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ON THE TOTAL VARIATION OF A FUNCTION

BY
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1. **Introduction.** There are a number of theories which assign to a function defined on the real line a measure that reflects somehow the variation of that function. The most familiar of these is, of course, the Lebesgue–Stieltjes measure associated with any monotonic function. The problem in general is to provide a construction of a measure from a completely arbitrary function in such a way that the values of this measure provide information about the total variation of the function over sets of real numbers and from which useful inferences can be drawn.

One standard strategy is to employ a construction of Carathéodory (known nowadays as "Munroe's method II" after Munroe [9]) which has been used to yield Lebesgue-Stieltjes and Hausdorff measures. One writes

$$\mu_{fn}(X) = \inf\left\{\sum |f(b_k) - f(a_k)| : \bigcup [a_k, b_k] \supseteq X, \, b_k - a_k < \frac{1}{n}\right\}$$

for each natural number n, and then

$$\mu_f(X) = \lim_{n\to\infty} \mu_{fn}(X).$$

Such a construction yields an outer measure with the property that every Borel set is measurable (i.e., a metric outer measure in the sense of Munroe [9]). If f has bounded variation on an interval $[a, b]$, then it can be checked that $\mu_f([a, b])$ is exactly the variation of f on that interval; this would lead one to expect that $\mu_f(X)$ describes the variation of f on the set X in some sense and that the study of μ_f in general should have some import.

In fact, though, the measure μ_f may vanish if the function f is too highly oscillatory: Ellis and Burry [5] have constructed an example of a continuous f that is not of bounded variation on $[0, 1]$ for which $\mu_f([0, 1]) = 0$. Bruckner [2] has gone on to show that this behaviour is typical, namely that except for a first category subset of the space $C[0, 1]$ every such μ_f must vanish.

Browne [1] uses essentially the same construction with some minor modifications. Bruneau [4] introduces many interesting ideas and can be considered a sourcebook on the subject; one of his main ideas is to construct the variation $v_f(K)$ for a function f on a compact set K by comparing f with functions of

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bounded variation and then extending v_f to a measure on the Borel sets by familiar methods.

There is really no unique or canonical way of assigning a variation measure to an arbitrary function and so the problem depends on the type of application one has in mind. Our main motivation rests in the study of the derivation properties of the function and within such a viewpoint there is a natural method of constructing some useful variation measures. In the study of the ordinary derivative certain concepts arise naturally: these are the notions of a Vitali cover of a set and related ideas. We can use such ideas to give variation measures that answer familiar problems in derivation theory.

We present immediately the definitions that lead to the theory.

Definition 1. A family $\mathcal{F}$ of closed subintervals of $[a, b]$ is said to be a *full cover* of $X \subseteq [a, b]$ if for every $x \in X$ there is a positive number $\delta(x)$ such that every interval of length less than $\delta(x)$ that has x as an endpoint necessarily belongs to $\mathcal{F}$.

Such a family $\mathcal{F}$ is said to be a *fine cover* of X if for every $x \in X$ and every positive number ε there is an interval $I \in \mathcal{F}$ with length less than ε and that has x as an endpoint.

Definition 2. Let f be a real-valued function on $[a, b]$ and $\mathcal{F}$ a family of closed subintervals of $[a, b]$; then we write

$$V(f, \mathcal{F}) = \sup \sum |f(b_k) - f(a_k)|$$

where the supremum is with regard to all sequences $\{[a_k, b_k]\} \subseteq \mathcal{F}$ with pairwise non-overlapping elements. (Write $V(f, \emptyset) = 0$.)

Definition 3. Let f be a real-valued function on $[a, b]$, then ψ_f and ψ^f denote the set functions

$$\psi^f(X) = \inf\{V(f, \mathcal{F}) : \mathcal{F} \text{ a full cover of } X\}$$

and

$$\psi_f(X) = \inf\{V(f, \mathcal{F}) : \mathcal{F} \text{ a fine cover of } X\}.$$

These set functions ψ_f and ψ^f are metric outer measures for any real-valued function f on $[a, b]$ and since they are constructed directly from concepts arising in derivation theory, it may be anticipated that they will reflect the derivation properties of the function f. We will investigate these measures within a more general framework that may obscure the simple constructions defined above.

