*Bull. Aust. Math. Soc.* (First published online 2024), page 1 of 9*
doi:10.1017/S0004972724000248
*Provisional—final page numbers to be inserted when paper edition is published

# THE WIGNER PROPERTY OF SMOOTH NORMED SPACES

**XUJIAN HUANG, JIABIN LIU and SHUMING WANG**

(Received 10 February 2024; accepted 16 March 2024)

**Abstract**

We prove that every smooth complex normed space $X$ has the Wigner property. That is, for any complex normed space $Y$ and every surjective mapping $f : X \to Y$ satisfying

$$\{\|f(x) + \alpha f(y)\| : \alpha \in \mathbb{T}\} = \{\|x + \alpha y\| : \alpha \in \mathbb{T}\}, \quad x, y \in X,$$

where $\mathbb{T}$ is the unit circle of the complex plane, there exists a function $\sigma : X \to \mathbb{T}$ such that $\sigma \cdot f$ is a linear or anti-linear isometry. This is a variant of Wigner's theorem for complex normed spaces.

2020 *Mathematics subject classification*: primary 46B04; secondary 39B05, 47J05.

*Keywords and phrases*: Wigner's theorem, Wigner property, phase-isometry, smooth space.

## 1. Introduction

Let $X$ and $Y$ be normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. Denote $\mathbb{T} = \{\alpha \in \mathbb{F} : |\alpha| = 1\}$. A function $\sigma : X \to \mathbb{T}$ whose values are of modulus one is called a *phase function* on $X$. A mapping $f : X \to Y$ is said to be *phase equivalent* to another mapping $g : X \to Y$ if there exists a phase function $\sigma : X \to \mathbb{T}$ such that $f = \sigma \cdot g$, that is, $f(x) = \sigma(x)g(x)$ for $x \in X$.

The celebrated Wigner's unitary–anti-unitary theorem is particularly important in the mathematical foundations of quantum mechanics. It states that for inner product spaces $(X, \langle\cdot,\cdot\rangle)$ and $(Y, \langle\cdot,\cdot\rangle)$ over $\mathbb{F}$, a mapping $f : X \to Y$ satisfies

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|, \quad x, y \in X \tag{1.1}$$

if and only if $f$ is phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$. There are several proofs of this result, see [1, 2, 4, 6, 13, 18, 22] to list just some of them. For further generalisations of this

The first and second authors were supported by the Natural Science Foundation of Tianjin Municipal Science and Technology Commission (Grant No. 22JCYBJC00420) and the National Natural Science Foundation of China (Grant No. 12271402). The third author was supported by the National Natural Science Foundation of China (Grant Nos. 12201459 and 12071358).

fundamental result, we mention the papers [3, 5, 15, 17]. Wigner's theorem is very important and therefore worthy of study from various points of view.

A mapping $f : X \to Y$ between normed spaces over $\mathbb{F}$ is called a *phase-isometry* if it satisfies the functional equation

$$\{\|f(x) + \alpha f(y)\| : \alpha \in \mathbb{T}\} = \{\|x + \alpha y\| : \alpha \in \mathbb{T}\}, \quad x, y \in X. \tag{1.2}$$

It is worth noting that if $X$ and $Y$ are inner product spaces, then $f : X \to Y$ satisfies (1.1) if and only if $f$ satisfies (1.2). Indeed, with the substitution $y = x$, we deduce from either (1.1) or (1.2) that $f$ is norm-preserving. Squaring the norms on both sides of (1.2), it follows that (1.2) holds if and only if

$$\{\mathrm{Re}(\alpha\langle f(x), f(y)\rangle) : \alpha \in \mathbb{T}\} = \{\mathrm{Re}(\alpha\langle x, y\rangle) : \alpha \in \mathbb{T}\}, \quad x, y \in X,$$

which happens if and only if (1.1) holds. Due to Wigner's theorem, a mapping between inner product spaces is a phase-isometry if and only if it is phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$.

