5

Condensates and zero modes on kinks

In this chapter we study the effect of a kink on other bosonic or fermionic fields that may be present in the system. Under certain circumstances, it might be energetically favorable for a bosonic field, denoted by χ, to become non-trivial within the kink. Then we say that there is a "bosonic condensate" which is trapped on the kink. On a domain wall, the condensate has dynamics that are restricted to lie on the world-sheet of the wall.

The situation is similar for a fermionic field though there are subtleties. For a fermionic field, denoted by ψ, the Dirac equation is solved in the presence of a kink background made up of bosonic fields. This determines the various quantum modes that the fermionic excitations can occupy. In several cases, there can be "zero modes" of fermions in the background of a kink and this leads to several new considerations. (Fermionic zero modes were first discovered in [27, 84] in the context of strings.) In addition to the zero mode, there may be fermionic bound states. The high energy states that are not bound to the wall are called "scattering (or continuum) states."

A difference between bosonic and fermionic condensates is that bosonic solutions can be treated classically but fermionic solutions can only be interpreted in quantum theory. For example, while there may be a bosonic solution with $\chi = 0$, the solution $\psi = 0$ of the Dirac equation has no meaning because this solution is not normalizable. Solutions of the Dirac equation are only meant to supply us with the modes that fermionic particles or antiparticles can occupy, and as such are required to be normalizable. It is a separate issue to decide if the modes are occupied or not. A mode contributes to the energy of the soliton only if it is occupied. This is quite different from the bosonic case in which there can be a classical condensate, on top of which there are modes that may or may not be occupied by bosonic particles. Fermions can form a classical condensate only after they have paired up to form bosons ("Cooper pairs"), and this leads to superfluidity or superconductivity.

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Fermionic zero modes can lend solitons some novel properties such as fractional quantum numbers (see Section 5.3).

5.1 Bosonic condensates

Consider the model

$$L = L_{\rm k}[\phi] + \frac{1}{2}(\partial_\mu \chi)^2 - U(\phi, \chi) \qquad (5.1)$$

where $L_{\rm k}[\phi]$ is the Lagrangian that leads to a kink solution in ϕ. For example, $L_{\rm k}$ can be the Lagrangian for the Z_2 or sine-Gordon models discussed in Chapter 1. χ is another scalar field that interacts with ϕ via some general interaction potential $U(\phi, \chi)$. Note that $U(\phi, \chi)$ does not contain any terms that are independent of χ – those are included in the potential, $V(\phi)$, that occurs in $L_{\rm k}$. As an example, we could have

$$U(\phi, \chi) = -\frac{m_\chi^2}{2}\chi^2 + \frac{\lambda_\chi}{4}\chi^4 + \frac{\sigma}{2}\phi^2\chi^2 \qquad (5.2)$$

We are assuming that the parameters in the model are chosen so that the minimum of the full potential, $V + U$, is at $\phi \neq 0$ but $\chi = 0$. This requirement also excludes terms that are linear in χ (e.g. $\phi^2\chi$).

In the fixed background of the kink, χ satisfies the classical equation of motion

$$\partial_t^2\chi - \partial_x^2\chi + U_\chi(\phi_{\rm k}(x), \chi) = 0 \qquad (5.3)$$

where U_χ denotes the derivative of U with respect to χ and $\phi_{\rm k}$ is the kink solution. Far from the wall, the lowest energy solution is $\chi(\pm\infty) = 0$.

A solution to Eq. (5.3) is $\chi(x) = 0$ and the energy of this solution is equal to the kink energy in the model $L_{\rm k}$. However, the trivial solution may not be the one of lowest energy. To show that a lower energy solution exists, we need only show that the trivial solution, $\chi = 0$, is unstable. Then we consider linearized perturbations of the form $\chi = \cos(\omega t) f(x)$ around the trivial solution. Inserting this form into Eq. (5.3) leads to the Schrödinger equation

$$-\partial_x^2 f + U_{\chi\chi}(\phi_{\rm k}(x)) f = \omega^2 f \qquad (5.4)$$

where $U_{\chi\chi}$ denotes the second derivative of V with respect to χ. If this equation has solutions with $\omega^2 < 0$, it implies that there are solutions for χ on the kink background that grow with time as $\cosh(+|\omega| t)$, denoting an instability of the state with $\chi = 0$. This means that the solution with least energy must have a non-trivial χ configuration. The lowest energy χ configuration is non-zero inside the kink and vanishing outside and is called a "bosonic condensate" (or simply "condensate").

5.1.1 Bosonic condensate: an example

A simple example in which there is a bosonic condensate on a Z_2 kink can be found in the model of Eq. (5.1), or explicitly,

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 + \frac{m_\chi^2}{2}\chi^2 - \frac{\lambda_\chi}{4}\chi^4 - \frac{\sigma}{2}\phi^2\chi^2 \tag{5.5}$$

Ignoring the condensate field χ, the kink solution is

$$\phi_{\rm k} = \eta \tanh\left(\frac{x}{w}\right) \tag{5.6}$$

and the Schrödinger equation corresponding to Eq. (5.4) is

$$-\partial_x^2 f + \left[-m_\chi^2 + \sigma\eta^2 \tanh^2 X \right] f = \omega^2 f \tag{5.7}$$

where $X = x/w$.

