Geometrical Illustrations of Cyclant Substitutions.

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§ 1. The following three illustrations of cyclant substitutions are supplementary to those given in § 5 of a previous paper.*

In the cyclant substitution in three variables with equimodular multipliers

$$\left.\begin{aligned} x_1' &= h_3 x_1 + h_2 x_2 + h_1 x_3 \\ x_2' &= h_1 x_1 + h_3 x_2 + h_2 x_3 \\ x_3' &= h_2 x_1 + h_1 x_2 + h_3 x_3 \end{aligned}\right\},$$

in which the coefficients are real and satisfy the relations

$$h_2 h_3 + h_3 h_1 + h_1 h_2 = 0, \quad h_1^2 + h_2^2 + h_3^2 = 1,$$

the following expressions, as may at once be verified, are absolute invariants:

$$x_1 + x_2 + x_3, \quad x_1^2 + x_2^2 + x_3^2, \quad x_2 x_3 + x_3 x_1 + x_1 x_2.$$

If x_1, x_2, x_3 are three concurrent edges of a rectangular solid, it follows that, for every substitution of the form specified, *the sum of the edges, the length of diagonal, and the superficial area are all invariant.*

§ 2. In a slightly more general form of the substitution, in which the multipliers are still equimodular, we have

$$h_2 h_3 + h_3 h_1 + h_1 h_2 = 0, \quad h_1^2 + h_2^2 + h_3^2 = h^2, \text{ say.}$$

Let the sides BC, CA, AB of any triangle be divided at A', B', C' in any the same ratio, which we may denote by

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$(1+l)/(1-l)$; and let the sides of the triangle $A'B'C'$ be denoted by a', b', c'. Then we easily prove

$$\left.\begin{aligned} a'^2 &= h_3 a^2 + h_2 b^2 + h_1 c^2 \\ b'^2 &= h_1 a^2 + h_3 b^2 + h_2 c^2 \\ c'^2 &= h_2 a^2 + h_1 b^2 + h_3 c^2 \end{aligned}\right\},$$

where

$$h_1 = \tfrac{1}{2}l(1+l), \quad h_2 = -\tfrac{1}{2}l(1-l), \quad h_3 = \tfrac{1}{4}(1-l^2);$$

whence

$$h_2 h_3 + h_3 h_1 + h_1 h_2 = 0, \quad h_1^2 + h_2^2 + h_3^2 = \tfrac{1}{16}(1+3l^2)^2 = h^2, \text{ say.}$$

It can now be shown that the following quantities are invariants, though not absolute, but with multipliers as indicated

$a^2+b^2+c^2$,	mult.	h
$a^4+b^4+c^4$,	,,	h^2
$b^2c^2+c^2a^2+a^2b^2$,	,,	h^2;

and it follows that the areas Δ, Δ' are in the ratio

$$\Delta'/\Delta = h.$$

Hence the *squares* of the sides of $A'B'C'$ are related to the *squares* of the sides of ABC by a cyclant substitution with equimodular multipliers, for all values of l.

By dividing the sides of $A'B'C'$ likewise at A'', B'', C'' in any the same ratio $(1+l')/(1-l')$, and so on, we can obtain an indefinite number of triangles, the squares of whose sides are all connected with those of ABC by substitutions of this form.

It is evident by Geometry that these triangles can all be obtained as orthogonal projections of a system of equilateral triangles in one plane; and it follows that they all possess the same Brocard angle.

If we put $AA' = \alpha$, $BB' = \beta$, $CC' = \gamma$, we shall find

$$\left.\begin{aligned} \alpha^2 &= \eta_3 a^2 + \eta_2 b^2 + \eta_1 c^2 \\ \beta^2 &= \eta_1 a^2 + \eta_3 b^2 + \eta_2 c^2 \\ \gamma^2 &= \eta_2 a^2 + \eta_1 b^2 + \eta_3 c^2 \end{aligned}\right\},$$

where $\quad \eta_1 = -\tfrac{1}{2}(1-l), \quad \eta_2 = \tfrac{1}{2}(1+l), \quad \eta_3 = -\tfrac{1}{4}(1-l^2);$
another substitution of the same type.

