Canad. Math. Bull. Vol. **23** (1), 1980

AUSLANDER–REITEN SEQUENCES FOR "NICE" TORSION THEORIES OF ARTINIAN ALGEBRAS

BY
K. W. ROGGENKAMP

Let $\mathfrak{k}$ be a field and $\mathfrak{A}$ a finite dimensional $\mathfrak{k}$-algebra. Auslander–Reiten sequences [AR] play a fundamental rôle in the representation theory of $\mathfrak{A}$; in particular, they can be used to construct new indecomposable modules from known ones. For the latter reason I think it worthwile to point out certain torsion theories $\mathfrak{T}$ on the category of $\mathfrak{A}$-modules, such that the category of $\mathfrak{T}$-torsionfree modules has Auslander–Reiten sequences; thus giving another construction of indecomposable modules. It should be noted, that these Auslander–Reiten sequences are different from the ordinary ones.

The impetus to the following observations came from discussions with C. M. Ringel [RR2].

Let $\mathfrak{T}$ be a hereditary torsion theory on $\mathfrak{A}$, generated by the simple $\mathfrak{A}$-modules $S'_1, \ldots, S'_n$ (cf. [S]); i.e. the $\mathfrak{T}$-torsionfree modules are the modules $U \in {}_{\mathfrak{A}}\mathfrak{M}_{\mathfrak{f}}$—the category of finitely generated left $\mathfrak{A}$-modules—such that the socle of $U-\mathrm{Soc}(U)$—has no direct summand in $\{S'_1, \ldots, S'_n\}$. If $S_1, \ldots, S_t$ are the $\mathfrak{T}$-torsionfree simple $\mathfrak{A}$-modules, we denote by $E_i = E(S_i)$, $1 = i = t$, the injective envelope of S_i. The $\mathfrak{T}$-torsionfree $\mathfrak{A}$-modules then can be described as those modules, whose injective envelope decomposes into a direct sum of E_i's. We denote the full subcategory of the torsion free $\mathfrak{A}$-modules by $\mathfrak{X}$. The complete ring of quotients of $\mathfrak{A}$ with respect to $\mathfrak{T}$ is then $\mathfrak{L} = \mathrm{End}_{\mathrm{End}(\oplus_{i=1}^t E)}(\oplus_{i=1}^t E_i)$, and we have a natural homomorphism

$$\mathfrak{f} : \mathfrak{A} \to \mathfrak{L}.$$

In order to make our construction work, we have to make the following assumption on:

(1) (i) $\mathrm{End}_{\mathfrak{A}}(E_i) = \mathfrak{k}_i$ is a skewfield, $1 \le i \le t$,

(ii) $\mathrm{Hom}_{\mathfrak{A}}(E_i, E_j) = 0$ for $1 \le i, j \le t$, $i \ne j$.

(2) REMARK. The above conditions make sure that

(i) $\mathfrak{L}$ is a semisimple $\mathfrak{k}$-algebra,

(ii) for every $U \in \mathfrak{X}$, the modules of quotients of U with respect to $\mathfrak{T}$ is $E(U)$, the injective envelope of U; in particular, $\mathfrak{L}$ is injective as left $\mathfrak{A}$-module (cf. [S, p. 202, 2.3]).

Received by the editors July 6, 1978.

On the other hand, given any homomorphism of $\mathfrak{k}$-algebras

$$\kappa : \mathfrak{A} \to \mathfrak{L}$$

such that $\mathfrak{L}$ is semisimple and $\mathfrak{L}$ is an injective $\mathfrak{A}$-module, then this homomorphism is induced by a torsion theory satisfying (1).

(3) Example. Let $\mathfrak{A}$ be the tensoralgebra of the graph $\to\!\!<$; i.e.

$$\mathfrak{A} = \left\{ \begin{pmatrix} \mathfrak{k}_1 & \mathfrak{k}_1 & \mathfrak{k}_1 & \mathfrak{k}_1 \\ 0 & \mathfrak{k}_1 & \mathfrak{k}_1 & \mathfrak{k}_1 \\ 0 & 0 & \mathfrak{k}_1 & \mathfrak{k}_1 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \mathfrak{k}_2 & \mathfrak{k}_2 & \mathfrak{k}_2 & \mathfrak{k}_2 \\ 0 & \mathfrak{k}_2 & \mathfrak{k}_2 & \mathfrak{k}_2 \\ 0 & 0 & \mathfrak{k}_2 & \mathfrak{k}_2 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \alpha \in \mathfrak{k} \right\}, \quad \mathfrak{k}_1 = \mathfrak{k}_2 = \mathfrak{k}.$$