With $\sigma\eta^2 > m_\chi^2$, we see that the Schrödinger potential is asymptotically positive, and hence $f(\pm\infty) = 0$. This is consistent with the requirement that χ not have a vacuum expectation value. At the origin, $U_{\chi\chi} < 0$, and hence the Schrödinger potential is a well that is centered at the origin. Since a potential well in one dimension always has a bound state [139], it follows that there is at least one bound state for χ. For a deep enough well i.e. large enough m_χ^2, the bound state has negative energy eigenvalue ($\omega^2 < 0$), and the trivial solution $\chi = 0$ is unstable. Hence there is a range of parameters (m_χ^2) for which a χ condensate exists.

To determine the range of m_χ^2 for which there is an instability, consider the critical case when there is a zero eigenvalue solution, f_0, of Eq. (5.7). Then we can write

$$-\partial_x^2 f_0 + \frac{\sigma\eta^2}{3}[3\tanh^2 X - 1] f_0 = \left[m_\chi^2 - \frac{\sigma\eta^2}{3} \right] f_0 \tag{5.8}$$

This is exactly the same form as Eq. (3.8), together with the potential in Eq. (3.10), provided we identify 3λ with σ, and ω^2 with the term within square brackets on the right-hand side. Since the lowest energy eigenvalue is zero for Eq. (3.8), there is a zero eigenvalue for Eq. (5.7) if

$$m_\chi^2 = \frac{\sigma\eta^2}{3} \tag{5.9}$$

For a larger value of m_χ^2, Eq. (5.7) has a negative eigenvalue, signaling an instability. Therefore a condensate solution exists in the range

$$\frac{\sigma\eta^2}{3} < m_\chi^2 < \sigma\eta^2 \tag{5.10}$$

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The exact profile for the condensate can be found by solving the full coupled equations of motion for ϕ and χ. This includes the non-linear terms in χ and the back-reaction of the condensate on the kink, and, in most cases, has to be done numerically. Let us denote the solution obtained in this way by $(\phi_\mathrm{k}(x), \chi_0(x))$. Then

$$\chi(t, x, y, z) = \chi_0(x)\cos(\omega t - k_y y - k_z z + \theta_0), \quad \omega = \sqrt{k_y^2 + k_z^2} \tag{5.11}$$

where θ_0 is a constant, is also a solution. The reason is simply that

$$\left(\partial_t^2 - \partial_y^2 - \partial_z^2\right)\cos(\omega t - k_y y - k_z z + \theta_0) = 0 \tag{5.12}$$

The trigonometric form of the solution in Eq. (5.11) was chosen so as to obtain a real solution. An identical analysis in the case when χ is a complex field leads to

$$\chi(t, x, y, z) = \chi_0(x)\mathrm{e}^{\pm\mathrm{i}(\omega t - k_y y - k_z z + \theta_0)} \tag{5.13}$$

The solution represents waves propagating in the (k_y, k_z) direction in the plane of the domain wall.

5.2 Fermionic zero modes

Fermionic fields may be coupled to the kink via terms that respect the discrete symmetries in the bosonic sector that are responsible for the existence of the kink. In the case of the Z_2 model, the coupling can be a Yukawa term and the Lagrangian may be written as

$$L = L_\phi + \mathrm{i}\bar{\psi}\,\partial\!\!\!/\psi - g\phi\bar{\psi}\psi \tag{5.14}$$

where L_ϕ denotes the scalar part of the Lagrangian, $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$, g is the coupling constant, ψ is a four-component fermionic field, and γ^μ are the Dirac matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ being the space-time metric. The explicit representation of the Dirac matrices that we adopt is

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{5.15}$$

where $i = 1, 2, 3$ (also sometimes written as $i = x, y, z$) and the Pauli spin matrices are defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.16}$$

The Yukawa interaction term in the model in Eq. (5.14) respects the $\phi \to -\phi$ symmetry of the Z_2 model provided we also transform the fermion field by

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$\psi \to \psi' = \gamma^5 \psi$ where

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{5.17}$$

This can be seen by using the properties $(\gamma^5)^\dagger = \gamma^5$, $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = 1$.[1]

If $\phi_{\mathrm{k}}(x)$ denotes the kink solution, the Dirac equation in the kink background is

$$\mathrm{i}\partial\!\!\!/\psi - g\phi_{\mathrm{k}}(x)\psi = 0 \tag{5.18}$$

Let us first try and solve Eq. (5.18) explicitly. Recognizing that ϕ_{k} does not depend on t, y, and z, we write the ansatz

$$\psi = f(t, y, z)\xi(x) \tag{5.19}$$

where $f(t, y, z)$ is a function while $\xi(x)$ is a four-component spinor. With this ansatz, the Dirac equation separates

$$\mathrm{i}\gamma^a\partial_a f = -\gamma^a k_a f \tag{5.20}$$

$$\mathrm{i}\gamma^x\partial_x\xi - g\phi_{\mathrm{k}}\xi = +\gamma^a k_a \xi \tag{5.21}$$

where $\gamma^a k_a$ is the constant matrix of separation and the index a runs over t, y, z. Requiring that the fermion be localized on the wall, we get the boundary conditions

$$\xi(\pm\infty) = 0 \tag{5.22}$$

These boundary conditions are valid only for bound states. If we wish to consider the scattering of fermions off a domain wall, we would choose incoming and reflected plane waves at $x = -\infty$.