When $X$ and $Y$ are normed spaces, one can easily see that if $f : X \to Y$ is phase equivalent to a linear or anti-linear isometry, then $f$ is a phase-isometry. For instance, if $f = \sigma \cdot U$, where $U$ is a linear isometry and $\sigma : X \to \mathbb{T}$ is a phase function, then for $x, y \in X$ and $\alpha \in \mathbb{T}$,

$$\begin{aligned}\|f(x) + \alpha f(y)\| &= \|\sigma(x)U(x) + \alpha\sigma(y)U(y)\| = \|U(\sigma(x)x + \alpha\sigma(y)y)\| \\ &= \|\sigma(x)x + \alpha\sigma(y)y\| = \|x + \alpha\overline{\sigma(x)}\sigma(y)y\|\end{aligned}$$

and then

$$\|x + \alpha y\| = \|x + (\alpha\sigma(x)\overline{\sigma(y)})\overline{\sigma(x)}\sigma(y)y\| = \|f(x) + \alpha\sigma(x)\overline{\sigma(y)}f(y)\|.$$

Similar reasoning applies when $U$ is an anti-linear isometry. Therefore, a natural problem posed by Maksa and Páles [13] (the case $\mathbb{F} = \mathbb{R}$), and Wang and Bugajewski [23] (the case $\mathbb{F} = \mathbb{C}$), can be restated as the following problem.

PROBLEM 1.1. Under what conditions is every phase-isometry between two normed spaces over $\mathbb{F}$ phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$?

A normed space $X$ over $\mathbb{F}$ is said to have the *Wigner property* if for any normed space $Y$ over $\mathbb{F}$, every surjective phase-isometry $f : X \to Y$ is phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$.

There have been several recent papers considering Problem 1.1 or the Wigner property in the case $\mathbb{F} = \mathbb{R}$. For relevant results, please refer to [7–9, 11–13, 19–21, 23]. In particular, Tan and Huang [19] proved that smooth real normed spaces have the Wigner property. Further, Ilišević *et al.* [9] proved that any real normed spaces have the Wigner property. However, to the best of our knowledge, apart from the case where $X$ and $Y$ are inner product spaces, there has been no progress in addressing Problem 1.1 in the case $\mathbb{F} = \mathbb{C}$. The aim of this paper is to give a partial solution for the case

$\mathbb{F} = \mathbb{C}$. Specifically, we show that every smooth complex normed space has the Wigner property. As a by-product, we give a Figiel-type result for phase-isometries. Although our paper is interesting in its own right, we hope that it will serve as a stepping stone to show that all complex normed spaces have the Wigner property.

## 2. Results

In the remainder of this paper, unless otherwise specified, all the normed spaces are over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Although the real case has been solved, for the sake of brevity and universality, we will present our lemmas, theorems and proofs in the united form $\mathbb{F}$ rather than the single form $\mathbb{C}$. For a normed space $X$, we use the notation $S_X, B_X$ and $X^*$ to represent the unit sphere, closed unit ball and dual space of $X$, respectively. The set of positive integers is denoted by $\mathbb{N}$.

We start this section with a simple and frequently-used property of phase-isometries between two normed spaces.

LEMMA 2.1. *Let $X$ and $Y$ be normed spaces and $f : X \to Y$ a phase-isometry. Then $f$ is a norm-preserving map. Moreover, if $f$ is surjective, then*

$$\{f(\alpha x) : \alpha \in \mathbb{T}\} = \{\alpha f(x) : \alpha \in \mathbb{T}\}, \quad x \in X.$$

PROOF. With the substitution $y = x$, it follows from (1.2) that

$$2\|f(x)\| = \max\{\|f(x) + \alpha f(x)\| : \alpha \in \mathbb{T}\} = \max\{\|x + \alpha x\| : \alpha \in \mathbb{T}\} = 2\|x\|,$$

which shows that $f$ is norm-preserving.