We choose

$$S_1 = \begin{pmatrix} \mathfrak{k}_1 \\ \mathfrak{k}_1 \\ 0 \\ 0 \end{pmatrix} \Big/ \begin{pmatrix} \mathfrak{k}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad S_2, \begin{pmatrix} \mathfrak{k}_2 \\ \mathfrak{k}_2 \\ 0 \\ 0 \end{pmatrix} \Big/ \begin{pmatrix} \mathfrak{k}_2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

then

$$E(S_1) = \begin{pmatrix} \mathfrak{k}_1 \\ \mathfrak{k}_1 \\ \mathfrak{k}_1 \\ \mathfrak{k}_1 \end{pmatrix} \Big/ \begin{pmatrix} E_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad E(S_2) = \begin{pmatrix} \mathfrak{k}_2 \\ \mathfrak{k}_2 \\ \mathfrak{k}_2 \\ \mathfrak{k}_2 \end{pmatrix} \Big/ \begin{pmatrix} \mathfrak{k}_2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $\mathfrak{L} = (\mathfrak{k}_1)_3 \Pi (\mathfrak{k}_2)_3$; moreover, $\kappa : \mathfrak{A} \to \mathfrak{L}$ has

$$\operatorname{Im}(\kappa) = \left\{ \begin{pmatrix} \mathfrak{k}_1 & \mathfrak{k}_1 & \mathfrak{k}_1 \\ 0 & \mathfrak{k}_1 & \mathfrak{k}_1 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \mathfrak{k}_2 & \mathfrak{k}_2 & \mathfrak{k}_2 \\ 0 & \mathfrak{k}_2 & \mathfrak{k}_2 \\ 0 & 0 & \alpha \end{pmatrix}, \alpha \in \mathfrak{k} \right\}.$$

(4) Proposition. $\mathfrak{X}$ has enough projective and injective objects.

Proof. Let $\mathfrak{a} = \operatorname{Ker}(\kappa)$, then the projective $\mathfrak{A}/\mathfrak{a}$-modules are precisely the projective objects in $\mathfrak{X}$. In order to construct the injective objects in $\mathfrak{X}$, we have to make a *detour.*

Let $R = \mathfrak{k}[\![X]\!]$ be the ring of formal power series over $\mathfrak{k}$, and let K be the quotient field of R. Then

$$\Gamma = R \otimes_{\mathfrak{k}} \mathfrak{L}$$

is a hereditary R-order in $A = K \oplus_R \Gamma$; moreover, Γ has exactly t non-isomorphic indecomposable lattices, one for each simple $\mathfrak{L}$-module. Let Λ be the pullback of the diagram

$$\begin{array}{ccc} \mathfrak{A}/\mathfrak{a} & \hookrightarrow & \mathfrak{L} \\ \uparrow & & \uparrow \\ \Lambda & \dashrightarrow & \Gamma \end{array} \tag{5}$$

https://doi.org/10.4153/CMB-1980-008-7 Published online by Cambridge University Press

If $I=\operatorname{Ker}(\Gamma\to\mathfrak{L})$, then I is a two-sided Γ-ideal and $\Lambda/I\cong\mathfrak{A}/\mathfrak{a}$. Since $I\subset\operatorname{rad}(\Lambda)$, we can lift projective modules, and so Λ has as many non-isomorphic indecomposable projective modules as has $\mathfrak{A}/\mathfrak{a}$. We put $\bar{\mathfrak{A}}=\mathfrak{A}/\mathfrak{a}$. Let $I_1,\ldots,I_m$ be the injective Λ-lattices; i.e. injective objects in ${}_\Lambda\mathfrak{M}^0$, the category of left Λ-lattices. Then these are of the form $\operatorname{Hom}_R(Q,R)$, for Q an indecomposable projective right Λ-module.

(6) The modules I_j/II_j, $i\le j\le m$, are the injective objects in $\mathfrak{X}$.

Proof of (6). The embedding $I_j\to\Gamma I_j$ induces an embedding

$$I_j/II_j\to\Gamma I_j/II_j,$$

however, $\Gamma I_j/II_j$ is a $\mathfrak{L}$-module; whence injective over $\mathfrak{A}$, and so I_j/II_j is in $\mathfrak{X}$. Using the injectivity of I_j as Λ-lattice, it is easily seen that I_j/II_j is an indecomposable injective object in $\mathfrak{X}$, if one observes that reduction modulo I is an exact functor (cf. [R2]), and that reduction modulo I is a representation equivalence between ${}_\Lambda\mathfrak{M}^0$ and $\mathfrak{X}$ (cf. [RR1]).