The Dirac equations have an infinite number of solutions, corresponding to all the fermion eigenmodes in the domain wall background. These include fermionic bound states and scattering states. There is one state, however, which is novel because it leads to some very interesting properties of the soliton, described in the sections below. This state is the one with zero energy eigenvalue, also called the "zero mode."

Equation (5.20) can be solved

$$f = \exp(\mathrm{i}k_a x^a) \equiv \exp(\mathrm{i}(\omega t - k_y y - k_z z)) \tag{5.23}$$

Zero energy is obtained by setting $\omega = 0 = k_y = k_z$ and then $f = 1$. Let us first look at this case ($k_a = 0$).

Multiplying Eq. (5.21) by $\mathrm{i}\gamma^x$ we see that $\mathrm{i}\gamma^x\xi$ satisfies the same equation of motion as ξ. Therefore if ξ is a solution, then so is $\mathrm{i}\gamma^x\xi$. Hence solutions to the

[1] The Yukawa term does not respect the $\phi \to \phi + 2\pi/\beta$ symmetry of the sine-Gordon model and hence our discussion of fermion zero modes cannot be used for that case. Nor do we consider the case of fermions with Majorana mass terms [147].

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Dirac equation come in distinct pairs unless ξ is an eigenstate of $i\gamma^x$, in which case the two solutions ξ and $i\gamma^x\xi$ are not distinct. The zero mode solution is found by choosing ξ to be an eigenstate of $i\gamma^x$

$$i\gamma^x\xi = c\xi \tag{5.24}$$

and, since $(i\gamma^x)^2 = 1$, we must have $c = \pm 1$. The ξ equation now becomes

$$\partial_x\xi = cg\phi_{\rm k}\xi \tag{5.25}$$

and the solution is

$$\xi(x) = \xi(0)\exp\left[cg\int_0^x \phi_{\rm k}(x')dx'\right] \tag{5.26}$$

Assuming $\phi_{\rm k}(+\infty) > 0$ and $g > 0$, the boundary conditions in Eq. (5.22) are only satisfied if $c = -1$. (The boundary condition at $x = -\infty$ is also satisfied provided $\phi_{\rm k}(-\infty) < 0$.) Therefore the zero mode solution is

$$\xi(x) = \xi(0)\exp\left[-g\int_0^x \phi_{\rm k}(x')dx'\right], \quad (g > 0) \tag{5.27}$$

If $\phi_{\rm k}(+\infty) < 0$ and $g > 0$, the solution is obtained by choosing $c = +1$.

To determine $\xi(0)$, we solve the eigenvalue equation $i\gamma^x\xi(0) = -\xi(0)$ and find

$$\xi(0) = \begin{pmatrix} \alpha \\ \beta \\ i\beta \\ i\alpha \end{pmatrix} \tag{5.28}$$

where α, β are any complex constants. Therefore there are two basis zero modes (with coefficients α and β) and the general zero mode is a linear superposition of these two modes. The constants, α and β, can be fixed by normalizing the wavefunction.

Next consider the case with $k_a \neq 0$. Then Eq. (5.27) is still a solution to Eq. (5.21) provided $k_a\gamma^a\xi(0) = 0$. By explicitly substituting the γ^a matrices and $\xi(0)$, this leads to the two equations

$$k_y\alpha + i(\omega + k_z)\beta = 0 \tag{5.29}$$

$$i(\omega - k_z)\alpha - k_y\beta = 0 \tag{5.30}$$

A solution for α and β exists only if

$$\omega = \pm\sqrt{k_y^2 + k_z^2} \tag{5.31}$$

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which is the dispersion relation for a massless particle (see Fig. 5.1). With this relation, the solutions fix the ratio of α and β to obtain

$$\psi = \frac{N e^{i(\omega t - k_y y - k_z z)}}{2\sqrt{\omega}} e^{-g \int_0^x \phi_k(x')dx'} \begin{pmatrix} \sqrt{\omega + k_z} \\ \mathrm{sgn}(k_y) i \sqrt{\omega - k_z} \\ -\mathrm{sgn}(k_y)\sqrt{\omega - k_z} \\ i\sqrt{\omega + k_z} \end{pmatrix} \tag{5.32}$$

where N is a normalization constant where $\mathrm{sgn}(k_y) \equiv k_y/|k_y|$.

So far we have not specified the exact form of the kink profile ϕ_k and Eq. (5.32) holds for any model in which the Yukawa interaction term respects the symmetries. Next, as an example, we use the solution for the Z_2 kink (see Eq. (1.9)). Then the integral over ϕ_k can be done explicitly to yield

$$\psi = \frac{N e^{i(\omega t - k_y y - k_z z)}}{2\sqrt{\omega}} \left[\mathrm{sech}\left(\frac{x}{w}\right)\right]^{g\sqrt{2/\lambda}} \begin{pmatrix} \sqrt{\omega + k_z} \\ \mathrm{sgn}(k_y) i \sqrt{\omega - k_z} \\ -\mathrm{sgn}(k_y)\sqrt{\omega - k_z} \\ i\sqrt{\omega + k_z} \end{pmatrix}, \quad (g > 0) \tag{5.33}$$

where w is the width of the kink as defined in Eq. (1.21). This is our final expression for the zero mode on the Z_2 kink.