2. **Derivation bases on the line.** The constructions given above arise within the context of ordinary differentiation on the real line. There are many different ways of generalizing the ordinary derivative and to each such way

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there would correspond a similar construction. For example, the symmetric derivative of a function f is defined to be

$$f^{[1]}(x_0) = \lim_{h\to 0} [f(x_0+h) - f(x_0-h)]/2h$$

and a study of this derivative or the related extreme derivatives would involve notions similar to the full and fine covers of Definition 1 above but with obvious changes (instead of x as an endpoint we would require that x be the midpoint of the intervals in the cover). To unify these ideas and generalize them further we introduce the following definitions.

DEFINITION 4. A *derivation basis* on the real line is any family $\mathfrak{A}$ of subsets of $\mathcal{I} \times \mathbf{R}$ where $\mathbf{R}$ is the real numbers and $\mathcal{I}$ the collection of all closed intervals, with the property that whenever $\mathbf{S}_1$ and $\mathbf{S}_2$ belong to $\mathfrak{A}$ there is an $\mathbf{S}_3 \in \mathfrak{A}$ with $\mathbf{S}_3 \subseteq \mathbf{S}_1 \cap \mathbf{S}_2$.

DEFINITION 5. If $\mathbf{S}$ is any subset of $\mathcal{I} \times \mathbf{R}$ and h is any real-valued function defined on $\mathcal{I} \times \mathbf{R}$ then we write

$$V(h, \mathbf{S}) = \sup \sum |h(I_i, x_i)|$$

where $\{(I_i, x_i)\}$ is a sequence of interval-point pairs from $\mathbf{S}$ with pairwise non-overlapping elements $\{I_i\}$. (Write $V(h, \emptyset) = 0$.)

If $\mathfrak{A}$ is any non-empty family of subsets of $\mathcal{I} \times \mathbf{R}$ (not necessarily a derivation basis) then we write as well

$$V(h, \mathfrak{A}) = \inf\{V(h, \mathbf{S}) : \mathbf{S} \in \mathfrak{A}\}.$$

These concepts provide the needed generalizations. To return to the earlier versions of our outer measures we will write

$$\mathbf{S}[X] = \{(I, x) \in \mathbf{S} : x \in X\}$$

whenever $\mathbf{S} \subseteq \mathcal{I} \times \mathbf{R}$ and $X \subseteq \mathbf{R}$, and

$$\mathfrak{A}[X] = \{\mathbf{S}[X] : \mathbf{S} \in \mathfrak{A}\}$$

whenever $\mathfrak{A}$ is a family of subsets of $\mathcal{I} \times \mathbf{R}$. Then the outer measures $\psi_f(X)$ and $\psi^f(X)$ can be realized as $V(h, \mathfrak{A}[X])$ where h is the function $h([a, b], x) = f(b) - f(a)$ and where $\mathfrak{A}$ can be chosen separately to yield either measure.

This construction is essentially due to Ralph Henstock [7] and arises really from considerations of Riemann sums in general settings. The families $\mathfrak{A}$ need not have many delicate properties in order to ensure that the set functions $X \to V(h, \mathfrak{A}[X])$ are outer measures. There are a number of properties that will appear in [11] and which can be found in the other literature of the subject. We cite only two of these properties.

DEFINITION 6. A family $\mathfrak{A}$ of subsets of $\mathcal{I} \times \mathbf{R}$ will be said to be:

.1 *decomposable* [respectively σ-*decomposable*] if to every family $\{X_a : a \in A\}$ of subsets of $\mathbf{R}$ [respectively countable family] that is pairwise disjoint and corresponding family $\{\mathbf{S}_a : a \in A\} \subseteq \mathfrak{A}$ there is an $\mathbf{S} \in \mathfrak{A}$ with $\mathbf{S}[X_a] \subseteq \mathbf{S}_a$ for every index a.

.2 *finer than the topology* on $\mathbf{R}$ if to every open set $G \subseteq \mathbf{R}$ there is an $\mathbf{S} \in \mathfrak{A}$ so that every $(I, x) \in \mathbf{S}[G]$ has $I \subseteq G$.

These definitions give rise to the following theorem which provides the basic measure theory that arises from a general family $\mathfrak{A}$.

THEOREM 1. *Let h be an arbitrary real-valued function on $\mathcal{I} \times \mathbf{R}$ and let $\mathfrak{A}$ be a σ-decomposable family of subsets of $\mathcal{I} \times \mathbf{R}$. Then the function $h^*(X) = V(h, \mathfrak{A}[X])$ is an outer measure on $\mathbf{R}$. If in additon $\mathfrak{A}$ is a derivation basis that is finer than the topology, h^* is a metric outer measure.*

Proof. This is proved in [11] but is straightforward in any case.