Now suppose that $f$ is surjective. Let us take a nonzero $x \in X$ and $\alpha \in \mathbb{T}$. The surjectivity guarantees that there exists some $y \in X$ such that $f(y) = \alpha f(x)$. Then

$$\min\{\|y + \beta x\| : \beta \in \mathbb{T}\} = \min\{\|f(y) + \beta f(x)\| : \beta \in \mathbb{T}\} = 0,$$

which implies that

$$\{\alpha f(x) : \alpha \in \mathbb{T}\} \subset \{f(\alpha x) : \alpha \in \mathbb{T}\}.$$

Moreover, we conclude from (1.2) that

$$\min\{\|f(\alpha x) + \beta f(x)\| : \beta \in \mathbb{T}\} = \min\{\|\alpha x + \beta x\| : \beta \in \mathbb{T}\} = 0,$$

which shows that

$$\{f(\alpha x) : \alpha \in \mathbb{T}\} \subset \{\alpha f(x) : \alpha \in \mathbb{T}\}.$$

This competes the proof. □

From [19, Lemma 2], it follows that every surjective phase-isometry between two real normed spaces is injective. The following example shows that a surjective phase-isometry between two complex normed spaces may not be injective.

EXAMPLE 2.2. Let $X$ be a complex normed space and $x_0 \in X\backslash\{0\}$. Define $f : X \to X$ by $f(\alpha x_0) = \alpha^2 x_0$ for all $\alpha \in \mathbb{T}$ and $f(x) = x$ otherwise. Then $f$ is a surjective phase-isometry, but it is not injective since $f(-x_0) = x_0 = f(x_0)$.

In Example 2.2, $f$ is phase equivalent to the identity mapping, letting the phase function $\sigma$ be $\sigma(\alpha x_0) = \alpha$ for all $\alpha \in \mathbb{T}$ and $\sigma(x) = 1$ otherwise.

Recall that a *support functional* $\phi$ at $x \in X\backslash\{0\}$ is a norm-one linear functional in $X^*$ such that $\phi(x) = \|x\|$. Denote by $D(x)$ the set of all support functionals at $x \neq 0$, that is,

$$D(x) = \{\phi \in S_{X^*} : \phi(x) = \|x\|\}.$$

The Hahn–Banach theorem implies that $D(x) \neq \emptyset$ for every $x \in X\backslash\{0\}$. A normed space $X$ is said to be *smooth* at $x \neq 0$ if there exists a unique supporting functional at $x$, that is, $D(x)$ consists of only one element. If $X$ is smooth at every $x \neq 0$, then $X$ is said to be *smooth*. It follows from [14, Proposition 5.4.20] that each subspace of a smooth normed space is smooth.

Recall also the concept of Gateaux differentiability. Let $X$ be a normed space, $x, y \in X$. Define

$$G_+(x, y) := \lim_{t\to 0^+, t\in\mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t\to +\infty, t\in\mathbb{R}} (\|tx + y\| - \|tx\|)$$

and

$$G_-(x, y) := \lim_{t\to 0^-, t\in\mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t\to +\infty, t\in\mathbb{R}} (\|tx\| - \|tx - y\|).$$

It is known [14, 16] that both $G_+(x, y)$ and $G_-(x, y)$ exist for each $x, y \in X$ and

$$G_+(x, y) = \max\{\operatorname{Re}\phi(y) : \phi \in D(x)\}, \quad G_-(x, y) = \min\{\operatorname{Re}\phi(y) : \phi \in D(x)\}.$$

We say that the norm of $X$ is *Gateaux differentiable* at $x \neq 0$ whenever $G_+(x, y) = G_-(x, y)$ for all $y \in X$, in which case the common value of $G_+(x, y)$ and $G_-(x, y)$ is denoted by $G(x, y)$. It is easy to see that a normed space $X$ is smooth at $x$ if and only if the norm is Gateaux differentiable at $x$.