In the asymptotic vacuum, where ϕ is constant, the Dirac equation derived from Eq. (5.14) yields four solutions all with the same momentum. These four states are referred to as spin up and down states for the particle and hole (or antiparticle). On the domain wall, however, there are only *two* zero mode solutions for fixed value of the momentum (k_y, k_z). One of these has positive energy (ω) and the other has negative energy. Therefore the two states may be called particle and hole states but the spin degree of freedom is not present. Consider the special case when $k_y = 0$ and $k_z \neq 0$. Then we have $\omega = \pm k_z$ and the spinor is proportional to $(1, 0, 0, i)^T$ if $\omega = +k_z$, and to $(0, i, -1, 0)^T$ if $\omega = -k_z$. If we also take $k_z = 0$, both these two states have $\omega = 0$ and become degenerate in energy.

The two-fold degeneracy of the zero mode ($\omega = 0$) occurs since we are working in three spatial dimensions where the Dirac spinors have four components. If we find the zero modes in one spatial dimension, the fermions are described by two-component spinors, and then there is only a single zero mode. If we use four-component spinors in one spatial dimension, it amounts to having two two-component spinors that do not interact with each other. Hence the degrees of freedom are doubled.

Note that the boundary conditions in Eq. (5.22) can only be satisfied if ϕ_k changes sign in going from $x = -\infty$ to $+\infty$. So the topological nature of the kink is essential to the existence of the zero mode.

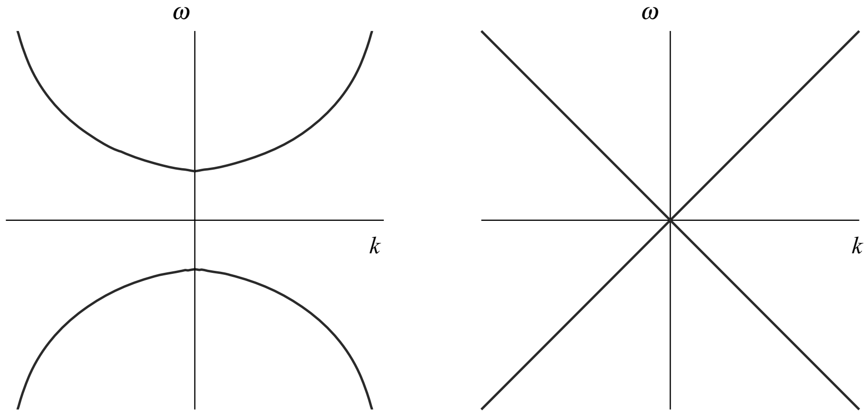

Figure 5.1 The dispersion curve for fermions in the vacuum is shown on the left and for fermion zero modes on the domain wall on the right.

In constructing the zero mode, we have postulated that ξ be an eigenstate of $i\gamma^x$. Therefore there is a possibility that there might be other zero mode solutions. However, it is possible to prove that this is not the case and the zero mode(s) that we have found are the only such solutions. The proof proceeds by choosing a set of orthogonal basis spinors

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \qquad \chi_3 = \begin{pmatrix} i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \chi_4 = \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix} \tag{5.34}$$

The first two spinors are eigenstates of $i\gamma^x$ with eigenvalue -1. These are the spinors that occur in the general solution we have already found subject to the condition that $i\gamma^x\xi = -\xi$. Since the Dirac equation is linear, any new solution must be a linear combination of χ_3 and χ_4. However, both these basis spinors are eigenstates of $i\gamma^x$ with eigenvalue $+1$ and we have seen that such states cannot be part of the solution since the boundary conditions cannot be met. Therefore there are no other zero mode solutions beside the ones that we have already constructed.

As mentioned in the introduction to this chapter, the interpretation of fermionic zero modes is quite distinct from that of bosonic condensates. Fermionic modes should be viewed as states in which the fermions can reside. A mode by itself does not carry energy density or charge or some other physical quantity. Only if the mode is occupied, can it contribute to the energy. However, the zero mode is special in some ways since, even if it is occupied, the fermion occupying the zero mode contributes zero energy. Likewise, if the zero mode is unoccupied, it also contributes zero energy and so the system has a degenerate ground state. Indeed

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the occurrence of a zero mode leads to some novel and important quantum field theoretic consequences that we shall outline in Section 5.3.

In the discussion of fermion zero modes above we have considered only a Yukawa interaction between the fermion field and the field that makes up the domain wall. More generally, there can also be Majorana interactions. Zero modes of Majorana fermions on domain walls have been discussed in [147].

Just like scalar field condensates and fermionic zero modes on domain walls, there can also be gauge field (or spin-1) condensates. These arise when the model has broken gauge symmetries in addition to broken discrete symmetries. This is precisely the situation in the $SU(5) \times Z_2$ model discussed in Chapter 2 and the kinks in the model have condensates of spin-1 fields as we describe in Section 5.5.

Finally we close this section by remarking that there are several mathematical "index theorems" that can be used to obtain information on the number of zero modes on a soliton [176]. In the case of domain walls that we have been discussing, however, the index theorems do not lead to a useful result.