Example (3) continued. $\Gamma=(R_1)_3\Pi(R_2)_3$ and

$$\Lambda=\left\{\begin{pmatrix} R_1 & R_1 & R_1\\ XR_1 & R_1 & R_1\\ XR_1 & XR_1 & \alpha\end{pmatrix}\begin{pmatrix} R_2 & R_2 & R_2\\ XR_2 & R_2 & R_2\\ XR_2 & XR_2 & \alpha+XR_2\end{pmatrix},\ \alpha\in R\right\}\quad R_1=R_2=R.$$

The injective objects in $\mathfrak{X}$ are

$$\begin{pmatrix}\mathfrak{k}_1\\ \mathfrak{k}_1\\ \mathfrak{k}_1\end{pmatrix},\ \begin{pmatrix}\mathfrak{k}_2\\ \mathfrak{k}_2\\ \mathfrak{k}_2\end{pmatrix},\ \begin{pmatrix}\mathfrak{k}_1\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}\mathfrak{k}_2\\ 0\\ 0\end{pmatrix},\ \left\{\begin{pmatrix}\mathfrak{k}_1\\ \mathfrak{k}_1\\ \alpha\end{pmatrix}\begin{pmatrix}\mathfrak{k}_2\\ \mathfrak{k}_2\\ \alpha\end{pmatrix},\ \alpha\in\mathfrak{k}\right\}.$$

It should be noted that not all of these modules are injective $\bar{\mathfrak{A}}$-modules.

(7) Definitions. (i) An exact sequence of objects in $\mathfrak{X}$ is said to be an *Auslander–Reiten sequence*

$$\varepsilon:0\to M\xrightarrow{\varphi}E\xrightarrow{\psi}N\to 0,\text{ if}$$

α.) ε is not split exact,

β.) M and N are indecomposable,

γ.) given any homomorphism $\alpha:X\to M(\beta:M\to Y)$, $X(Y)\in\mathfrak{X}$ such that α is not a split epimorphism (β is not a split monomorphism), then there exists $\sigma:X\to E(\tau:E\to Y)$ with $\alpha=\sigma\psi(\beta=\varphi\tau)$.

(ii) $\mathfrak{X}$ is said to have Auslander–Reiten sequences, if given any $N\in\mathfrak{X}$, N indecomposable not projective in $\mathfrak{X}$ ($M\in\mathfrak{X}$, M indecomposable not injective in $\mathfrak{X}$), then there exists an Auslander–Reiten sequence

$$0\to M\to E\to N\to 0.$$

https://doi.org/10.4153/CMB-1980-008-7 Published online by Cambridge University Press

By (i) this sequence is uniquely determined up to isomorphism—if it exists. We write $M=\Delta_{\mathfrak{X}}^{+}(N)$ and $N=\Delta_{\mathfrak{X}}^{-}(M)$, and define $\Delta_{\mathfrak{X}}^{\pm s}$ inductively.

(9) Proposition. $\mathfrak{X}$ has Auslander–Reiten sequences.

Proof. We shall use the notation of (4). If N is indecomposable non-projective in $\mathfrak{X}$, then there exists an indecomposable non-projective Λ-lattice L with $L/IL\cong N$. ${}_{\Lambda}\mathfrak{M}^0$ has Auslander–Reiten sequences [A, R1, e.a.], so we choose one

$$0\to L'\to L_0\to L\to 0.$$

Since reduction modulo I is exact and preserves non-split sequences, one sees that

$$0\to L'/IL'\to L_0/IL_0\to N\to 0$$

is an Auslander–Reiten sequence in $\mathfrak{X}$. A similar construction is done if M is indecomposable non-injective in $\mathfrak{X}$. (Observe that the injective objects in $\mathfrak{X}$ are in bijection with the injective Λ-lattices.)

It is possible to give an explicit description of the Auslander–Reiten sequences in $\mathfrak{X}$, similar to the one given in [R3].

We conclude with some remarks on hereditary $\mathfrak{k}$-algebras.