More results for the measures h^* can be obtained by varying the hypotheses on the family $\mathfrak{A}$. For the remainder of the paper, however, our concern is with specific applications of the theory and we drop the general approach returning to concrete examples of measures that are generated by real-valued functions f of a real variable.

3. **The Peano–Jordan "measures."** The classical Peano–Jordan measure, or Jordan content as it is sometimes called, can be defined as an example of the theory of the previous section. In fact, the definition given by Stolz in 1884 (cf. [10, p. 30]) was essentially in terms of limits of Riemann sums and so fits into this framework. Here we give the construction of set functions m_f corresponding to any function f on $\mathbf{R}$: if $f(x) = x$ then m_f is precisely this Peano–Jordan measure and accordingly we may consider these set functions as generalizations of this classical concept.

DEFINITION 7. The family $\mathfrak{R}$ is defined to be the collection of all subsets $\mathbf{S}_\delta$ of $\mathcal{I} \times \mathbf{R}$ where

$$\mathbf{S}_\delta = \{(I, x) : I \in \mathcal{I},\ x \in I,\ |I| < \delta\}$$

and δ is an arbitrary positive number.

For any function f on $\mathbf{R}$ write

$$m_f(X) = V(f, \mathfrak{R}[X]) \qquad (X \subseteq \mathbf{R}).$$

(Here f is considered to be defined on $\mathcal{I} \times \mathbf{R}$ by the device $f([a, b], x) = f(b) - f(a)$.)

This $\mathfrak{R}$ is a derivation basis but does not have either of the properties of Definition 6. The set function m_f is obviously non-negative, monotone, finitely subadditive, and additive over topologically separated sets. It is not a measure or outer measure in the usual sense of those words, of course. Some of the specific properties of these set functions are given in the next theorem.

THEOREM 2. *The Peano–Jordan "measures" m_f have the following properties:*

.1 *if f is monotone then*

$$m_f(a, b) = m_f[a, b] = |f(b+) - f(a-)|$$

.2 *the value of m_f at a singleton $\{x\}$ is*

$$m_f(\{x\}) = \limsup_{h\to 0+} |f(x+h) - f(x)| + \limsup_{h\to 0+} |f(x-h) - f(x)|$$

.3 *for monotone f and bounded $X \subseteq \mathbf{R}$,*

$$m_f(X) = \inf\{m_f(G) : G \supseteq X,\ G \text{ a finite union of open intervals}\}$$

.4 *for monotone f and bounded $X \subseteq \mathbf{R}$,*

$$m_f(X) = m_f(\bar{X})$$

.5 *for monotone f and $\{K_n\}$ a decreasing sequence of compact sets,*

$$m_f\left(\bigcap K_n\right) = \lim_{n\to\infty} m_f(K_n)$$

.6 *the value of m_f on an arbitrary interval $[a, b]$ is*

$$m_f[a, b] = CV(f, [a, b]) + \limsup_{h\to 0+} |f(b+h) - f(b)| + \limsup_{h\to 0+} |f(a-h) - f(a)|$$

where CV is the classical variation of the function f on the interval.

The proofs are elementary and will be omitted.

Some elementary examples show that certain of these computations cannot be improved: (i) let f be defined by setting $f(x) = 0$ for negative x, $f(x) = 1$ for positive x and $f(0) = 2$; then $m_f(\{0\}) = 3$, $m_f(0, 1) = 1$, and $m_f[0, 1] = 3$ so that if f is not monotone (.4) might fail; (ii) let f be continuous and nowhere differentiable, then while $m_f(F) = 0$ for every finite set F, $m_f(G) = +\infty$ for every open set G so that (.3), (.4), and (.5) may all fail for non-monotone f.

The final estimate we wish to provide for this measure relates it to a classical variational idea of Lusin. For any positive number δ and any perfect set $P \subseteq \mathbf{R}$ define

$$v_{f,\delta}(P) = \sup\Big\{\sum O(f, [a_k, b_k]) : \bigcup [a_k, b_k] \supseteq P,\ \{[a_k, b_k]\} \text{ non-overlapping},$$
$$P \cap (a_k, b_k) \neq \emptyset,\ b_k - a_k < \delta\Big\}$$

and

$$v_f(P) = \lim_{\delta\to 0+} v_{f,\delta}(P).$$

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Our estimate shows that $v_f(P)$ is finite if and only if $m_f(P)$ is finite. By the expression $O(f, I)$ here we mean the oscillation of the function f on the interval I defined as

$$\sup\{|f(x)-f(y)| : x, y \in I\}.$$

LEMMA. *Let f be a continuous function on* $\mathbf{R}$ *and P a perfect set; then* $m_f(P) \leq v_f(P) \leq 3m_f(P)$.