5.3 Fractional quantum numbers

To quantize a fermionic field we find all the modes (solutions of the Dirac equation) and then associate creation and annhilation operators with each of the modes.[2] The same procedure may be followed in the presence of zero modes [83]. As discussed in the previous section, there is a single zero mode on the Z_2 kink (in one spatial dimension), which is denoted by ψ_0. Then the expansion of the field operator in modes is

$$\psi = a_0\psi_0 + \sum_p \left[b_p\psi_{p+} + d_p^\dagger \psi_{p-}^c\right] \tag{5.35}$$

The second term is the usual sum over the positive energy modes, ψ_{p+}, and fermion-number conjugates of the negative energy modes, ψ_{p-}^c.[3] There is no sum over spin because there is no spin degree of freedom in one spatial dimension. The first term in Eq. (5.35) contains the zero mode, ψ_0, and a_0 is the operator associated with this mode. The term may seem strange because the zero mode does not have a corresponding conjugated term. This is because $\psi_0^c = \psi_0$ and so the mode functions associated with a_0 and its conjugate operator are identical. However, one still has the usual equal time anticommutation relations for the field and its canonical momentum

$$\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y})\delta_{ab} \tag{5.36}$$

[2] We work in one spatial dimension in this section and hence spinors have two components.
[3] That is, ψ_{p-}^c is the wavefunction of a hole in the Dirac sea that has momentum p.

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while other anticommutators vanish. Using the expansion in terms of creation and annihilation operators this gives

$$\{a_0, a_0\} = \{a_0^\dagger, a_0^\dagger\} = 0, \qquad \{a_0, a_0^\dagger\} = 1 \tag{5.37}$$

Since the Dirac Lagrangian in Eq. (5.14) is invariant if the fermion fields are rotated by a phase, the model has a conserved fermion number current. The Noether current is given by $\bar{\psi}\gamma^\mu\psi$ ($\mu = 0, 1$). In the quantum theory the physical current operator needs to be normal ordered. This is equivalent to defining the fermion number operator as

$$Q_{\rm f} = \int {\rm d}x : j^0 := \frac{1}{2}\int {\rm d}x(\psi_\alpha^\dagger\psi_\alpha - \psi_\alpha\psi_\alpha^\dagger) \tag{5.38}$$

We can act by this operator on any state to determine the fermion number of that state. Let us denote the state with no positive energy particles and empty zero mode by $|0; -\rangle$ and the state with no positive energy particles and filled zero mode by $|0; +\rangle$. Then the fermion numbers of these two states are

$$\begin{aligned} Q_{\rm f}|0; \pm\rangle &= \frac{1}{2}[a_0^\dagger a_0 - a_0 a_0^\dagger]|0; \pm\rangle \\ &= \frac{1}{2}[2a_0^\dagger a_0 - 1]|0; \pm\rangle \\ &= \pm\frac{1}{2}|0; \pm\rangle \end{aligned} \tag{5.39}$$

Therefore the kink carries a half-integer fermion number of either sign. If the fermion carries electric charge, the electric charge on the kink is also half-integral.

It is critical to not think of the kink as being "kink *plus* fermion." Instead the kink is made of both the bosonic and fermionic fields. Then there are only two states for the kink: one with filled zero mode and the second with the zero mode empty.

Surprising as the half-integer fermion number is, further work in [150, 68] obtained different fractional charges in other systems (see Section 9.1). Indeed, [68] shows that the charges can even be irrational.

5.4 Other consequences

If the bosonic condensate is electrically charged, the domain wall becomes superconducting. To see this in some more detail, consider the case of a complex, electrically charged, scalar field, χ, interacting with the field ϕ that forms a domain wall

$$L = L[\phi] + L[A_\mu] + \frac{1}{2}|D_\mu\chi|^2 - \frac{m_\chi^2}{2}|\chi|^2 - \frac{\lambda_\chi}{4}|\chi|^4 - \frac{\sigma}{2}\phi^2|\chi|^2 \tag{5.40}$$

The first term is the Lagrangian for the Z_2 model and the second is the usual Maxwell Lagrangian for the gauge field A_μ. The covariant derivative is defined by

$$D_\mu = \partial_\mu - iqA_\mu \tag{5.41}$$

The propagating modes of the condensate are

$$\chi = \chi_0(x)e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{5.42}$$

where $\chi_0(x)$ is the condensate profile and $\mathbf{k}$ is the wave-vector restricted to lie in the plane of the wall, the yz-plane. Since χ carries electric charge, q, the electric current is

$$\mathbf{j}_\chi = \frac{iq}{2}(\chi^\dagger \nabla \chi - \chi \nabla \chi^\dagger) \tag{5.43}$$

Inserting Eq. (5.42) into (5.43), we find that the current is along the $\mathbf{k}$ direction

$$\mathbf{j}_\chi = q|\chi_0|^2\mathbf{k} \tag{5.44}$$

The simplest way to see that the wall with the condensate is superconducting is to write

$$\chi = \chi_0(x)e^{i\theta} \tag{5.45}$$

where χ_0 is the condensate solution and θ is the phase variable. Then the low energy Lagrangian for the θ degree of freedom can be derived by integrating the full Lagrangian density, Eq. (5.40), over x to get

$$L(\theta) = \frac{1}{2}(\partial_\mu\theta - eA_\mu)^2 \tag{5.46}$$

where we have omitted an overall constant factor. This effective Lagrangian is the relativistic generalization of the Lagrangian in the Ginzburg-Landau theory of superconductivity. Assuming that the relativistic generalization does not make any qualitative difference, results from the Ginzburg-Landau theory can then be applied directly to the present case. In particular, the domain wall with charged condensate can be expected to carry persistent electric currents, have magnetic vortices, and exhibit the Meissner effect (expulsion of magnetic fields) [61].

