D_n(s))| \, ds$$

$$= \frac{2}{\pi} \int_o^\pi \left| \frac{1}{n+\theta} \Sigma_0^{n-1} D_k(s) + \frac{\theta+\varepsilon}{n+\theta} D_n(s) \right| ds$$

$$= \frac{2}{\pi} \frac{n}{n+\theta} \int_o^\pi \left| \frac{1}{n} \Sigma_0^{n-1} D_k(s) \right| ds + 0 \left\{ \frac{1}{n} \cdot \frac{2}{\pi} \int_o^\pi | D_n(s) | \, ds \right\} .$$

Now Fejer [2] (see also Lorch [3]) has shown that

$$\frac{2}{\pi} \int_0^\pi | D_n(s) | \, ds = \frac{4}{\pi^2} \log n + 0(1) , \quad n \to \infty ,$$

so that the last term in (5) is $o(1)$, $n \to \infty$. The first term, on the other hand, is $\frac{n}{n+\theta} \cdot L_{n-1}(C;1)$, that is, it is a multiple of the (n-1)th Lebesgue constant for the (C;1) means, the factor $\frac{n}{n+\theta}$. tending to unity as $n \to \infty$. In [2], Fejér has also shown that $L_n(C;1)$ tends to unity as $n \to \infty$. Thus

$$(6) \qquad L_n(\sigma) = \frac{n}{n+\theta} L_{n-1}(C;1) + o(1) , \quad n \to \infty$$

$$\to 1, \quad n \to \infty .$$

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