§ 3. Consider the cyclic quadrilaterals which can be formed with sides equal to four given lines a, b, c, d. According as

a, b, or c is opposite d, there are three such, whose common area S and circumradius R are given below.

If we write

$$\alpha^2,\ \beta^2,\ \gamma^2 = ad + bc,\quad bd + ca,\quad cd + ab,$$

$$x,\ y,\ z = \beta\gamma/\alpha,\quad \gamma\alpha/\beta,\quad \alpha\beta/\gamma,$$

and denote by X, Y, Z the angles subtended at the circumference of a circle radius R by chords of lengths x, y, z, then the following formulae, closely resembling those for a triangle, are found to hold:

$$S = (s - a \,.\, s - b \,.\, s - c \,.\, s - d)^{\frac{1}{2}},\quad (2s = a + b + c + d),$$

$$R = \frac{xyz}{4S} = \frac{x}{2\sin X} = \frac{y}{2\sin Y} = \frac{z}{2\sin Z}.$$

Taking as standard the case in which a is opposite d, the diagonals are of lengths y, z; the angle between them is X; while the angles of the quadrilateral are Y, Z and their supplements. The other cases come by cyclic interchange of X, Y, Z; x, y, z. The same three angles appear each time, and the diagonals are the lengths x, y, z in pairs.

§ 4. Let the lengths of the sides be subjected to a cyclant substitution of either of the forms (A), (B), given in the *Proceedings*, Vol. XXXIV., p. 25. The following elements will then, by the results there proved, remain unchanged:

(i) the perimeter,

(ii) the sum of squares of sides,

(iii) the product of diagonals,

(iv) the product of sums of opposite sides; *i.e.*, in the standard case, $(a + d)(b + c)$.

§ 5. If we seek the conditions that the new diagonals be equal to the old, each to each, we are reduced virtually to a single transformation,

$$a' = s - a,\quad b' = s - b,\quad c' = s - c,\quad d' = s - d,$$

a very special case of a cyclant substitution in four variables. The relation of the new quadrilateral to the old is remarkable. In the two figures, in addition to the diagonals, we find that the perimeters, the sum of squares of sides, the quantities α, β, γ, the squared differences of each pair of sides, are respectively equal. The third or *external* diagonals are also equal; for in the standard case the square of this diagonal is (*Cf.* Hobson, *Plane Trig.*, § 167)

$$(bd+ca)(cd+ab)\left\{\frac{bc}{(b^2-c^2)^2}+\frac{ad}{(a^2-d^2)^2}\right\};$$

which can be expressed, in terms of the invariants just enumerated, thus:

$$\frac{\beta\gamma}{(\beta^2-\gamma^2)^2}\left\{\beta\gamma\Sigma a^2-\alpha^2(\beta^2+\gamma^2)\right\}.$$

The areas of the two are in the ratio

$$\frac{S'}{S}=\left(\frac{abcd}{s-a\,.\,s-b\,.\,s-c\,.\,s-d}\right)^{\frac{1}{3}};$$

their circumradii in the inverse of this ratio; and the point of intersection of diagonals divides the radius in a ratio which also is unchanged.

§ 6. By changing the linear dimensions of the second figure in the ratio R/R' and superposing it on the first, we obtain two quadrilaterals inscribed in the same circle; with the same point of intersection of diagonals; their respective perimeters, and internal and external diagonals, being in the same fixed ratio. The sums of squares of sides also are in the duplicate, and the areas in the triplicate, of that ratio.

§ 7. What is the condition that the quadrilaterals formed with sides a', b', c', d' shall be congruent, in some order, to those formed with a, b, c, d? It is easily proved that this will happen only when the sum of one pair of sides is equal to the sum of the remaining pair; and then one of the quadrilaterals formed by a, b, c, d will admit of an inscribed circle.