We now discuss fermionic superconductivity on domain walls. The relevant modes are given in Eq. (5.33) and the (normal ordered) current is

$$\mathbf{j}_\psi = q : \psi^\dagger \gamma \psi : \tag{5.47}$$

Using the expansion of ψ in terms of creation and annihilation operators (Eq. (5.35)), the current in any Fock state of fermions can be evaluated. Similarly, the electric charge on a domain wall can also be evaluated.

Fermions on domain walls can only make the wall superconducting if they form Cooper pairs and condense. It is believed that the slightest attractive interaction between the fermions on the wall will lead to condensation below some critical temperature. On a domain wall, there are possible channels for attractive interactions. For example, the fermions interact with each other via exchange of ϕ quanta and this can lead to an attractive force. The problem of rigorously showing fermionic superconductivity of domain walls has not been investigated. In particular, the Meissner effect, which is the hallmark of superconductivity, has not been shown. Indeed, the response of non-interacting fermion zero modes to an external magnetic field has been discussed with the conclusion that the walls are diamagnetic [173] (also see [82, 172]).

In the particle-physics/cosmology literature, the existence of electrically charged zero modes is simply assumed to imply superconductivity (though see [15]). A reason for this assumption is that a current on a wall persists even without the application of an external electric field. Once the current carrying fermionic zero mode states have been populated there are very few processes by which these states can be emptied [184]. Two such dissipative processes are the scattering of counter-propagating fermion zero modes, and the scattering of fermion zero modes with fluctuations of the domain wall field itself. Generally these processes occur at a very slow rate, at least in astrophysical situations of interest. Hence, strictly speaking, domain walls in particle physics/cosmology have only been shown to be excellent conductors and not superconductors.

The equilibrium current on a domain wall in any setting depends on the balance of the rates of current increase owing to an external electric field and decrease owing to dissipation. Note that an external magnetic field in which a domain wall is moving is, effectively, an electric field in the rest frame of the wall. Since magnetic fields are ubiquitous in astrophysics, any cosmological domain walls with fermion zero modes can be expected to be current carrying. Superconducting domain walls in realistic grand unification models have been discussed in [98].

The fermion zero mode states that we have discussed above are single particle eigenstates. The true states of the domain wall are also affected by fermion-fermion interactions. The many-body problem falls in the class of two-dimensional systems of interacting fermions. In the presence of a strong external magnetic field, so that the Landau level spacing is large compared to other energy scales, the fermions on the wall are similar to electrons in a quantum Hall system.

5.5 Condensates on $SU(5) \times Z_2$ kinks

In Chapter 2 we have discussed kinks in an $SU(5) \times Z_2$ model, which is the simplest example of a Grand Unified Theory. Even though $SU(5)$ grand unification is known not to be phenomenologically viable, the model is still pedagogically useful.

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The Lagrangian for the model is

$$L = L_\text{b}[\Phi, \chi, X_\mu] + L_\text{f}[\chi, \psi, X_\mu] \tag{5.48}$$

where the $SU(5)$ adjoint field, Φ, does not couple directly to the fermionic fields (denoted generically by ψ). Only an additional $SU(5)$ fundamental field, χ, couples to the fermions. The vacuum expectation value of χ is responsible for electroweak symmetry breaking and the masses of all the observed quarks and leptons arise from this symmetry breaking. The $SU(5)$ symmetry breaking has no consequences for the fermionic sector, except via the χ field. This indirect effect can be important in the presence of kinks, since χ can form a condensate on the kink, which can then interact with some of the fermions. We will discuss this further below.

The bosonic part of the Lagrangian is

$$L_\text{b} = \text{Tr}(\text{D}_\mu \Phi)^2 + |\text{D}_\mu \chi|^2 - V(\Phi, \chi) - \frac{1}{2}\text{Tr}(X_{\mu\nu} X^{\mu\nu}) \tag{5.49}$$

The covariant derivative is defined by $\text{D}_\mu \equiv \partial_\mu - \text{i}g X_\mu$ and its action on the scalar fields is

$$\text{D}_\mu \Phi \equiv \partial_\mu \Phi - \text{i}g[X_\mu, \Phi], \quad \text{D}_\mu \chi \equiv (\partial_\mu - \text{i}g X_\mu)\chi \tag{5.50}$$

The potential is given by

$$V(\Phi, \chi) = V(\Phi) + V(\chi) + \lambda_4(\text{Tr}\Phi^2)\chi^\dagger \chi + \lambda_5(\chi^\dagger \Phi^2 \chi) \tag{5.51}$$

with

$$V(\Phi) = -m_1^2(\text{Tr}\Phi^2) + \lambda_1(\text{Tr}\Phi^2)^2 + \lambda_2(\text{Tr}\Phi^4) \tag{5.52}$$

$$V(\chi) = -m_2^2 \chi^\dagger \chi + \lambda_3(\chi^\dagger \chi)^2 \tag{5.53}$$

Successful grand unification requires that the global minimum of the potential leaves an $SU(3) \times U(1)$ symmetry unbroken.[4] As already described in Section 2.2, the minimum of the potential with χ set equal to zero, occurs at

$$\Phi_0 = \frac{\eta}{2\sqrt{15}}\text{diag}(2, 2, 2, -3, -3) \tag{5.54}$$

(up to $SU(5) \times Z_2$ transformations) in the parameter range

$$\lambda \geq 0, \qquad \lambda' \equiv h + \frac{7}{30}\lambda \geq 0 \tag{5.55}$$

The vacuum expectation value of Φ breaks $SU(5) \times Z_2$ to $[SU(3) \times SU(2) \times U(1)]/Z_6$. If we assume that the back-reaction of a vacuum expectation value of χ

[4] Symmetry breaking patterns have been discussed quite generally in [99].