Proof. We obtain the inequality $m_f(P) \leq v_f(P)$ by showing that for any $\delta > 0$ we must have $V(f, \mathbf{S}_{\delta/2}[P]) \leq v_{f,\delta}(P)$. Since f is continuous and P is perfect, this is just a matter of adjusting the intervals in any partition so that they are of the type appearing in the definition of $v_{f,\delta}(P)$; we omit the details.

To obtain the other inequality let $\sum O(f, [a_k, b_k])$ be one of the sums that appear in the definition of $v_{f,\delta}$ and choose $x_k, y_k \in [a_k, b_k]$ as respectively the maximum and minimum points for f in that interval. Let $I_k = [a_k, x_k]$ if $[a_k, x_k] \cap P \neq \emptyset$ and let $I_k = [x_k, b_k]$ otherwise; similarly let $J_k = [a_k, y_k]$ or $[y_k, b_k]$ again depending on which meets P. Then

$$O(f, [a_k, b_k]) = f(x_k) - f(y_k) \leq |f(b_k) - f(a_k)| + |f(I_k)| + |f(J_k)|$$

and so any such sum used to estimate $v_{f,\delta}(P)$ is dominated by $3V(f, \mathbf{S}_\delta[P])$ and this gives $v_f(P) \leq 3m_f(P)$ as required.

4. **The Lebesgue–Stieltjes measures.** McShane [8] has studied a derivation basis on the real line that can be used to characterize the Lebesgue integral in terms of limits of Riemann sums; indeed he has announced that a book is in press that contains an exposition of this accessible to undergraduates. It is thus appropriate to discuss the corresponding total variation measures that arise from this derivation basis. For a function f of bounded variation these generate the usual Lebesgue–Stieltjes measures and so we have labelled them as such in general.

DEFINITION 8. The family $\mathfrak{M}$ is defined to be the collection of all subsets $\mathbf{S}_\delta^*$ of $\mathcal{I} \times \mathbf{R}$ where δ is an arbitrary positive function on $\mathbf{R}$ and

$$\mathbf{S}_\delta^* = \{(I, x) : I \in \mathcal{I}, x \in \mathbf{R}, \mathrm{I} \subseteq (x - \delta(x), x + \delta(x))\}.$$

For any function f on $\mathbf{R}$ we write

$$\varphi_f(X) = V(f, \mathfrak{M}[X])$$

again according to the convention that f is defined on $\mathcal{I} \times \mathbf{R}$ as an interval function, $f(I, x) = f(I) = f(b) - f(a)$ if $I = [a, b]$.

Although we will not here discuss the differentiation theory that arises in connection with this basis it should be noted that this construction occurs in McShane's study of the Lebesgue integral and occurs in differentiation theory under the term "unstraddled" derivative (see for example Bruckner [3, p. 69]) or Peano derivative.

The next theorem summarizes the results available for these measures.

THEOREM 3. *The Lebesgue-Stieltjes total variation measures φ_f have the following properties:*

.1 *φ_f is a metric outer measure on* $\mathbf{R}$

.2 *φ_f has the increasing sets property, namely that for any increasing sequence of sets $\{X_n\}$,*

$$\varphi_f\left(\bigcup_{n=1}^{\infty} X_n\right)=\lim_{n\to\infty} \varphi_f(X_n).$$

.3 *the value of φ_f on an open interval (a, b) is exactly the classical variation of f on (a, b),*

$$\varphi_f(a, b)=CV(f, (a, b))$$

.4 $\varphi_f(X)=\inf\{\varphi_f(G): G\supseteq X, G \text{ open}\}$

.5 *if f is monotone and K is compact then $m_f(K)=\varphi_f(K)$*

.6 *$\varphi_f(\{x_0\})<+\infty$ if and only if f has bounded variation in some neighbourhood of x_0.*

Proof. Assertions (.1), (.2), and (.4) are best seen within the context of the general theory; [11] contains proofs. Assertion (.3) is a straightforward consequence of the fact that partitions of any interval can be extracted from any $\mathbf{S}_\delta^*\in\mathfrak{M}$ (again see [11]). Assertion (.5) follows from (.4) and Theorem 2, and Assertion (.6) is only a direct computation.