A point $\phi \in S_{X^*}$ is said to be a *$w^*$-exposed point of $B_{X^*}$* provided that $\phi$ is the only supporting functional for some smooth point $u \in S_X$. Recently, Tan and Huang [19] showed that for every phase-isometry $f$ of a real normed space $X$ into another real normed space $Y$ and every $w^*$-exposed point $\phi$ of $B_{X^*}$, there exists $\varphi \in S_{Y^*}$ such that $\phi(x) = \pm\varphi(f(x))$ for all $x \in X$. This result can be viewed as an extension of Figiel's theorem, which plays an important role in the study of isometric embedding. We will present a similar result for a phase-isometry between two normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

LEMMA 2.3. *Let $X$ and $Y$ be normed spaces and $f : X \to Y$ a phase-isometry. Then for every $w^*$-exposed point $\phi$ of $B_{X^*}$, there exists $\varphi \in S_{Y^*}$ such that*

$$|\phi(x)| = |\varphi(f(x))|, \quad x \in X.$$

PROOF. Let $u \in S_X$ be a smooth point such that $\phi(u) = 1$. For every $n \in \mathbb{N}$, the Hahn–Banach theorem guarantees the existence of $\varphi_n \in S_{Y^*}$ such that

$$\varphi_n(f(nu)) = \|f(nu)\| = \|nu\| = n.$$

https://doi.org/10.1017/S0004972724000248 Published online by Cambridge University Press

For $t \in [0, n]$, there exists some $\alpha_{t,n} \in \mathbb{T}$ such that

$$\|f(nu) - \alpha_{t,n} f(tu)\| = \|nu - tu\| = n - t.$$

Consequently, we deduce that

$$\begin{aligned} 2n &= |\varphi_n(f(nu) - \alpha_{t,n} f(tu)) + \varphi_n(f(nu) + \alpha_{t,n} f(tu))| \\ &\le |\varphi_n(f(nu) - \alpha_{t,n} f(tu))| + |\varphi_n(f(nu) + \alpha_{t,n} f(tu))| \\ &\le \|f(nu) - \alpha_{t,n} f(tu)\| + \|f(nu) + \alpha_{t,n} f(tu)\| \\ &\le (n-t) + (n+t) = 2n, \end{aligned}$$

which implies that $\varphi_n(\alpha_{t,n} f(tu)) = t$. This means that for each $t \in (0, n]$, there exists a unique $\alpha_{t,n} \in \mathbb{T}$ such that $\varphi_n(f(tu)) = \overline{\alpha_{t,n}} t$. By Alaoglu's theorem, the sequence $\{\varphi_n\}$ has a cluster point $\varphi \in S_{Y^*}$ in the $w^*$ topology. It follows that for each $t > 0$, there exists $\alpha_t \in \mathbb{T}$ depending only on $t$ such that $\varphi(f(tu)) = \alpha_t t$.

For each $x \in X$, there exist $\alpha_x, \beta_x \in \mathbb{T}$ such that $\alpha_x \phi(x) = |\phi(x)|$ and $\beta_x \varphi(f(x)) = |\varphi(f(x))|$. For each $n \in \mathbb{N}$, there exists $\alpha_{x,n}, \beta_{x,n} \in \mathbb{T}$ such that

$$\begin{aligned} \|nu - \alpha_x x\| = \|f(nu) - \alpha_{x,n}\alpha_n f(x)\| &\ge |\varphi(f(nu)) - \alpha_{x,n}\alpha_n \varphi(f(x))| \\ &= |\alpha_n n - \alpha_{x,n}\alpha_n \varphi(f(x))| = |n - \alpha_{x,n}\varphi(f(x))| \end{aligned}$$

and

$$\begin{aligned} |n + \beta_x \varphi(f(x))| = |\alpha_n n + \alpha_n \beta_x \varphi(f(x))| &= |\varphi(f(nu)) + \alpha_n \beta_x \varphi(f(x))| \\ &\le \|f(nu) + \alpha_n \beta_x f(x)\| = \|nu + \beta_{x,n} x\|. \end{aligned}$$