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on that of Φ is small, we can write down a reduced potential for χ alone

$$V_{\rm red}(\chi;\Phi_0) = \left(-m_2^2 + \lambda_4 {\rm Tr}\Phi_0^2 + \frac{\lambda_5}{15}\eta^2\right)\chi_a^\dagger \chi_a + \left(-m_2^2 + \lambda_4 {\rm Tr}\Phi_0^2 + \frac{3\lambda_5}{20}\eta^2\right)\chi_b^\dagger \chi_b + \lambda_3(\chi^\dagger\chi)^2 \qquad (5.56)$$

where $a = 1, 2, 3$ and $b = 4, 5$. The symmetry is broken to $[SU(3) \times U(1)]/Z_3$ only if the vacuum expectation value of χ is along the χ_4 or χ_5 directions. This further restricts the range of parameters to

$$\frac{\lambda_5}{15}\eta^2 > m_2^2 - \lambda_4 {\rm Tr}\Phi_0^2 > \frac{3\lambda_5}{20}\eta^2, \qquad \lambda_3 > 0 \qquad (5.57)$$

which also implies $\lambda_5 < 0$ and $m^2 < \lambda_4 {\rm Tr}\Phi_0^2$. We assume that these conditions on the parameters are satisfied. Then a minimum of the reduced potential occurs at

$$\chi^T = \eta_{\rm ew}(0, 0, 0, 1, 0) \qquad (5.58)$$

where

$$\eta_{\rm ew}^2 = \frac{1}{2\lambda_3}\left(m_2^2 - \lambda_4 {\rm Tr}\Phi_0^2 - \frac{3\lambda_5}{20}\eta^2\right) \qquad (5.59)$$

is the electroweak symmetry breaking scale. The final $[SU(3) \times U(1)]/Z_3$ symmetry corresponds to the color and electromagnetic symmetries present today.

Next we describe the fermionic sector.[5] There are two fermion fields: ψ, which is in the fundamental (5-dimensional) representation of $SU(5)$ and ζ, which is in the antisymmetric 10-dimensional representation. The known quarks and leptons fit within the components of these fields. With the choice of vacuum expectation values in Eq. (5.58)

$$(\psi^i)_{\rm L} = (d^{c1}, d^{c2}, d^{c3}, e^-, -\nu_e)_{\rm L} \qquad (5.60)$$

$$(\psi^i)_{\rm R} = \left(d_1, d_2, d_3, e^+, -\nu_e^c\right)_{\rm R} \qquad (5.61)$$

$$(\zeta_{ij})_{\rm L} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & u^{c3} & -u^{c2} & u_1 & d_1 \\ -u^{c3} & 0 & u^{c1} & u_2 & d_2 \\ u^{c2} & -u^{c1} & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L \qquad (5.62)$$

(see Eq. (14.9) in [30]). The numerical index on the u and d fields refers to color charge, and the placement (subscript or superscript) depends on the representation

[5] Actually we describe only one of the three families of the standard model fermionic sector, and then too the neutrino is taken to be massless.

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(fundamental or fundamental conjugate) in which the field transforms under the unbroken $SU(3)$. The c superscript denotes charge conjugation:

$$\psi^c \equiv i\gamma^2\psi^* \tag{5.63}$$

The L and R subscripts refer to left- and right-handed spinors

$$\psi_L \equiv \frac{1-\gamma^5}{2}\psi, \qquad \psi_R \equiv \frac{1+\gamma^5}{2}\psi \tag{5.64}$$

The Dirac γ matrices are defined in Eqs. (5.15) and (5.17).

Now we are ready to describe the interactions of the various fields with the $SU(5) \times Z_2$ kink, described as the $q = 2$ kink in Chapter 2.

- In the presence of a ($q = 2$) kink, the vacuum expectation values are

$$\begin{aligned}
\Phi(-\infty) &= +\frac{\eta}{2\sqrt{15}}\text{diag}(2, 2, 2, -3, -3) \\
\chi^T(-\infty) &= \eta_{\text{ew}}(0, 0, 0, 1, 0) \\
\Phi(+\infty) &= -\frac{\eta}{2\sqrt{15}}\text{diag}(2, -3, -3, 2, 2) \\
\chi^T(+\infty) &= \eta_{\text{ew}}(0, 0, 1, 0, 0)
\end{aligned}$$

  Note that the non-trivial entry of χ has to coincide with one of the -3 entries of Φ since this is what minimizes the potential $V(\Phi, \chi)$. Therefore χ must rotate through the kink. Inside the kink, the fields are not pure rotations of the asymptotic values.
- The component Φ_{11} goes from $+2$ to -2 as the wall is crossed. Hence it must vanish in the wall. This is very similar to the Z_2 case, where the field vanishes at the center of the wall. The field χ interacts with Φ as given by the potential in Eq. (5.51). Note the interaction term $\lambda_5 \text{Tr}(\chi^\dagger \Phi^2 \chi)$, which directly couples χ_1 to Φ_{11}. (The other term couples all components of χ to $\text{Tr}\Phi^2$ only.) By explicit construction it can be seen that χ_1 can condense on the wall for a certain range of parameters [146]. Hence the $SU(5) \times Z_2$ model allows for scalar condensates on the wall (see Section 5.1). In addition, since χ_1 is a complex scalar field, the condensate has an associated phase. The choice of phase on different parts of the wall may be different, leading to vortices in χ_1 that can only exist on the wall. Since χ_1 transforms non-trivially under the unbroken $SU(3)$, the vortices carry color magnetic field. This is similar to our discussion below Eq. (5.46).
- Next we consider fermion interactions with the wall [146]. The fermions do not couple directly to Φ. Hence the only coupling to the wall is due to the rotation of χ in passage through the wall and to the condensate in the χ_1 component. Consider the scattering of the fifth component, ψ_5, which corresponds to a neutrino on the left side of the wall but a d-quark on the right. This fifth component has non-zero