(10) If $\mathfrak{A}$ is a hereditary $\mathfrak{k}$-algebra and $\mathfrak{T}$ a torsiontheory satisfying (1), then $\mathfrak{A}/\mathfrak{a}$ is again a hereditary $\mathfrak{k}$-algebra; in fact, $\mathfrak{a}$ is the torsion submodule of $\mathfrak{A}$, and so $\mathfrak{a}$ is the maximal left $\mathfrak{A}$-ideal which has composition factors only in $\{S_1',\ldots,S_n'\}$. In order to see that $\mathfrak{A}/\mathfrak{a}$ is hereditary, let $P/\mathfrak{a}P$ be a projective $\mathfrak{A}/\mathfrak{a}$-module, and let U be an $\mathfrak{A}/\mathfrak{a}$-submodule of $P/\mathfrak{a}P$. We form the pullback

$$\begin{array}{ccc} P & \longrightarrow & P/\mathfrak{a}P \\ \uparrow & & \uparrow \\ Q & \dashrightarrow & U \end{array}$$

Then Q is a projective $\mathfrak{A}$-module, and it remains to show that $\mathfrak{a}Q$ is its torsion submodule. But $\mathfrak{a}Q$ is the kernel of the map

$$Q\to\mathfrak{L}\oplus_{\mathfrak{A}}Q,$$

our torsion theory being perfect, and so $\mathfrak{a}Q=\mathfrak{a}P$, and U is a projective $\mathfrak{A}/\mathfrak{a}$-module.

Modifying the proof of [AP] one thus obtains (cf. [R3]);

(11) Let $\mathfrak{A}$ be a hereditary $\mathfrak{k}$-algebra and $\mathfrak{T}$ a torsion theory satisfying (1), then the following are equivalent for the category $\mathfrak{X}$:

(i) $\mathfrak{X}$ has a finite number of non-isomorphic indecomposable objects.

(ii) For every indecomposable object $U\in\mathfrak{X}$ there exists an $s\in\mathbb{N}$ with $U\cong\Delta_{\mathfrak{X}}^{-s}(P)$, where P is indecomposable projective in $\mathfrak{X}$.

Finally we turn to the situation where $\mathfrak{A}$ is the tensoralgebra of a $\mathfrak{k}$-species with valued graph $\mathfrak{S}$.

(12) We say that an oriented graph G with valuation and without oriented loops is *reducible*, if there exists an edge $a \xrightarrow{(1,1)} b$ such that the graph $G \setminus \langle a \xrightarrow{(1,1)} b \rangle$ which is obtained from G by removing the edge $a \xrightarrow{(1,1)} b$ is the disjoint union $G' \cup A_s$ of some graph G' and a second graph of type A_s, in such a way that a is a sink in G' and b is a sink in A_s. We then denote by $G_{a=b}$ the graph obtained from G by identifying a and b and omitting the edge between them, and we say that $G_{a=b}$ is *obtained from G by reduction.*

(13) Let $\mathfrak{A}$ be the tensoralgebra of a $\mathfrak{k}$-species for $\mathfrak{S}$; then $\mathfrak{A}/\mathfrak{a}$ is the tensoralgebra of a $\mathfrak{k}$-species corresponding to a graph G. The following statements are equivalent:

(i) $\mathfrak{X}$ has a finite number of non-isomorphic indecomposable objects,

(ii) G can be reduced (by the process described in (12)) to a Dynkin diagram.

This statement is proved in [RR2] under nearly the same hypotheses.

References

[A] Auslander, M., *Existence theorems for almost split sequences*, Oklahoma Ring theory conference, March 1976.

———, *Representation theory of Artin algebras* I, II, Comm. Algebra, **1,** 177–268 (1974), **1,** 239–294 (1975).

[AP] Auslander, M. and I. M. Platzeck, *Representation theory of hereditary Artin algebras*, Preprint 1976.

[AR] Auslander, M. and I. Reiten, *Representation theory of Artin algebras III*, Comm. Algebra **3,** 239–294 (1975).

[DR] Dlab, V. and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Memoir Am. Math. Soc. **173,** 1976.

[R1] Roggenkamp, K. W., *The construction of almost split sequences for integral group rings and orders*, Comm. Algebra **5,** 1363–1373.

[R2] ———, *Some examples of orders of global dimension two*, Math. Z. **154,** 225–238 (1977).

———, *Orders of global dimension two*, Math. Z. 160, 63–67, (1978).

[R3] ———, *Indecomposable representations of orders of global dimension two II*, to appear J. Algebra.

[RR1] Ringel, C. M. and K. W. Roggenkamp, *Diagrammatic methods in representation theory of orders*, to appear J. Algebra.

[RR2] ———, *Indecomposable representations of orders of global dimension two I*, to be published.

[S] Stenström, B., *Rings of quotients*, Grundl. Math. **217,** Springer-Verlag 1975.

Mathematisches Institut B
Universitat Stuttgart
7 Stuttgart 1,
Postfach 560, West Germany