One can view these measures as a reflection of the variation of the function in the crude sense that if the function f fails to have a finite variation somewhere, the measure φ_f will indicate this by having an infinite value. Thus for most functions the measure φ_f assigns infinite value to every set and so is of little interest.

5. **The Henstock total variation measures.** The standard derivation basis on the real line (in our version that is) that corresponds to ordinary differentiation was used by Henstock [7] to yield a characterization of the Denjoy–Perron integral in terms of limits of Riemann sums. The corresponding total variation measures that arise from this basis have been defined in Section 1 above, and will be redefined from a slightly altered viewpoint here. Because they arise from a study of ordinary differentiation, it is to be expected that they will have applications to that study.

DEFINITION 9. The family $\mathfrak{H}$ is defined to be the collection of all subsets $\mathbf{H}_\delta$ of $\mathcal{I}\times\mathbf{R}$ where δ is an arbitrary positive function on $\mathbf{R}$ and where

$$\mathbf{H}_\delta=\{(I, x): I\in\mathcal{I}, x \text{ is an endpoint of } I, |I|<\delta(x)\}.$$

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For any function f on $\mathbf{R}$ we will write

$$\psi^f(X) = V(f, \mathfrak{H}[X]).$$

The measures m_f, φ_f, and ψ^f have been defined as set functions

$$X \to V(f, \mathfrak{A}[X])$$

for an appropriate derivation basis $\mathfrak{A}$ and their theory is similar. To obtain the measure ψ_f of Section 1 in the same spirit we need a family that is not a derivation basis but which has many desirable properties, for example, the decomposable property of Definition 6. The pattern we use in the next definition applies to any derivation basis; for a more detailed exposition of this idea in a general setting see [11].

DEFINITION 10. A subset $\mathbf{H}^*$ of $\mathscr{I} \times \mathbf{R}$ is said to be *$\mathfrak{H}$-fine* if for every $\mathbf{H} \in \mathfrak{H}$ and every $x \in \mathbf{R}$,

$$\mathbf{H}^* \cap \mathrm{H}[\{x\}] \neq \emptyset.$$

The family $\mathfrak{H}_v$ is the collection of all $\mathfrak{H}$-fine subsets $\mathbf{H}^*$ of $\mathscr{I} \times \mathbf{R}$, and for any function f on $\mathbf{R}$ we will write

$$\psi_f(X) = V(f, \mathfrak{H}_v[X]).$$

Henstock refers to the measure ψ^f as the "variation" and has developed a concept similar to ψ_f using the term "inner variation." We shall refer to both of these as the Henstock total variation measures, and consider that an understanding of the nature of f (for example, the nature of the oscillation behaviour, the level set structure, and so on) is promoted by the study of the two measures together. The basic theorem follows.

THEOREM 4. *The Henstock total variation measures have the following properties:*

.1 ψ_f *and* ψ^f *are metric outer measures on* $\mathbf{R}$

.2 $\psi_f \leq \psi^f$

.3 ψ^f *has the increasing sets property*

.4 $\psi^f \leq m_f$ *and* $\psi^f \leq \varphi_f$

.5 $\psi^f(G) = \varphi_f(G)$ *if G is open*

.6 $$\psi^f(\{x\}) = \limsup_{h \to 0+} |f(x+h) - f(x)| + \limsup_{h \to 0+} |f(x-h) - f(x)|$$

and

$$\psi_f(\{x\}) = \min\left\{\limsup_{h \to 0+} |f(x+h) - f(x)|, \limsup_{h \to 0+} |f(x-h) - f(x)|\right\}.$$

Proof. Both the families $\mathfrak{H}$ and $\mathfrak{H}_v$ are decomposable and so Theorem 1 proves that ψ_f and ψ^f are outer measures; also $\mathfrak{H}$ is finer than the topology on $\mathbf{R}$ so that ψ^f is even a metric outer measure, and similar considerations apply to

ψ_f. The increasing sets property (.3) is most easily proved within the general theory of [11].