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reflection and transmission probability. If it reflects, the particle is still a neutrino. If it transmits, it must change into a d-quark. If a neutrino becomes a d-quark in passing through the wall, it must pick up electric and color charge from the wall. Hence we are forced to conclude that there must be electric and color excitations that live entirely on the wall. If a χ_1 condensate is not present, the only available excitations are the charged gauge field components. Hence charged gauge fields must condense on the wall.

To see the presence of a charged gauge field condensate, it is most convenient to go to a gauge where the scalar field vacuum expectation values are oriented in the same directions on both sides of the wall, as we now discuss.

- Consider a very thin wall, so that

$$\Phi(x<0) = +\frac{\eta}{2\sqrt{15}}\text{diag}(2,2,2,-3,-3) \equiv \Phi_0$$
$$\Phi(x>0) = -\frac{\eta}{2\sqrt{15}}\text{diag}(2,-3,-3,2,2) \tag{5.65}$$

Now we perform a local gauge transformation that rotates Φ into the direction of Φ_0 (up to a sign) everywhere. Such a gauge rotation is local since it is equal to the identity transformation for $x<0$ but is non-trivial for $x>0$ since it exchanges the 23- and 45-blocks of Φ. In both regions, $x<0$ and $x>0$, the gauge rotation is constant. The rotation is non-constant only at $x=0$ i.e. on the wall. Hence the gauge fields after the rotation vanish everywhere except on the wall itself and there are gauge degrees of freedom residing on the wall. A more explicit calculation shows that the gauge fields living on the wall carry electric and color charge.

5.6 Possibility of fermion bound states

In addition to fermionic zero modes on a kink, there may also be fermionic bound states. Such bound states would have a non-vanishing energy eigenvalue ω with $0<\omega<m$. Since the energy eigenvalue is less than the asymptotic mass, the fermion would be bound to the wall. We examine whether the model in Eq. (5.14) leads to fermionic bound states on a Z_2 kink.

For convenience we work in one spatial dimension. Then spinors have two components and there are only two gamma matrices, which can be taken to be

$$\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \mathrm{i}\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.66}$$

Then the Dirac equation $\mathrm{i}\partial\!\!\!/\psi - g\phi_\mathrm{k}\psi = 0$ together with $\psi = \exp(-\mathrm{i}\omega t)\xi$ gives

$$\partial_x\xi_1 = -(\omega + g\phi_\mathrm{k})\xi_2$$
$$\partial_x\xi_2 = +(\omega - g\phi_\mathrm{k})\xi_1 \tag{5.67}$$

where

$$\xi = \begin{pmatrix} \xi_1(x) \\ \xi_2(x) \end{pmatrix} \tag{5.68}$$

and we are interested in solutions with

$$0 < \omega < m_{\mathrm{f}} \equiv g\eta \tag{5.69}$$

The boundary conditions at the origin for ξ_1 and ξ_2 may be determined by noting that we are free to rescale both ξ_1 and ξ_2 by a constant factor. So we can set $\xi_1(0) = +1$. Further, using the symmetry $\phi_{\mathrm{k}}(x) = -\phi_{\mathrm{k}}(-x)$, we find that the equations are invariant if we replace $\xi_1(x)$ by $c\xi_2(-x)$ and $\xi_2(x)$ by $c\xi_1(-x)$, where c is a constant. Hence

$$\xi_1(x) = c\xi_2(-x), \qquad \xi_2(x) = c\xi_1(-x) \tag{5.70}$$

This gives

$$\xi_1(x) = c\xi_2(-x) = c^2\xi_1(x) \tag{5.71}$$

Since $\xi_1(x)$ cannot vanish for all x, we get

$$c = \pm 1 \tag{5.72}$$

Therefore there are two possible boundary conditions at the origin

$$\xi_2(0) = \pm\xi_1(0) = \pm 1 \tag{5.73}$$

At infinity we require $\xi_1 \to 0$ and $\xi_2 \to 0$.

A numerical search for a solution with non-zero ω did not reveal any bound states for the range of parameters $0.1 < m_{\mathrm{f}}w < 20$, where w is the width of the kink. However this does not exclude the existence of fermion bound states (beside the zero mode) on kinks in other systems, and it remains an open problem to find systems where such bound states exist.

5.7 Open questions

1. Explore the classical and quantum physics of a domain wall with electrically charged bosonic and fermionic zero modes placed in an external magnetic field. What happens if the domain wall is moving?
2. Calculate the reflection of photons off a superconducting domain wall. Is the wall a good mirror? (See [184].)
3. Construct a system in which the kink has both a zero mode and a fermionic bound state.