Given that $\mathbb{T}$ is compact, there must be a strictly increasing sequence $\{n_j : j \in \mathbb{N}\}$ in $\mathbb{N}$ and $\alpha'_x, \beta'_x \in \mathbb{T}$ such that $\lim_{j\to\infty} \alpha_{x,n_j} = \alpha'_x$ and $\lim_{j\to\infty} \beta_{x,n_j} = \beta'_x$. Since $\phi$ is the only supporting functional at $u$,

$$\begin{aligned} |\phi(x)| &= \operatorname{Re} \phi(\alpha_x x) = \lim_{j\to\infty} (\|n_j u\| - \|n_j u - \alpha_x x\|) \\ &\le \lim_{j\to\infty} (n_j - |n_j - \alpha_{x,n_j} \varphi(f(x))|) = \lim_{j\to\infty} (n_j - |n_j - \alpha'_x \varphi(f(x))|) \\ &= \operatorname{Re} (\alpha'_x \varphi(f(x))) \le |\varphi(f(x))| \end{aligned}$$

and

$$\begin{aligned} |\varphi(f(x))| &= \operatorname{Re} (\beta_x \varphi(f(x))) = \lim_{j\to\infty} (|n_j + \beta_x \varphi(f(x))| - n_j) \\ &\le \lim_{j\to\infty} (\|n_j u + \beta_{x,n_j} x\| - \|n_j u\|) = \lim_{j\to\infty} (\|n_j u + \beta'_x x\| - \|n_j u\|) \\ &= \operatorname{Re} \phi(\beta'_x x) \le |\phi(x)|. \end{aligned}$$

This completes the proof. □

Let $V$ be a vector space. For $M \subset V$, $[M]$ denotes the subspace generated by $M$. If $x, y \in V$, then we write $[x] := [\{x\}]$ and $[x, y] := [\{x, y\}]$ for simplicity.

https://doi.org/10.1017/S0004972724000248 Published online by Cambridge University Press

LEMMA 2.4. *Let $X$ and $Y$ be normed spaces with $X$ being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. Then for every $x \in X$,*

$$f([x]) = [f(x)].$$

PROOF. We first prove that $[f(x)] \subset f([x])$ for each $x \in X$. Assume, for a contradiction, that $tf(x) \notin f([x])$ for some nonzero $x \in X$ and $t \in \mathbb{F}$. Since $f$ is surjective, there exists $y \in X$ such that $f(y) = tf(x)$. The function $s \mapsto \|y - sx\|$ is continuous and its value tends to infinity when $|s|$ tends to infinity. Hence, there is at least one point $s_0 \in \mathbb{F}$ such that

$$d := d(y, [x]) = \min\{\|y - sx\| : s \in \mathbb{F}\} = \|y - s_0x\| > 0.$$

Set $E := [x, y]$. By the Hahn–Banach theorem, there exists $\phi \in S_{E^*}$ which satisfies $\phi(y) = d$ and $\phi(x) = 0$. Note that $E$ being a two-dimensional subspace of $X$ is reflexive. This guarantees the existence of some $z \in S_E$ such that $\phi(z) = 1$. Since $X$ is smooth, so is its subspace $E$. Therefore, $\phi$ is the only supporting functional at $z \in S_E$. We apply Lemma 2.3 to $f|_E : E \to Y$ to obtain $\varphi \in S_{Y^*}$ such that $|\phi| = |\varphi \circ f|$ on $E$. Then

$$0 < d = |\phi(y)| = |\varphi(f(y))| = |\varphi(tf(x))| = |t||\varphi(f(x))| = |t||\phi(x)| = 0,$$

which is a contradiction. This proves $[f(x)] \subset f([x])$.