For (.2) note that $\mathfrak{H}_v[X] \supseteq \mathfrak{H}[X]$ for every $X \subseteq \mathbf{R}$ and hence $\psi_f(X) = V(f, \mathfrak{H}_v[X]) \leq V(f, \mathfrak{H}[X]) = \psi^f(X)$. Assertion (.4) is similar.

Assertion (.5) follows from the fact that for any G open, for any $\mathbf{H} \in \mathfrak{H}[G]$, and for any interval $[a, b]$ inside G there must be a partition of $[a, b]$ chosen from $\mathbf{H}$. From this we can derive that the variation $\psi^f(G)$ is precisely the classical variation of f on G and this is equal to $\varphi_f(G)$ as required. Finally, the estimate in (.6) is just a direct computation.

We will complete our investigation here with the general derivation results that can be proved with the help of these measures. The extreme relative derivatives of a function f with respect to a function g are defined, as in Saks [12], by writing

$$\bar{f}_g(x) = \limsup_{h \to 0} [f(x+h) - f(x)]/[g(x+h) - g(x)]$$

$$\underline{f}_g(x) = \liminf_{h \to 0} [f(x+h) - f(x)]/[g(x+h) - g(x)]$$

but according to the interpretation of a quotient $c/0$ subject to the convention that $c/0 = 0$ if $c = 0$ and $c/0 = +\infty$ or $-\infty$ if c is positive or negative. It should be observed that these definitions are exactly the same as

$$\bar{f}_g(x) = \inf_{\mathbf{H} \in \mathfrak{H}} \sup_{(I,x) \in \mathbf{H}} f(I)/g(I)$$

and

$$\underline{f}_g(x) = \sup_{\mathbf{H} \in \mathfrak{H}} \inf_{(I,x) \in \mathbf{H}} f(I)/g(I)$$

expressed in terms of the derivation basis $\mathfrak{H}$. The family $\mathfrak{H}_v$ enters in as estimates are considered that give rise to Vitali coverings, and, of course, this is the source of the concept.

THEOREM 5. *Let f and g be real-valued functions on $\mathbf{R}$ and let X denote a subset of $\mathbf{R}$; then*

.1 *if ψ^f is σ-finite on X, then for ψ_g-almost every x in X the extreme derivatives $\underline{f}_g(x)$ and $\bar{f}_g(x)$ are finite.*

.2 *if $\psi^f(X) = 0$ then for ψ_g-almost every x in X, $\underline{f}_g(x) = \bar{f}_g(x) = 0$.*

.3 *if ψ^g is σ-finite on X and everywhere in X, $\underline{f}_g(x) = \bar{f}_g(x) = 0$ then $\psi^f(X) = 0$.*

.4 *if ψ^g is σ-finite on X and at every point x in X either $\underline{f}_g(x)$ or $\bar{f}_g(x)$ vanishes, then $\psi_f(X) = 0$.*

.5 *if ψ^g is σ-finite on X and both extreme derivatives $\underline{f}_g(x)$ and $\bar{f}_g(x)$ are finite everywhere in X, then ψ^f is σ-finite on X.*

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Proof. To prove (.1) suppose that ψ^f is σ-finite on X and write $Y_1 = \{x \in X : \bar{f}_g(x) = +\infty\}$ and $Y_2 = \{x \in X : \underline{f}_g(x) = -\infty\}$. The proof amounts to showing that $\psi_g(Y_1 \cup Y_2) = 0$. We give the details for $\psi_g(Y_1) = 0$. For any $\mathbf{H} \in \mathfrak{H}[X]$ and any natural number n write

$$\mathbf{H}^* = \{(I, x) \in \mathbf{H} : f(I)/g(I) > n\}$$

and observe that $\mathbf{H}^*$ must belong to $\mathfrak{H}_v[Y_1]$ and that every $(I, x) \in \mathbf{H}^*$ has $n\,|g(I)| \leq |f(I)|$. From this we can compute $n\psi_g(Y_1) \leq nV(g, \mathbf{H}^*) \leq V(f, \mathbf{H})$ and hence that $n\psi_g(Y_1) \leq \psi^f(X)$. If $\psi^f(X) < +\infty$ then certainly $\psi_g(Y_1) = 0$; if ψ^f is σ-finite on X, then the same conclusion is obtained by splitting X into a sequence of sets of finite measure.

This completes the proof of (.1). It is recognizably similar to the classical proofs of this type of statement in which the Vitali theorem plays a role and this is the source of the idea. Since the remaining proofs in this theorem employ precisely the same devices, we may omit them.

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