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ON AN IDENTITY RELATING TO PARTITIONS AND REPETITIONS OF PARTS

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This note is concerned with a simple but rather surprising identity which emerged unexpectedly from the work of one of the authors on the characterisation of characters. Consider, for example, the seven partitions of 5. These are

(1) $5, 4\ 1, 3\ 2, 3\ 1^2, 2^2\ 1, 2\ 1^3, 1^5$

and with each of these we can associate a product of factorials of the numbers of repetitions, respectively

(2) $1!, (1!)(1!), (1!)(1!), (1!)(2!), (2!)(1!), (1!)(3!), 5!$

It is then seen that the product of all the numbers occurring in (1) coincides with that of all the numbers in (2).

Generally, for any particular natural number n, the partitions can be written in the form

$$1^{a_1}\, 2^{a_2}\, 3^{a_3} \dots,$$

in which a_k is the frequency of repetition of the part k, and are enumerated by the distinct sets $\{\alpha\} = \{a_1, a_2, \dots,\}$ with $a_k \geqq 0$ and $\sum ka_k = n$. The equality noted above is then a particular case (for $n = 5$) of the general identity

$$(3)\quad \prod_{\{\alpha\}} \left[\prod_k (k^{a_k})\right] = \prod_{\{\alpha\}} \left[\prod_k (a_k!)\right].$$

Concerning the proof of this in the context in which it emerged [**1**], it will here suffice to remark that, in the case of the symmetric group, the group of integer-valued class functions contains, as a subgroup of finite index, the group of generalised characters. The order of the corresponding factor group can be calculated by two different methods, leading to (3).

We give here instead a direct proof of the identity based on generating functions. Taking logarithms of equation (3) we have to establish that

$$\sum_{\{\alpha\}} \left[\sum_k (a_k \log k)\right] = \sum_{\{\alpha\}} \left[\sum_k \left(\sum_{j=1}^{a_k} \log j\right)\right]$$

which will be true, and only likely to be true, if the coefficients of the

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different logarithms are the same on either side; that is

$$\sum_{\{\alpha\}} [a_k] = \sum_{\{\alpha\}} \left[\sum_{j, a_j \geqq k} 1 \right]$$

or

$$(4) \quad \sum_{\{\alpha\}} [a_k] = \sum_{j=1}^{\infty} \left[\sum_{\{\alpha\}, a_j \geqq k} 1 \right]$$

for each $k > 1$. In fact, as we now show, (4) is true for $k \geqq 1$, and all n.

It is well known that the total number of partitions of n, $p(n) = \sum_{\{\alpha\}} 1$, is given by the coefficient of x^n in the expansion of the generating function

$$G(x) = \prod_{m=1}^{\infty} (1 - x^m)^{-1}.$$

The sum on the left side of (4) is obtained from a generating function in which the factor $(1 - x^k)^{-1}$ of $G(x)$ is replaced by $x^k(1 - x^k)^{-2}$, so that the term $(x^k)^{a_k}$ is weighted by the factor a_k.

On the right side, for each j, we have to sum over partitions for which $a^j \geqq k$. These are enumerated by replacing the factor $(1 - x^j)^{-1}$ of $G(x)$ by $(x^j)^k(1 - x^j)^{-1}$. Therefore the expression on the right side of (4) is generated by

$$\sum_{j=1}^{\infty} (x^j)^k G(x) = \frac{x^k}{1 - x^k} G(x)$$

as is the left side, establishing (4) for every $k \geqq 1$.

We have thus established the identity (3) by showing that the frequency with which any integer factor appears in the detailed product expansion is the same on both sides and given by (4).

Reference

1. M. S. Kirdar, *On the factor group of integer-valued class functions modulo the group of the generalized characters*, in preparation.

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