Conversely, fix a nonzero $x \in X$. For each $r \in (0, +\infty)$, by the above inclusion and the norm preserving property of $f$, there exists some $\alpha_r \in \mathbb{T}$ such that $r^{-1}f(rx) = f(\alpha_r x)$. For each $\alpha \in \mathbb{T}$, by Lemma 2.1, there exist $\beta_{r,\alpha}, \alpha'_r \in \mathbb{T}$ such that

$$f(r\alpha x) = \beta_{r,\alpha}f(rx) = \beta_{r,\alpha}rf(\alpha_r x) = r\beta_{r,\alpha}\alpha'_r f(x),$$

which implies that $f([x]) \subset [f(x)]$. The proof is complete. □

Note that the conclusion of Lemma 2.4 is equivalent to

$$\{f(r\alpha x) : \alpha \in \mathbb{T}\} = \{r\alpha f(x) : \alpha \in \mathbb{T}\}, \quad x \in X,\ r \in [0, +\infty).$$

LEMMA 2.5. *Let $X$ and $Y$ be normed spaces with $X$ being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. Then for every $x, y \in X$,*

$$\{G_+(f(x), \alpha f(y)) : \alpha \in \mathbb{T}\} = \{G(x, \alpha y) : \alpha \in \mathbb{T}\} = \{G_-(f(x), \alpha f(y)) : \alpha \in \mathbb{T}\}.$$

PROOF. We only prove the first equality, the second being similar. Let $x, y \in X$ be nonzero and $\alpha \in \mathbb{T}$. For each $n \in \mathbb{N}$, Lemma 2.4 and (1.2) imply that there exist $\alpha_n, \beta_n, \gamma_n \in \mathbb{T}$ such that $f(nx) = \alpha_n nf(x)$ and

$$\|f(nx) + \alpha_n\alpha f(y)\| = \|nx + \beta_n y\|, \quad \|f(nx) + \alpha_n\gamma_n f(y)\| = \|nx + \alpha y\|.$$

By the compactness of $\mathbb{T}$, there is a strictly increasing sequence $\{n_j : j \in \mathbb{N}\}$ in $\mathbb{N}$ and $\beta, \gamma \in \mathbb{T}$ such that $\lim_{j\to\infty} \beta_{n_j} = \beta$ and $\lim_{j\to\infty} \gamma_{n_j} = \gamma$. Then

https://doi.org/10.1017/S0004972724000248 Published online by Cambridge University Press

$$\begin{aligned} G_+(f(x), \alpha f(y)) &= \lim_{j\to\infty}(\|n_j f(x) + \alpha f(y)\| - \|n_j f(x)\|) \\ &= \lim_{j\to\infty}(\|f(n_j x) + \alpha_{n_j}\alpha f(y)\| - \|n_j f(x)\|) \\ &= \lim_{j\to\infty}(\|n_j x + \beta_{n_j} y\| - \|n_j x\|) = \lim_{j\to\infty}(\|n_j x + \beta y\| - \|n_j x\|) = G(x, \beta y) \end{aligned}$$

and

$$\begin{aligned} G(x, \alpha y) &= \lim_{j\to\infty}(\|n_j x + \alpha y\| - \|n_j x\|) \\ &= \lim_{j\to\infty}(\|f(n_j x) + \alpha_{n_j}\gamma_{n_j} f(y)\| - \|f(n_j x)\|) \\ &= \lim_{j\to\infty}(\|n_j f(x) + \gamma_{n_j} f(y)\| - \|n_j f(x)\|) \\ &= \lim_{j\to\infty}(\|n_j f(x) + \gamma f(y)\| - \|n_j f(x)\|) = G_+(f(x), \gamma f(y)). \end{aligned}$$

The proof is complete. □

LEMMA 2.6. *Let $X$ and $Y$ be normed spaces with $X$ being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. Then $Y$ is smooth.*

PROOF. Let $x \in X$ be a nonzero element with the unique supporting functional $\phi_x \in D(x)$. It suffices to prove that $D(f(x))$ is a singleton set. Let $\varphi, \psi \in D(f(x))$ and $f(y) \in \ker\varphi$. For each $\alpha \in \mathbb{T}$, Lemma 2.5 implies that there exists $\beta, \gamma \in \mathbb{T}$ such that

$$\mathrm{Re}(\alpha\phi_x(y)) = \mathrm{Re}\phi_x(\alpha y) = G(x, \alpha y) = G_+(f(x), \beta f(y)) \geq \mathrm{Re}\varphi(\beta f(y)) = 0$$

and

$$\mathrm{Re}(\alpha\psi(f(y))) = \mathrm{Re}\psi(\alpha f(y)) \leq G_+(f(x), \alpha f(y)) = G(x, \gamma y) = \mathrm{Re}\phi_x(\gamma y).$$

Using the arbitrariness of $\alpha \in \mathbb{T}$ twice gives $\phi_x(y) = 0$ by the first inequality and therefore $\psi(f(y)) = 0$ by the second inequality. This shows that $\ker\varphi \subset \ker\psi$. Thus, $\psi = \lambda\varphi$ for some $\lambda \in \mathbb{F}$. Considering that $\psi, \varphi \in D(f(x))$, we find that $\lambda = 1$. This implies that $\psi = \varphi$, which completes the proof. □

Recently, Ilišević and Turnšek [10, Theorem 2.2 and Remark 2.1] generalised Wigner's theorem to smooth normed spaces via semi-inner products. This can be translated into the following theorem in the language of supporting functionals.

THEOREM 2.7. *Let $X$ and $Y$ be smooth normed spaces over $\mathbb{F}$ and $f : X \to Y$ a surjective mapping satisfying, for all nonzero $x, y \in X$,*

$$|\phi_{f(x)}(f(y))| = |\phi_x(y)|.$$

*Then $f$ is phase equivalent to a linear or anti-linear surjective isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear surjective isometry in the case $\mathbb{F} = \mathbb{R}$.*

Combining the above results gives our main theorem.

THEOREM 2.8. *Every smooth normed space has the Wigner property.*

PROOF. Let $X$ and $Y$ be normed spaces with $X$ being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. By Lemma 2.6, $Y$ is smooth. Then Lemma 2.5 implies that for all nonzero $x, y \in X$,

$$\{\mathrm{Re}\phi_{f(x)}(\alpha f(y)) : \alpha \in \mathbb{T}\} = \{\mathrm{Re}\phi_x(\alpha y) : \alpha \in \mathbb{T}\}.$$

Taking the maximum on both sides, for all nonzero $x, y \in X$,

$$|\phi_{f(x)}(f(y))| = |\phi_x(y)|.$$

By Theorem 2.7, $f$ is phase equivalent to a linear or anti-linear surjective isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear surjective isometry in the case $\mathbb{F} = \mathbb{R}$. This completes the proof. □

It is well known that $L^p(\mu)$ is a smooth normed space, where $\mu$ is a measure and $1 < p < \infty$. The following corollary is immediate.

COROLLARY 2.9. *$L^p(\mu)$ has the Wigner property, where $\mu$ is a measure and $1 < p < \infty$.*

## References


[1] D. F. Almeida and C. S. Sharma, 'The first mathematical proof of Wigner's theorem', *J. Nat. Geom.* **2** (1992), 113–123.

[2] V. Bargmann, 'Note on Wigner's theorem on symmetry operations', *J. Math. Phys.* **5** (1964), 862–868.

[3] G. Chevalier, 'Wigner's theorem and its generalizations', in: *Handbook of Quantum Logic and Quantum Structures* (eds. K. Engesser, D. M. Gabbay and D. Lehmann) (Elsevier Science B.V., Amsterdam, 2007), 429–475.

[4] G. P. Gehér, 'An elementary proof for the non-bijective version of Wigner's theorem', *Phys. Lett. A* **378**(30–31) (2014), 2054–2057.

[5] G. P. Gehér, 'Wigner's theorem on Grassmann spaces', *J. Funct. Anal.* **273**(9) (2017), 2994–3001.

[6] M. Győry, 'A new proof of Wigner's theorem', *Rep. Math. Phys.* **54** (2004), 159–167.

[7] X. Huang and D. Tan, 'Wigner's theorem in atomic $L_p$-spaces ($p > 0$)', *Publ. Math. Debrecen* **92**(3–4) (2018), 411–418.

[8] X. Huang and D. Tan, 'Min-phase-isometries and Wigner's theorem on real normed spaces', *Results Math.* **77**(4) (2022), Article no. 152, 15 pages.

[9] D. Ilišević, M. Omladič and A. Turnšek, 'Phase-isometries between normed spaces', *Linear Algebra Appl.* **612** (2021), 99–111.

[10] D. Ilišević and A. Turnšek, 'On Wigner's theorem in smooth normed spaces', *Aequationes Math.* **94**(6) (2020), 1257–1267.

[11] D. Ilišević and A. Turnšek, 'On Wigner's theorem in strictly convex normed spaces', *Publ. Math. Debrecen* **97**(3–4) (2020), 393–401.

[12] Y. Li and D. Tan, 'Wigner's theorem on the Tsirelson space $T$', *Ann. Funct. Anal.* **10**(4) (2019), 515–524.

[13] G. Maksa and Z. Páles, 'Wigner's theorem revisited', *Publ. Math. Debrecen* **81**(1–2) (2012), 243–249.

[14] R. E. Megginson, *An Introduction to Banach Space Theory* (Springer-Verlag, New York, 1998).

[15] L. Molnár, 'Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem', *J. Funct. Anal.* **194**(2) (2002), 248–262.

[16] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd edn (Springer-Verlag, Berlin, 1993).


https://doi.org/10.1017/S0004972724000248 Published online by Cambridge University Press

[17] W. Qian, L. Wang, W. Wu and W. Yuan, ‘Wigner-type theorem on transition probability preserving maps in semifinite factors’, *J. Funct. Anal.* **276**(6) (2019), 1773–1787.
[18] J. Rätz, ‘On Wigner’s theorem: remarks, complements, comments, and corollaries’, *Aequationes Math.* **52**(1–2) (1996), 1–9.
[19] D. Tan and X. Huang, ‘Phase-isometries on real normed spaces’, *J. Math. Anal. Appl.* **488**(1) (2020), Article no. 124058, 17 pages.
[20] D. Tan and X. Huang, ‘The Wigner property for CL-spaces and finite-dimensional polyhedral Banach spaces’, *Proc. Edinb. Math. Soc. (2)* **64**(2) (2021), 183–199.
[21] D. Tan and F. Zhang, ‘Min-phase-isometries in strictly convex normed spaces’, *Bull. Aust. Math. Soc.* **109**(1) (2024), 152–160.
[22] A. Turnšek, ‘A variant of Wigner’s functional equation’, *Aequationes Math.* **89**(4) (2015), 949–956.
[23] R. Wang and D. Bugajewski, ‘On normed spaces with the Wigner Property’, *Ann. Funct. Anal.* **11**(3) (2020), 523–539.

XUJIAN HUANG, Institute of Operations Research and Systems Engineering,
College of Science, Tianjin University of Technology, Tianjin 300384, PR China
e-mail: huangxujian86@sina.com

JIABIN LIU, College of Science,
Tianjin University of Technology, Tianjin 300384, PR China
e-mail: liujb98@163.com

SHUMING WANG, College of Science,
Tianjin University of Technology, Tianjin 300384, PR China
e-mail: smwang@email.tjut.edu.cn